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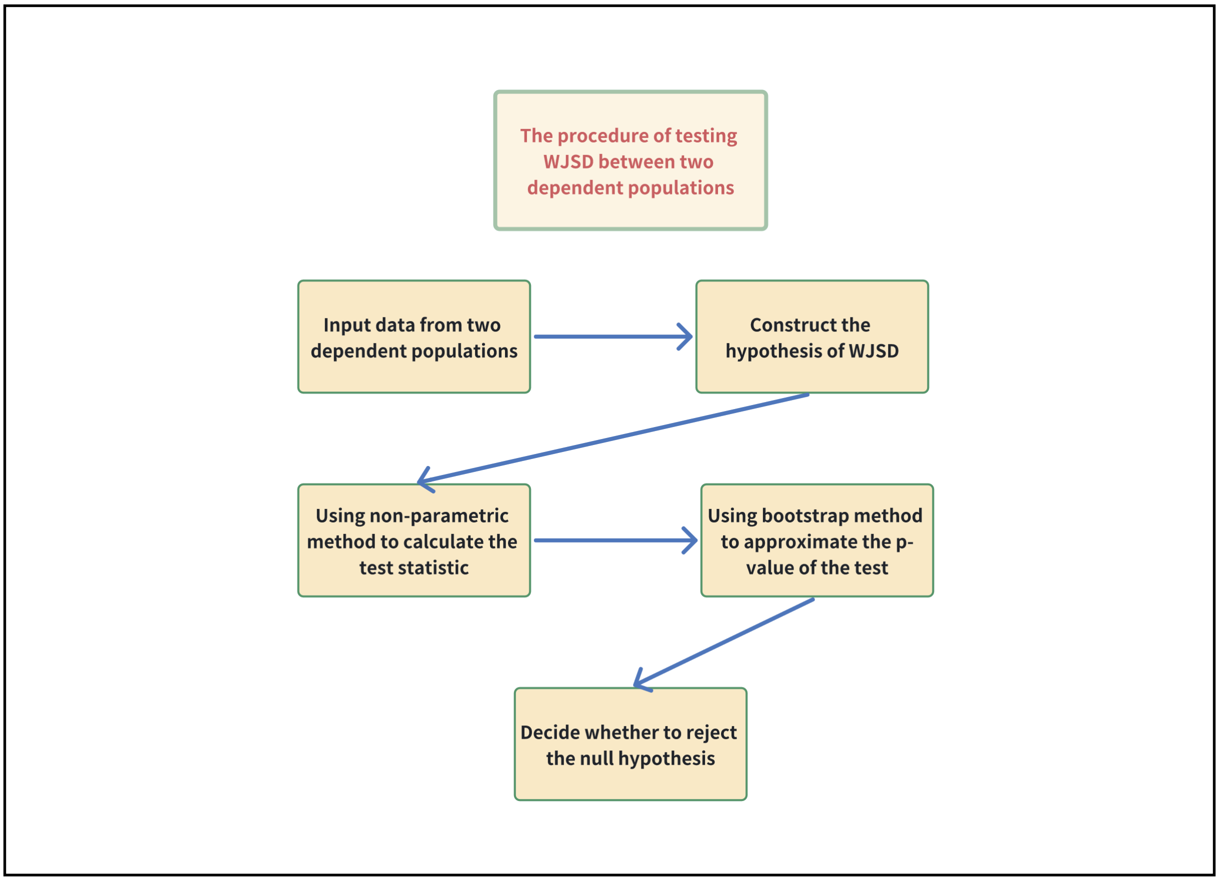
Improved nonparametric bootstrap tests for weak joint stochastic dominance

Weiqi Yang[1], Weiwei Zhuang[2] ✉, and Kehua Cheng[1]

[1]*School of Management, University of Science and Technology of China, Hefei 230026, China;*
[2]*International Institute of Finance, School of Management, University of Science and Technology of China, Hefei 230601, China*

✉Correspondence: Weiwei Zhuang, E-mail: weizh@ustc.edu.cn

Graphical abstract



This is the procedure of testing WJSD between two dependent populations.

Public summary

- Present a comparative analysis of populations with a dependency structure, utilizing a criterion known as weak joint stochastic dominance.
- Establish a consistent test for weak joint stochastic dominance, deriving the asymptotic properties of the statistics.
- Propose an improved bootstrap method for estimating precise *p*-values, ensuring the consistency of the bootstrap method and guaranteeing the accuracy of our approach.

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[2]International Institute of Finance, School of Management, University of Science and Technology of China, Hefei 230601, China

✉Correspondence: Weiwei Zhuang, E-mail: weizh@ustc.edu.cn

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Abstract: The comparison of dependent random variables is pivotal across various domains, including poverty measurement and welfare evaluation. However, traditional stochastic dominance (SD) criteria neglect the dependency structure between variables, making them inadequate for comparing two dependent variables. To address this limitation, the concept of weak joint stochastic dominance (WJSD) has been introduced, offering deeper insights into the comparative magnitudes of dependent random variables. Nevertheless, previous studies have failed to established the accurate asymptotic properties of the statistics used for WJSD. Instead, these studies have relied on Monte Carlo method to estimate the upper boundaries of critical values, which may introduce potential inaccuracies. To mitigate these issues, this paper derives the asymptotic distribution of the WJSD statistic, verifies its consistency, and proposes an improved bootstrap methodology for precise p-value estimation. Our simulation studies reveal the enhanced performance of our method over traditional Monte Carlo methods, and its application to real data further highlights its practical applicability and relevance.

Keywords: stochastic dominance; nonparametric test; dependent random variables; asymptotic distribution; bootstrap method

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1 Introduction

The comparison of random variables plays a pivotal role in both societal and research contexts, finding applications across diverse fields such as poverty measurement and welfare evaluation[1,2]. Stochastic dominance (SD) has emerged as a more robust tool than the traditional mean-variance comparison method and has been extensively applied to economics, finance, and risk analysis.

The body of literature on statistical inference for testing the SD criterion is vast. Ref. [3] developed tests for various forms of SD using t-statistics to compare measures in two independent samples. Ref. [4] suggested comparing variables at a set number of arbitrarily selected points; however, these methods have shown inconsistency. To overcome this drawback, Ref. [5] introduced a Kolmogorov-Smirnov test for comparisons over a compact set, enhancing the robustness of SD tests. Subsequent improvements in testing methods for SD have been proposed by researchers including Refs. [6–10].

While the traditional SD criterion theoretically allows for the comparison of two dependent variables, it frequently fails to account for the underlying dependency structure, rendering it less effective for such analyses[11]. Although there have been proposed criteria to address stochastic dominance among dependent random variables[12,13], these primarily assess whether one variable consistently exceeds the other without considering the precise magnitude of the differences. To bridge this gap, Refs. [11,14] independently introduced a slight modification to the conventional stochastic dominance framework. This adjustment maintains the core principles of SD while integrating the dependency structure, leading to the concept of weak joint stochastic dominance (WJSD) as a novel benchmark for evaluating populations with dependent structures.

This proposed criterion evaluates the disparities between the random variables $X-Y$ and $Y-X$ through stochastic dominance. The approach leverages the differences $X-Y$ and $Y-X$ to inherently incorporate aspects of the joint distribution of (X,Y). Essentially, if the cumulative distribution function of $X-Y$ is less than that of $Y-X$, $X-Y$ generates larger differences with a higher probability than $Y-X$ does, thus making X more desirable than Y. This becomes particularly evident in the context of countermonotone random variables X and Y. When X increases, Y decreases, leading to a larger value of $X-Y$. This phenomenon is not captured by traditional stochastic dominance, which relies solely on marginal distributions.

Ref. [11] developed a nonparametric test for weak joint stochastic dominance but did not accurately determine the asymptotic properties of the statistic, relying on a Monte Carlo method to estimate the upper boundary of the critical value. This approach can lead to erroneous results, such as an elevated rejection rate when the populations are nearly identical, deviating from the expected nominal level α.

This paper addresses these limitations by deriving the asymptotic distribution of the statistic and establishing the consistency of the test. We introduce an improved bootstrap method for estimating p-values, ensuring both the method's consistency and the accuracy of our results. Compared with the basic Monte Carlo method, this enhanced bootstrap approach yields more precise p-value estimates and improves the testing power. We designate the collection of bounded real-valued functions defined on the set Λ, endowed with the uniform metric, as $l^\infty(\Lambda)$. Weak convergence in both the norm space and the product norm space is denoted by $\xrightarrow{\mathrm{w}}$.

The remainder of this paper is organized as follows: Section 2 introduces weak joint stochastic dominance and proposes a novel nonparametric statistic for its testing, including a comprehensive analysis of its asymptotic properties. Section 3 describes our bootstrap procedure and presents associated statistics, exploring their asymptotic behavior. Section 4 includes a simulation study to evaluate our method's performance against the Monte Carlo approach by Ref. [11]. Section 5 applies our methodology to empirical data. Finally, Section 6 concludes the stduy.

2 Hypothesis test of weak joint stochastic dominance

2.1 Test statistics

Suppose X and Y are two dependent random variables with continuous distribution functions. We say that X is smaller than Y in terms of weak joint stochastic dominance, denoted by $X \le_{st:wj} Y$, if and only if

$$\mathrm{P}(X-Y>t) \le \mathrm{P}(Y-X>t) \quad \text{for all} \quad t \in \mathbb{R}.$$

Let $\{(X_i, Y_i)\}_{i=1}^n$ be independent random samples from (X, Y). We aim to test

$$H_0 : X \le_{st:wj} Y \quad \text{vs.} \quad H_1 : X \not\le_{st:wj} Y.$$

Furthermore, we denote the distributions of $Y-X$ and $X-Y$ as $\mathcal{F}_1$ and $\mathcal{F}_2$, and their common support as Λ. Then the above hypotheses can be transferred to

$$H_0 : \mathcal{F}_1(x) \le \mathcal{F}_2(x) \quad \text{for all } x \in \Lambda \quad \text{vs.}$$
$$H_1 : \mathcal{F}_1(x) > \mathcal{F}_2(x) \quad \text{for some } x \in \Lambda.$$

Therefore, the main objective is to estimate the difference between $\mathcal{F}_1(x)$ and $\mathcal{F}_2(x)$. Using the empirical distribution function, we can naturally obtain estimators $\hat{\mathcal{F}}_i$ of $\mathcal{F}_i$, $i = 1, 2$, as

$$\hat{\mathcal{F}}_1(x) = \frac{1}{n}\sum_{i=1}^{n} I_{\{Y_i - X_i \le x\}}, \quad \hat{\mathcal{F}}_2(x) = \frac{1}{n}\sum_{i=1}^{n} I_{\{X_i - Y_i \le x\}}.$$

Thus, the difference $\varphi(x) = \mathcal{F}_1(x) - \mathcal{F}_2(x)$ can be estimated as $\hat{\varphi}(x) = \hat{\mathcal{F}}_1(x) - \hat{\mathcal{F}}_2(x)$.

Note that H_0 is true if and only if $\sup_{x\in\Lambda}\varphi(x) \le 0$. Therefore, we define the statistic based on the supremum of $\hat{\varphi}(x)$, as

$$\hat{T} = \sqrt{n}\sup_{x\in\Lambda}\{\hat{\varphi}(x)\} = \sqrt{n}\mathcal{T}(\hat{\varphi}).$$

Here, $\mathcal{T}(l) = \|l\|_\infty$ is a functional from $\ell^\infty(\Lambda)$ to $\mathbb{R}$, where $\|.\|_\infty$ represents the uniform norm in the space $\ell^\infty(\Lambda)$. We subsequently study the asymptotic properties of this statistic.

2.2 Asymptotic properties

Before presenting the asymptotic property of the statistic, we provide the following assumption.

Assumption 1. For the populations X and Y, we assume that their cumulative distribution functions are continuously differentiable on the interior of its support with a strictly positive derivative, and their common support is compact.

To satisfy the condition that two distributions possess identical support, one typically selects two populations that are of the same category, for instance, both pertaining to income distributions. On the basis of Assumption 1, for $i = 1, 2$, $\mathcal{F}_i$ is continuously differentiable on the interior of its support with a strictly positive derivative. In addition, their common support, denoted as Λ, is compact.

First, we ascertain the asymptotic weak convergence of $\sqrt{n}(\hat{\varphi} - \varphi)$.

Theorem 1. Under Assumption 1, as $n \to \infty$,

$$\sqrt{n}(\hat{\varphi} - \varphi) \xrightarrow{\mathrm{w}} \mathcal{G} \ \text{ in } \ell^\infty(\Lambda),$$

where $\mathcal{G}$ is a tight process with mean zero and covariance

$$\mathrm{Cov}(\mathcal{G}(u_1), \mathcal{G}(u_2)) = \sum_{i=1}^{2}\sum_{j=1}^{2}\mathcal{F}_i(u_1)\mathcal{F}_j(u_2) + \sum_{i=1}^{2}\mathcal{F}_i(u_1 \wedge u_2) + \mathrm{I}_{\{u_1+u_2\ge 0\}}\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\mathcal{F}_i(u_j) - 2\right),$$

for any $u_1, u_2 \in \Lambda$.

Proof. Note that

$$\sqrt{n}(\hat{\varphi} - \varphi) = \sqrt{n}(\hat{\mathcal{F}}_1 - \mathcal{F}_1) - \sqrt{n}(\hat{\mathcal{F}}_2 - \mathcal{F}_2).$$

Applying the classical Donsker theorem yields the weak convergence

$$\sqrt{n}(\hat{\mathcal{F}}_i - \mathcal{F}_i) \xrightarrow{\mathrm{w}} \mathcal{G}_i \text{ in } \ell^\infty(\Lambda), \quad i = 1, 2,$$

where $\mathcal{G}_i$ is a Gaussian process with a zero mean and a covariance function given by

$$\mathrm{Cov}(\mathcal{G}_i(u_1), \mathcal{G}_i(u_2)) = \mathcal{F}_i(u_1 \wedge u_2) - \mathcal{F}_i(u_1)\mathcal{F}_i(u_2) \quad \text{for any } u_1, u_2 \in \Lambda.$$

On the basis of Theorem 1.4.8 in Ref. [15], $\sqrt{n}(\hat{\varphi} - \varphi)$ converges to a zero-mean tight process denoted as $\mathcal{G}$. Given any $u \in \Lambda$, we observe that

$$\begin{aligned}
&\mathrm{E}\left\{\sqrt{n}(\hat{\mathcal{F}}_1(u) - \mathcal{F}_1(u))\sqrt{n}(\hat{\mathcal{F}}_2(u) - \mathcal{F}_2(u))\right\} = \\
&n\mathrm{E}\left\{\hat{\mathcal{F}}_1(u)\hat{\mathcal{F}}_2(u) - \mathcal{F}_1(u)\mathcal{F}_2(u)\right\} = \\
&\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\mathrm{E}\mathrm{I}_{\{Y_j - X_j \le u, X_k - Y_k \le u\}} - n\mathcal{F}_1(u)\mathcal{F}_2(u) = \\
&\mathrm{I}_{\{u\ge 0\}}\left(\sum_{i=1}^{2}\mathcal{F}_i(u) - 1\right) - \mathcal{F}_1(u)\mathcal{F}_2(u).
\end{aligned} \tag{1}$$

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Thus, from Eq. (1), the variance function of $\mathcal{G}$ can be articulated as

$$\begin{aligned}&\mathrm{Var}\left\{\sqrt{n}(\widehat{\varphi}(u)-\varphi(u))\right\}=\\&\mathrm{Var}\left\{\sqrt{n}(\hat{\mathcal{F}}_1(u)-\mathcal{F}_1(u)\right\}+\mathrm{Var}\left\{\sqrt{n}(\hat{\mathcal{F}}_2(u)-\mathcal{F}_2(u)\right\}-\\&2\mathrm{E}\left\{\sqrt{n}(\hat{\mathcal{F}}_1(u)-\mathcal{F}_1(u))\sqrt{n}(\hat{\mathcal{F}}_2(u)-\mathcal{F}_2(u)\right\}=\\&\mathcal{F}_1(u)+\mathcal{F}_2(u)-(\mathcal{F}_1(u)-\mathcal{F}_2(u))^2-2I_{\{u\geq 0\}}\left(\sum_{i=1}^{2}\mathcal{F}_i(u)-1\right).\end{aligned}$$

To compute the covariance function of $\mathcal{G}$, for any $u_1,u_2\in\Lambda$, $i,j=1,2$, we define

$$\mathrm{E}_{ij}(u_1,u_2)=\mathrm{E}\left\{\sqrt{n}(\hat{\mathcal{F}}_i(u_1)-\mathcal{F}_i(u_1))\sqrt{n}(\hat{\mathcal{F}}_j(u_2)-\mathcal{F}_j(u_2))\right\}.$$

Notably, for $i=1,2$,

$$\begin{aligned}\mathrm{E}_{ii}(u_1,u_2)=&\mathrm{Cov}\left\{\sqrt{n}(\hat{\mathcal{F}}_i(u_1)-\mathcal{F}_i(u_1)),\ \sqrt{n}(\hat{\mathcal{F}}_i(u_2)-\mathcal{F}_i(u_2))\right\}=\\&\mathcal{F}_i(u_1\wedge u_2)-\mathcal{F}_i(u_1)\mathcal{F}_i(u_2).\end{aligned}$$

Consequently,

$$\begin{aligned}\mathrm{E}_{12}(u_1,u_2)=&n\mathrm{E}\left\{\hat{\mathcal{F}}_1(u_1)\hat{\mathcal{F}}_2(u_2)-\mathcal{F}_1(u_1)\mathcal{F}_2(u_2)\right\}=\\&\frac{1}{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\mathrm{EI}_{\{Y_l-X_l\leq u_1,X_k-Y_k\leq u_2\}}-n\mathcal{F}_1(u_1)\mathcal{F}_2(u_2)=\\&\mathrm{P}(-u_2\leq Y-X\leq u_1)-\mathcal{F}_1(u_1)\mathcal{F}_2(u_2).\end{aligned}$$

Similarly, $\mathrm{E}_{21}(u_1,u_2)=P(-u_1\leq Y-X\leq u_2)-\mathcal{F}_1(u_2)\mathcal{F}_2(u_1)$, then we can conclude that for any $u_1,u_2\in\Lambda$,

$$\begin{aligned}&\mathrm{Cov}\left(\sqrt{n}(\widehat{\varphi}(u_1)-\varphi(u_1)),\ \sqrt{n}(\widehat{\varphi}(u_2)-\varphi(u_2))\right)=\\&\mathrm{E}_{11}(u_1,u_2)-\mathrm{E}_{12}(u_1,u_2)-\mathrm{E}_{21}(u_1,u_2)+\mathrm{E}_{22}(u_1,u_2)=\\&-(\mathcal{F}_1(u_1)-\mathcal{F}_2(u_1))(\mathcal{F}_1(u_2)-\mathcal{F}_2(u_2))+\\&\sum_{i=1}^{2}\mathcal{F}_i(u_1\wedge u_2)-\mathrm{I}_{\{u_1+u_2\geq 0\}}\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\mathcal{F}_i(u_j)-2\right).\end{aligned}$$

Owing to the compactness of Λ, the null hypothesis H_0 is satisfied if and only if $\mathcal{T}(\varphi)=0$. Consequently, we can express the statistic $\hat{T}$ as

$$\hat{T}=\sqrt{n}\mathcal{T}(\hat{\varphi})=\sqrt{n}(\mathcal{T}(\hat{\varphi})-\mathcal{T}(\varphi)).$$

This form motivates the application of the functional delta method. However, the functional $\mathcal{T}$ is not Hadamard differentiable, preventing us from using the standard approaches of the functional delta method to obtain the limiting distribution of $\hat{T}$. However, Lemma S.4.9 of Ref. [16] demonstrates that the functional $\mathcal{T}$ is Hadamard directionally differentiable. Consequently, by means of the more general version of the functional delta method[17,18], the subsequent theorem can be directly obtained from our Theorem 1 and Theorem 2.1 of Ref. [16].

Theorem 2. Under Assumption 1, as $n\to\infty$, then

$$\sqrt{n}(\mathcal{T}(\widehat{\varphi})-\mathcal{T}(\varphi))\xrightarrow{\mathrm{w}}\sup_{x\in\Omega(\varphi)}\{\mathcal{G}(x)\}\ \text{ in } \mathbb{R},$$

where $\Omega(\varphi)=\arg\max_{x\in\Lambda}\varphi(x)$ and $\mathcal{G}$ is the process defined in Theorem 1.

We are now ready to explore the selection of critical values for our test. When H_0 is true, $\Omega(\varphi)=\{x\in\Lambda:\varphi(x)=0\}$. Theorem 2 indicates that $\hat{T}$ converges weakly to $\overline{T}=\sup_{x\in\Omega(\varphi)}\{\mathcal{G}(x)\}=O_p(1)$. For a desired nominal level $\alpha\in(0,0.5]$, we choose the critical value c_α as the corresponding upper αth quantile of $\overline{T}$. The decision rule is to "reject H_0 if $\hat{T}>c_\alpha$". The following theorem presents the asymptotic powers of our test using the above decision rule.

Theorem 3. Under the satisfaction of Assumption 1, for a given nominal level $\alpha\in(0,0.5]$, the following conclusions hold:

(i) If H_0 is false, then $\lim_{n\to\infty}\mathrm{P}(\text{reject } H_0)=1$;

(ii) if H_0 is true then $\lim_{n\to\infty}\mathrm{P}(\text{reject } H_0)\leq\alpha$.

Proof. (i) When the null hypothesis H_0 is false, there exists a $x^*\in\Lambda$ for which $\varphi(x^*)=\delta>0$. Consequently,

$$\hat{T}=\sqrt{n}\mathcal{T}(\hat{\varphi})\geq\sqrt{n}(\hat{\varphi}(x^*)-\varphi(x^*))+\sqrt{n}\delta.$$

Invoking Theorem 1, we observe that

$$\sup_{x\in\Lambda}\left|\sqrt{n}(\hat{\varphi}(x)-\varphi(x))\right|=O_p(1).$$

As $n\to\infty$, $\sqrt{n}\delta\to\infty$ follows. Therefore, $\hat{T}\to\infty$ in probability, while c_α remains finite. This observation solidifies the conclusion.

(ii) Under the truth of H_0, the asymptotic behavior of $\hat{T}$ is characterized by Theorem 2, where it is shown not to converge to zero. This outcome is consistent with the definition of c_α, thereby validating the result.

The above theorem states that if such a c_α is available, the proposed test performs accordingly. However, direct implementation of the test in Theorem 3 is not feasible as the computation of c_α requires knowledge of the true $\mathcal{F}_1$ and $\mathcal{F}_2$. Since we lack knowledge of the true distributions, we employ bootstrap methods to resolve this logical impasse in the following section.

3 Bootstrap procedure

Let $(X_i,Y_i)_{i=1}^n$ be independent observations. For $b=1,\cdots,B$, we draw the bootstrap samples $(X_i^{*b},Y_i^{*b})_{i=1}^n$ from $(X_i,Y_i)_{i=1}^n$ with replacement. These bootstrap samples are independent of both the data and each other. From the bth sample, we calculate the empirical distributions as follows:

$$\hat{\mathcal{F}}_1^{*b}(x)=\frac{1}{n}\sum_{i=1}^{n}\mathrm{I}_{\{Y_i^{*b}-X_i^{*b}\leq x\}},\quad \hat{\mathcal{F}}_2^{*b}(x)=\frac{1}{n}\sum_{i=1}^{n}\mathrm{I}_{\{X_i^{*b}-Y_i^{*b}\leq x\}}.$$

Subsequently, we estimate the difference between the two distributions as

$$\varphi^{*b}(x)=\hat{\mathcal{F}}_1^{*b}(x)-\hat{\mathcal{F}}_2^{*b}(x).$$

The subsequent result establishes that, conditional on $(X_i,Y_i)_{i=1}^n$, the bootstrap process $\sqrt{n}(\varphi^{*b}-\hat{\varphi})$ consistently approximates the weak limit of $\sqrt{n}(\hat{\varphi}-\varphi)$.

Theorem 4. Under Assumption 1, as $n\to\infty$, conditional on $(X_i,Y_i)_{i=1}^n$,

$$\sqrt{n}(\varphi^{*b}-\hat{\varphi})\xrightarrow{\mathrm{w}}\mathcal{G}\ \text{ in } \ell^\infty(\Lambda).$$

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Proof. From Theorem 23.7 in Ref. [17], we can deduce that conditional on $(X_i, Y_i)_{i=1}^n$,

$$\sqrt{n}\left(\hat{\mathcal{F}}_i^{*b} - \hat{\mathcal{F}}_i\right) \xrightarrow{\mathrm{w}} \mathcal{G}_i \text{ in } \ell^\infty(\Lambda),$$

where $\mathcal{G}_i$ matches the one utilized in the proof of Theorem 2. Likewise, we establish that $\sqrt{n}(\varphi^{*b} - \hat{\varphi})$ converges to a tight process, conditioned on $(X_i, Y_i)_{i=1}^n$. For any $u \in \Lambda$,

$$\begin{aligned}&\mathrm{E}\left\{\sqrt{n}(\hat{\mathcal{F}}_1^{*b}(u) - \hat{\mathcal{F}}_1(u))\sqrt{n}(\hat{\mathcal{F}}_2^{*b}(u) - \hat{\mathcal{F}}_2(u)) \mid (X_i, Y_i)_{i=1}^n\right\} = \\ &n\left\{\mathrm{E}\left(\hat{\mathcal{F}}_1^{*b}(u)\hat{\mathcal{F}}_2^{*b}(u) \mid (X_i, Y_i)_{i=1}^n\right) - \hat{\mathcal{F}}_1(u)\hat{\mathcal{F}}_2(u)\right\} = \\ &\frac{1}{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\mathrm{E}\left\{\mathrm{I}_{\{Y_l^{*b}-X_l^{*b}\le u, X_k^{*b}-Y_k^{*b}\le u\}} \mid (X_i, Y_i)_{i=1}^n\right\} - n\hat{\mathcal{F}}_1(u)\hat{\mathcal{F}}_2(u) = \\ &\mathrm{I}_{\{u\ge 0\}}\left(\sum_{i=1}^{2}\hat{\mathcal{F}}_i(u) - 1\right) - \hat{\mathcal{F}}_1(u)\hat{\mathcal{F}}_2(u) \xrightarrow{\mathrm{w}} \\ &\mathrm{I}_{\{u\ge 0\}}\left(\sum_{i=1}^{2}\mathcal{F}_i(u) - 1\right) - \mathcal{F}_1(u)\mathcal{F}_2(u).\end{aligned}$$

Therefore, the variance is computed as

$$\begin{aligned}&\mathrm{Var}\left\{\sqrt{n}(\varphi^{*b} - \hat{\varphi}) | (X_i, Y_i)_{i=1}^n\right\} \xrightarrow{\mathrm{w}} \\ &\mathcal{F}_1(u) + \mathcal{F}_2(u) - (\mathcal{F}_1(u) - \mathcal{F}_2(u))^2 - 2\mathrm{I}_{\{u\ge 0\}}\left(\sum_{i=1}^{2}\mathcal{F}_i(u) - 1\right).\end{aligned}$$

The covariance calculation is subsequently carried out. For any u_1, $u_2 \in \Lambda$, i, $j = 1, 2$, let $\mathrm{E}_{ij}^*(u_1, u_2)$ denote

$$\mathrm{E}\left\{n(\hat{\mathcal{F}}_1^{*b}(u_1) - \hat{\mathcal{F}}_1(u_1))(\hat{\mathcal{F}}_2^{*b}(u_2) - \hat{\mathcal{F}}_2(u_2)) \mid (X_i, Y_i)_{i=1}^n\right\}.$$

Specifically, note that $\mathrm{E}_{ii}^*(u_1, u_2) \xrightarrow{\mathrm{w}} \mathrm{Cov}(\mathcal{G}_i(u_1), \mathcal{G}_i(u_2)) = \mathcal{F}_i(u_1 \wedge u_2) - \mathcal{F}_i(u_1)\mathcal{F}_i(u_2)$ for $i = 1, 2$. Consequently,

$$\begin{aligned}&\mathrm{E}_{12}^*(u_1, u_2) = \\ &n\left\{\mathrm{E}\left(\hat{\mathcal{F}}_1^{*b}(u_1)\hat{\mathcal{F}}_2^{*b}(u_2) \mid (X_i, Y_i)_{i=1}^n\right) - \hat{\mathcal{F}}_1(u_1)\hat{\mathcal{F}}_2(u_2)\right\} = \\ &\frac{1}{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\mathrm{E}\left\{\mathrm{I}_{\{Y_l^{*b}-X_l^{*b}\le u_1, X_k^{*b}-Y_k^{*b}\le u_2\}} \mid (X_i, Y_i)_{i=1}^n\right\} - n\hat{\mathcal{F}}_1(u_1)\hat{\mathcal{F}}_2(u_2) = \\ &\mathrm{I}_{\{u_1+u_2\ge 0\}}\left(\hat{\mathcal{F}}_1(u_1) + \hat{\mathcal{F}}_2(u_2) - 1\right) - \hat{\mathcal{F}}_1(u_1)\hat{\mathcal{F}}_2(u_2).\end{aligned}$$

Similar calculations yield

$$\mathrm{E}_{21}^*(u_1, u_2) = \mathrm{I}_{\{u_1+u_2\ge 0\}}\left(\hat{\mathcal{F}}_1(u_2) + \hat{\mathcal{F}}_2(u_1) - 1\right) - \hat{\mathcal{F}}_1(u_2)\hat{\mathcal{F}}_2(u_1),$$

which leads to the conclusion that for any $u_1, u_2 \in \Lambda$,

$$\begin{aligned}&\mathrm{Cov}\left\{\sqrt{n}(\widehat{\varphi}(u_1) - \varphi(u_1)), \sqrt{n}(\widehat{\varphi}(u_2) - \varphi(u_2)) \mid (X_i, Y_i)_{i=1}^n\right\} = \\ &E_{11}^*(u_1, u_2) - E_{12}^*(u_1, u_2) - E_{21}^*(u_1, u_2) + E_{22}^*(u_1, u_2) \xrightarrow{\mathrm{w}} \\ &-(\mathcal{F}_1(u_1) - \mathcal{F}_2(u_1))(\mathcal{F}_1(u_2) - \mathcal{F}_2(u_2)) + \\ &\sum_{i=1}^{2}\mathcal{F}_i(u_1 \wedge u_2) - \mathrm{I}_{\{u_1+u_2\ge 0\}}\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\mathcal{F}_i(u_j) - 2\right).\end{aligned}$$

With $\mathcal{G}$ estimated by bootstrapping (irrespective of whether under H_0 or H_1), we can construct the new critical value. We start with H_0 and aim to find a surrogate of $\Omega(\varphi)$. To do so, we let c_n be a positive constant and go to infinity in order $O_p(n^{1/2})$ as $n \to \infty$. We define our surrogate as

$$\tilde{\Omega}(\varphi) = \left\{x \in \Lambda : \sqrt{n}\hat{\varphi}(x) > -c_n\right\}.$$

For each $x \in \Omega(\varphi)$, Theorem 2 indicates that $\sqrt{n}\hat{\varphi}(x) = \sqrt{n}(\hat{\varphi}(x) - \varphi(x)) = O_p(1)$. This further implies that

$$\lim_{n\to\infty}\mathrm{P}\left(x \in \tilde{\Omega}(\varphi)\right) = \lim_{n\to\infty}\mathrm{P}\left(\sqrt{n}\hat{\varphi}(x) > -c_n\right) = 1. \tag{2}$$

On the other hand, for each $x' \in \Lambda - \Omega(\varphi)$, $\varphi(x') < 0$, which leads to the conclusion that

$$\sqrt{n}\hat{\varphi}(x') = \sqrt{n}(\hat{\varphi}(x') - \varphi(x')) + \sqrt{n}\varphi(x') = O_p(1) - \sqrt{n}|\varphi(x')|.$$

Note that $-c_n$ goes to negative infinity of order $O_p(n^{1/2})$, then we can conclude that

$$\lim_{n\to\infty}\mathrm{P}(x \notin \tilde{\Omega}(\varphi)) = 1. \tag{3}$$

Combined with Eqs. (2) and (3), we obtain, under H_0, $\lim_{n\to\infty}\mathrm{P}\left(\Omega(\varphi) = \tilde{\Omega}(\varphi)\right) = 1$. Therefore, combined with Theorem 4, we can derive that under H_0, conditional on $(X_i, Y_i)_{i=1}^n$,

$$\hat{T}^{*b} = \sup_{x\in\tilde{\Omega}(\varphi)}\sqrt{n}(\hat{\varphi}^{*b}(x) - \hat{\varphi}(x)) \xrightarrow{\mathrm{w}} \sup_{x\in\Omega(\varphi)}\{\mathcal{G}(x)\} \text{ in } \mathbb{R}.$$

Let the αth upper quantile of $\{\hat{T}^{*b}\}_{b=1}^B$ be c_α^*. We reject H_0 when $\hat{T} > c_\alpha^{*b}$. Now suppose H_1 is true, and we follow the above to compute c_α^*. Then, $\tilde{\Omega}(\varphi)$ is no longer a consistent estimate of $\Omega(\varphi)$. However, $\sqrt{n}(\varphi^{*b} - \hat{\varphi})$ still consistently estimate $\mathcal{G}$ (under H_1), and $\tilde{\Omega}(\varphi)$ is always a subset of Λ. Consequently, we have

$$\left|\hat{T}^{*b}\right| \le \sup_{x\in\Lambda}\sqrt{n}\left|\varphi^{*b}(x) - \hat{\varphi}(x)\right|,$$

which converges weakly to $\sup_{x\in\Lambda}|\mathcal{G}(x)|$. Therefore, the quantile c_α^* of $\hat{T}^{*b}$ is bounded in probability. On the other hand, under H_1, $\hat{T}$ goes to positive infinity in probability. Therefore, $\lim_{n\to\infty}\mathrm{P}\left(\hat{T} > c_\alpha^*\right) = 1$ holds for H_1.

Equivalently, we may calculate the p-value as

$$p\text{-value} = \frac{1}{B}\sum_{b=1}^{B}\mathrm{I}\left(\hat{T}^{*b} \ge \hat{T}\right).$$

Given a significance level α, we reject the null hypothesis if the p-value $< \alpha$. Therefore, combined with Theorem 4, we have

Theorem 5. With Assumption 1 satisfied, for a given nominal level $\alpha \in (0, 1/2]$, the following results hold:

(i) $\lim_{n\to\infty}\mathrm{P}\left(\hat{T} > c_\alpha^*\right) = 1$, if H_0 is false;

(ii) $\lim_{n\to\infty}\mathrm{P}\left(\hat{T} > c_\alpha^*\right) \le \alpha$, if H_0 is true.

4 Simulation

This section is divided into two main segments. The first segment scrutinizes the influence of various c_n selections on simulation outcomes and identifies the optimal c_n value for future simulations. The second segment entails a comparative analysis of the efficacy of our refined bootstrap methodology against the approach introduced in Ref. [11]. Notably, while Assumption 1 requires that the common support Λ, for $Y - X$ and $X - Y$, is compact, this requirement can be relaxed in

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simulation settings. This relaxation is justified on the grounds that, within the context of a finite sample size, the observations are inherently bounded, thereby allowing us to consider these observations as residing within a compact set.

4.1 Choice of the bound parameter

To initiate our study, we investigate how the behavior is influenced by the order of c_n in $\tilde{\Omega}(\varphi)$ under the given conditions. We employ various c_n values to conduct tests in different scenarios, enabling performance comparisons.

Case 1.1. (X,Y) follows a bivariate normal distribution with mean vector $(2,4)$ with covariance matrix $\begin{pmatrix} 2 & 1{,}5 \\ 1.5 & 1.5 \end{pmatrix}$. In this case, the null hypothesis is true and the rejection rate should be close to zero.

Case 1.2. (X,Y) follows a bivariate normal distribution with mean vector $(3,1)$ with covariance matrix $\begin{pmatrix} 2 & 1{,}5 \\ 1.5 & 1.5 \end{pmatrix}$. Then the null hypothesis is strictly false. Thus, H_0 should be rejected, and the rejection rate should be close to one as the sample size increases.

Case 1.3. (X,Y) follows a bivariate normal distribution with mean vector $(2,2)$ with covariance matrix $\begin{pmatrix} 1.5 & 1.3 \\ 1.3 & 1.5 \end{pmatrix}$. Under this setting, the two distributions are equal and the rejection rate should be close to the significance level α.

We further divide these three examples into two scenarios: the large-sample scenario with $n = 500$, and the small-sample scenario with $n = 50$. We approximate the p-value by setting the number of bootstrap replications to 500, and we perform 2000 iterations for each experiment.

Based on the analysis presented in Table 1, we observe that different values of c_n can yield favorable results as long as they meet the specified requirements. This finding aligns with our theoretical expectations. Therefore, for the subsequent experiments, we will set c_n as $n^{1/3}$.

4.2 Comparison with Monte Carlo method

In this subsection, we will use our improved bootstrap method to test for weak joint stochastic dominance. To provide more convincing results, we will compare its performance with the method proposed in Ref. [11]. Therefore, we consider the following cases:

Case 2.1. (X,Y) follows a bivariate normal distribution with mean vector $(2,2.01)$ with covariance matrix $\begin{pmatrix} 1.5 & 1.3 \\ 1.3 & 1.5 \end{pmatrix}$.

Case 2.2. Let $X \sim \text{Pareto}(5,4)$ and $Y \sim \text{Pareto}(1.5,1)$.

Case 2.3. Let $X \sim \text{Weibull}(6,2)$ and $Y \sim \text{Weibull}(1.5,1.5)$.

Case 2.4. Let $X \sim \text{Weibull}(0.75,4)$ and

$$Y \sim \text{Weibull}(0.25,1.5).$$

Case 2.5. Let $X \sim \text{Weibull}(0.5,2)$ and

$$Y \sim \text{Weibull}(0.9,1.5).$$

Case 2.6. Let $X \sim \text{Pareto}(2,1)$ and $Y \sim \text{Pareto}(1.5,1)$.

In Cases 2.1, X and Y are very close in terms of distribution; thus, the rejection rate should be close to the nominal level α and slightly lower. In Cases 2.2–2.6, we consider a Clayton copula with a parameter of 0.5 as the dependence structure. The marginal distributions are either Pareto or Weibull, denoted by $X \sim \text{Pareto}(a,k)$ and $X \sim \text{Weibull}(a,b)$, respectively. The Pareto distribution is defined as $F(x) = 1-(k/x)^a$, for all $x \geq k$, and the Weibull distribution is defined as $F(x) = 1-\exp(-(x/b)^a)$, for all $x \geq 0$.

Fig. 1 clearly shows that the two lines intersect distinctly in

Table 1. Combined rejection rates of Cases 1.1–1.3.

Case	α	Sample size	c_n					
			$n^{1/3}$	$n^{1/5}$	$n^{1/7}$	$n^{1/9}$	$\log(n)$	$\log\log(n)$
Case 1.1	0.1	50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
		500	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.05	50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
		500	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.01	50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
		500	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Case 1.2	0.1	50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
		500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.05	50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
		500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.01	50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
		500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Case 1.3	0.1	50	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
		500	0.0935	0.0935	0.0935	0.0935	0.0935	0.0935
	0.05	50	0.0560	0.0560	0.0560	0.0560	0.0560	0.0560
		500	0.0535	0.0535	0.0535	0.0535	0.0535	0.0535
	0.01	50	0.0135	0.0135	0.0135	0.0135	0.0135	0.0135
		500	0.0125	0.0125	0.0125	0.0125	0.0125	0.0125

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Cases 2.2–2.4. Hence, the null hypothesis can be easily rejected in these cases. In Case 2.5, the intersection between the two lines is marginal, indicating a relatively more challenging rejection of the null hypothesis. Similarly, in Case 2.6, the curve of $Y-X$ slightly extends beyond the curve of $X-Y$, suggesting a relatively more difficult rejection of the null hypothesis as well.

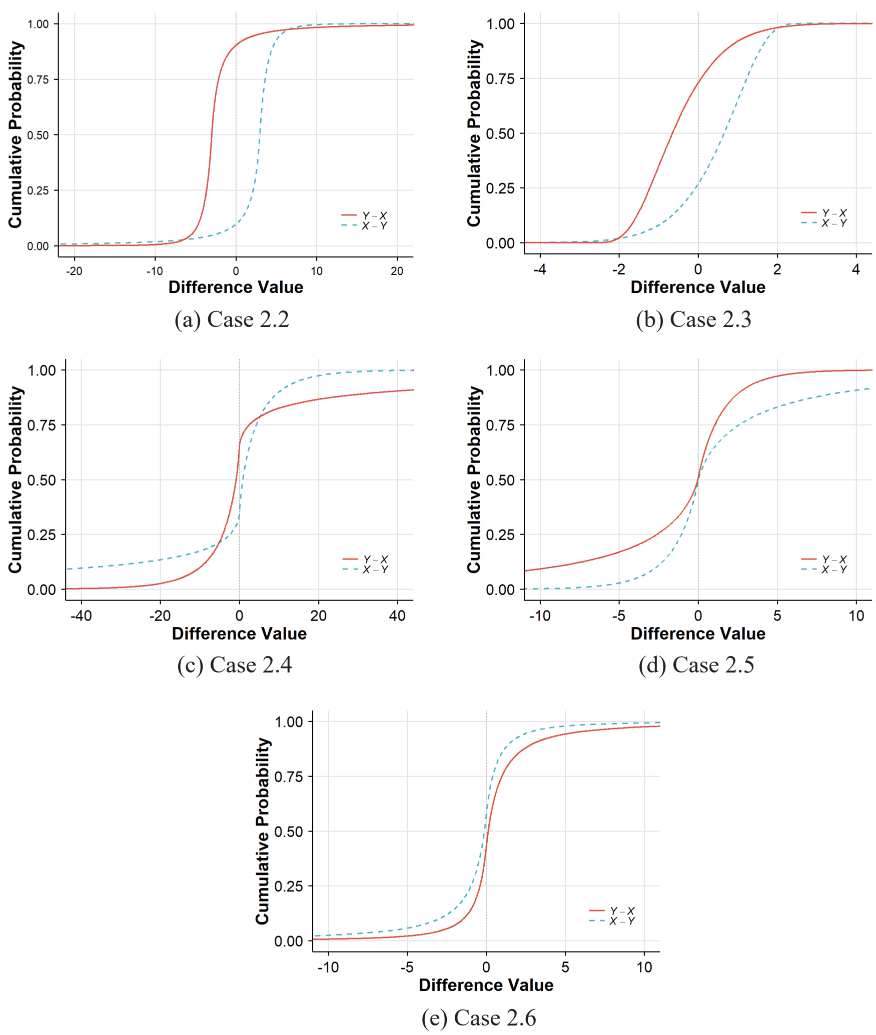
Like the configurations outlined in Ref. [11], we classify them into four distinct scenarios: $n=50$, $n=100$, $n=200$, and $n=500$. We subsequently test these cases at significance levels of $\alpha=0.05$ and 0.01. To estimate the p-value, we designate the number of bootstrap replications as 500, and each experiment is executed over 2000 iterations.

The results are presented in Table 2. From the table, it is evident that in Case 2.1, the outcomes of our bootstrap method align with the theoretical results, indicating values smaller than α. As the sample size increases, the difference between the rejection rates and α becomes more pronounced. In Cases 2.2–2.4, our rejection rates closely approximate 1, demonstrating consistency with the theoretical results. This can be attributed to the evident intersection of the two lines in these cases, making it easy to reject the null hypotheses. In Case 2.5, our rejection rates are lower for smaller sample sizes but steadily increase toward unity. This observation is reasonable as the two lines only slightly intersect. With a small sample size, it becomes difficult for the nonparametric method to detect and reject the null hypothesis, whereas a larger sample size facilitates the detection of this slight intersection. Case 2.6 exhibits a similar pattern to that of Case 2.5.

The results obtained through the Monte Carlo method in Ref. [11] exhibit a lack of agreement with the theoretical outcomes, particularly in Cases 2.1, 2.5, and 2.6. This discrepancy can be attributed to two factors. First, Ref. [11] fails to derive accurate asymptotic results for the statistics employed, relying instead on the Monte Carlo method. However, this method is unable to faithfully represent the true distribution of the statistic in this specific case. These two factors likely collectively contribute to the observed disparity between the theoretical results and those obtained through the Monte Carlo method.

4.3 Other cases

In this subsection, we will use our improved bootstrap method to test the data with other common copulas to further



(a) Case 2.2
(b) Case 2.3
(c) Case 2.4
(d) Case 2.5
(e) Case 2.6

Fig. 1. CDFs for X–Y and Y–X according to Cases 2.1–2.6.

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Table 2. Rejection rates of Cases 2.1–2.6.

Cases	α		50	100	200	500
Case 2.1	0.05	Boot	0.0460	0.0345	0.0370	0.0305
		MC	0.6110	0.4540	0.2870	0.0830
	0.01	Boot	0.0105	0.0090	0.0065	0.0065
		MC	0.5200	0.3360	0.1640	0.0250
Case 2.2	0.05	Boot	1.0000	1.0000	1.0000	1.0000
		MC	1.0000	1.0000	1.0000	1.0000
	0.01	Boot	0.9980	1.0000	1.0000	1.0000
		MC	1.0000	1.0000	1.0000	1.0000
Case 2.3	0.05	Boot	0.9995	1.0000	1.0000	1.0000
		MC	1.0000	1.0000	1.0000	1.0000
	0.01	Boot	0.9970	1.0000	1.0000	1.0000
		MC	1.0000	1.0000	1.0000	1.0000
Case 2.4	0.05	Boot	0.8985	0.9880	1.0000	1.0000
		MC	0.9850	0.9990	1.0000	1.0000
	0.01	Boot	0.6790	0.9385	1.0000	1.0000
		MC	0.9820	0.9970	1.0000	1.0000
Case 2.5	0.05	Boot	0.2015	0.4075	0.7390	1.0000
		MC	0.9850	0.9990	1.0000	1.0000
	0.01	Boot	0.0610	0.1500	0.4315	0.9710
		MC	0.9840	0.9960	1.0000	1.0000
Case 2.6	0.05	Boot	1.0000	1.0000	1.0000	1.0000
		MC	0.1600	0.0340	0.0000	0.0000
	0.01	Boot	0.0000	0.0000	0.0000	0.0000
		MC	0.0980	0.0170	0.0000	0.0000

verify our method.

In the next cases, the dependence structure is given by a gumbel copula with parameter 2, that is $C(u,v) = \exp\left\{-[(-\ln u)^2+(-\ln v)^2]^{1/2}\right\}$, for all $(u,v)\in[0,1]^2$, and the marginal distributions are either Pareto or Weibull.

Case 3.1. Let $X \sim \text{Pareto}(5,4)$ and $Y \sim \text{Pareto}(1.5,1)$.

Case 3.2. Let $X \sim \text{Weibull}(6,2)$ and $Y \sim \text{Weibull}(1.5,1.5)$.

Case 3.3. Let $X \sim \text{Weibull}(0.75,4)$ and

$$Y \sim \text{Weibull}(0.25,1.5).$$

Case 3.4. Let $X \sim \text{Weibull}(0.5,2)$ and

$$Y \sim \text{Weibull}(0.9,1.5).$$

Case 3.5. Let $X \sim \text{Pareto}(2,1)$ and $Y \sim \text{Pareto}(1.5,1)$.

Based on Fig. 2, it is evident that the two lines intersect distinctly in Cases 3.2, 3.3, and 3.4, thereby warranting a straightforward rejection of the null hypothesis. However, in Case 3.5, where the intersection between the lines is marginal, rejecting the null hypothesis becomes more challenging. Moreover, in Case 3.1, the curve of $Y-X$ surpasses that of $X-Y$, further supporting the decision to reject the null hypothesis.

In these examples, we categorize them into four distinct scenarios: $n=50$, $n=100$, $n=200$, and $n=500$. To approximate the p-value, we set the number of bootstrap replications to 500, and for each experiment, we perform 2000 iterations.

The obtained results are presented in Table 3. From these tables, an observation can be made: In Cases 3.1–3.4, our bootstrap method consistently aligns with the theoretical results, yielding values close to 1; however, in Case 3.5, we observe smaller rejection rates when the sample size and α are both small, gradually increasing toward reaching an approximate value of one. This behavior is reasonable, given the slight overlap between the two lines. For small sample sizes, it becomes challenging for the nonparametric method to detect and reject the null hypothesis. Conversely, with larger sample sizes, the nonparametric method becomes more adept at detecting this marginal overlap.

5 Real examples

To demonstrate the practical application of our methodology, we conducted an empirical analysis focusing on the income and expenditure patterns of residents in urban and rural areas in China. Our primary objective was to assess whether residents in these areas exhibited a greater inclination toward saving or comsumption.

To carry out this analysis, we utilized data from the China Family Panel Studies (CFPS) database (https://opendata.pku.edu.cn/dataverse/CFPS), which specifically targeted the income and expenditure data collected in 2020.

The dataset consists of 12 months' worth of total household consumption and income. To condense the data into a

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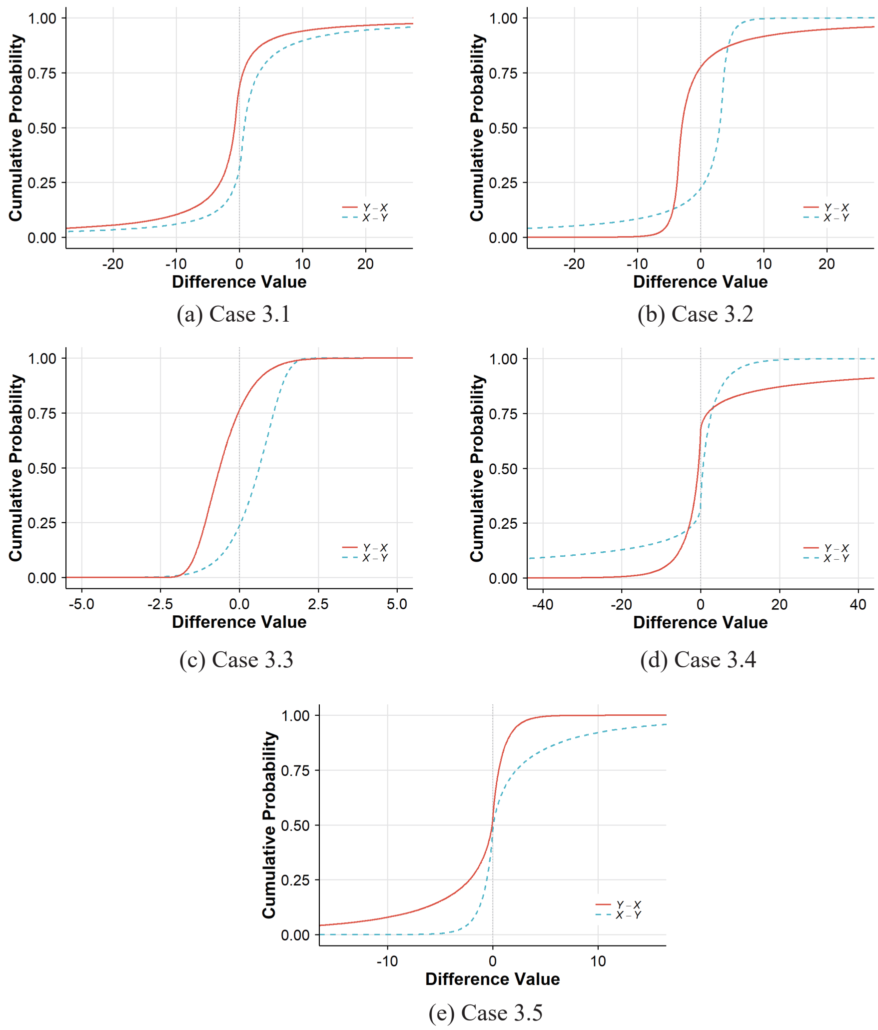

(a) Case 3.1 (b) Case 3.2 (c) Case 3.3 (d) Case 3.4 (e) Case 3.5

Fig. 2. CDFs for $X-Y$ and $Y-X$ according to Cases 3.1–3.5.

Table 3. Rejection rates of Cases 3.1–3.5.

Cases	α		50	100	200	500
Case 3.1	0.1	Boot	1.0000	1.0000	1.0000	1.0000
	0.05	Boot	1.0000	1.0000	1.0000	1.0000
	0.01	Boot	1.0000	1.0000	1.0000	1.0000
Case 3.2	0.1	Boot	1.0000	1.0000	1.0000	1.0000
	0.05	Boot	1.0000	1.0000	1.0000	1.0000
	0.01	Boot	1.0000	1.0000	1.0000	1.0000
Case 3.3	0.1	Boot	1.0000	1.0000	1.0000	1.0000
	0.05	Boot	1.0000	1.0000	1.0000	1.0000
	0.01	Boot	1.0000	1.0000	1.0000	1.0000
Case 3.4	0.1	Boot	1.0000	1.0000	1.0000	1.0000
	0.05	Boot	1.0000	1.0000	1.0000	1.0000
	0.01	Boot	1.0000	1.0000	1.0000	1.0000
Case 3.5	0.1	Boot	1.0000	1.0000	1.0000	1.0000
	0.05	Boot	0.5035	0.9790	1.0000	1.0000
	0.01	Boot	0.0000	0.0000	0.3915	1.0000

more concise format, we initially preprocessed it by taking the base-2 logarithm of the values. Table 4 presents summary statistics for the residents. Additionally, Fig. 3 provides evidence for the dependence between income and expenditure, which aligns perfectly with the scenario in which our method is applicable. We depict the empirical cumulative distribution functions (CDFs) for income–expenditure and expenditure–income for both urban and rural residents. Fig. 4 clearly

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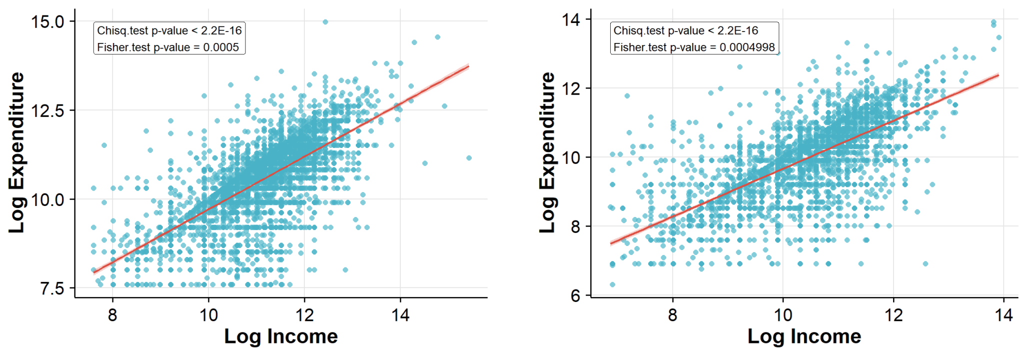
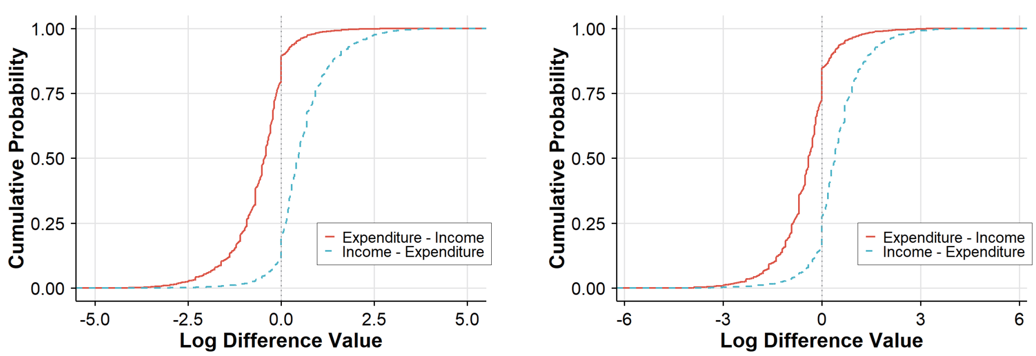
Table 4. Summary statistics for residents.

	Urban income	Urban expenditure	Rural income	Rural expenditure
Mean	11.128	10.548	10.444	9.97
SD	0.948	1.029	1.046	1.061
Median	11.156	10.597	10.597	10.086
Min	7.601	7.601	6.867	6.310
Max	15.425	14.979	13.911	13.911
Range	7.824	7.378	7.044	7.601
Skewness	−0.449	−0.404	−0.605	−0.385
Kurtosis	1.039	0.514	0.585	0.248
Numbers	5459	5459	4663	4663



(a) Urban (b) Rural

Fig. 3. Scatter plots and p-values of independence tests.



(a) Urban (b) Rural

Fig. 4. Empirical CDFs.

shows that the lines representing expenditure–income lie above those representing income–expenditure. Consequently, in both the urban and rural cases, we can reject the null hypothesis H_0 : income $\preceq_{st:wj}$ expenditure.

We employ our method to test the null hypothesis H_0 : income $\preceq_{st:wj}$ expenditure. For each population, we utilize the entire dataset, resulting in $n = 5459$ for the urban case and $n = 4663$ for the rural case. We set the number of replicates to 2000 and the number of bootstrap iterations to 500 ($B = 500$), with $c_n = n^{1/3}$. Consequently, the estimated p-values for both cases are 0, indicating that we should reject H_0 across all significance levels. This implies that both rural and urban residents in China are inclined to save rather than spend more in 2020. This trend may be attributed to the COVID-19 pandemic, which has diminished people's future expectations and heightened their awareness of risks, thus leading them to prioritize saving as a means of safeguarding against unknown uncertainties.

6 Conclusions

In this paper, we introduce an improved bootstrap method based on the Monte Carlo method proposed in Ref. [11] to test weak joint stochastic dominance. Furthermore, we derive the asymptotic properties of the statistics and bootstrap to theoretically validate the feasibility of our method. Simulation results demonstrate that our method outperforms the Monte Carlo method. Moving forward, our method holds potential for extension to multivariable cases in future research. However, several challenges may arise, such as characteriz-

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ing the dependent structure of two multidimensional populations and applying SD criteria to compare the differences between these populations.

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Conflict of interest

The authors declare that they have no conflict of interest.

Biographies

Weiqi Yang is currently pursuing the Ph.D. degree in Statistics and Finance from the University of Science and Technology of China. His research mainly focuses on stochastic dominance and statistical tests.

Weiwei Zhuang is currently an Associate Professor with the Department of Statistics and Finance, University of Science and Technology of China (USTC). She received her Ph.D. degree in Probability and Statistics from USTC in 2006. Her research mainy focuses on statistical dependence, stochastic comparisons, semiparametric model, and their applications.

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