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Solutions to SU($n + 1$) Toda systems with cone singularities via toric curves on compact Riemann surfaces

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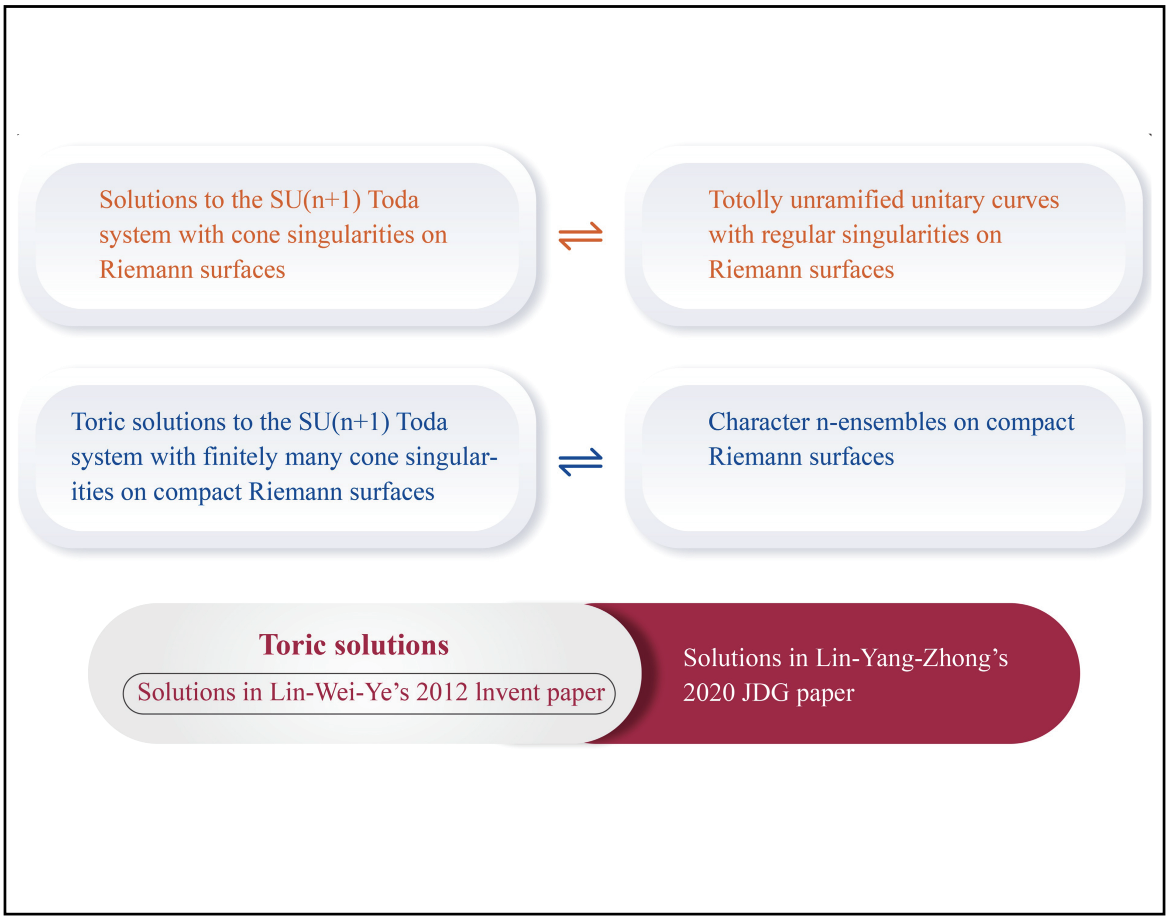
Graphical abstract



Toric solutions to SU(n+1) Toda system.

Public summary

- Toric solutions to SU(n+1) Toda system have monodromy in maximal tori of SU(n+1).
- Establish a correspondence between toric solutions and character ensembles on Riemann surfaces.
- Solutions to SU(n+1) Toda system generalize Kaehler–Einstein metrics on complex manifolds.

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Solutions to SU($n+1$) Toda systems with cone singularities via toric curves on compact Riemann surfaces

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Abstract: On a compact Riemann surface X with finite punctures $P_1,\cdots,P_k$, we define toric curves as multivalued, totally unramified holomorphic maps to $\mathbb{P}^n$ with monodromy in a maximal torus of $\mathrm{PSU}(n+1)$. Toric solutions to $\mathrm{SU}(n+1)$ Toda systems on $X\setminus\{P_1,\cdots,P_k\}$ are recognized by the associated toric curves in $\mathbb{P}^n$. We introduce character n-ensembles as n-tuples of meromorphic one-forms with simple poles and purely imaginary periods, generating toric curves on X minus finitely many points. On X, we establish a correspondence between character n-ensembles and toric solutions to the $\mathrm{SU}(n+1)$ system with finitely many cone singularities. Our approach not only broadens seminal solutions with two cone singularities on the Riemann sphere, as classified by Jost–Wang (*Int. Math. Res. Not.*, 2002, (6): 277–290) and Lin–Wei–Ye (*Invent. Math.*, 2012, 190(1): 169–207), but also advances beyond the limits of Lin–Yang–Zhong's existence theorems (*J. Differential Geom.*, 2020, 114(2): 337–391) by introducing a new solution class.

Keywords: SU(n+1) Toda system; regular singularity; unitary curve; toric solution; character ensemble

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2020 Mathematics Subject Classification: Primary 37K10; Secondary 35J47

1 Introduction

Gervais and Matsuo[1, Section 2.2] were the first to observe that totally unramified holomorphic curves[2, p. 270, Proposition] in $\mathbb{P}^n$ yield local solutions to $\mathrm{SU}(n+1)$ Toda system, rewriting these systems as the infinitesimal Plücker formula[2, p. 269] for these curves. Doliwa[3] expanded upon this foundation, extending their findings to Toda systems associated with nonexceptional simple Lie algebras. Over the ensuing years, Battaglia, Lin, and their collaborators[4–8] have devoted significant research efforts to classifying and empirically substantiating existence theorems for solutions to Toda systems. These systems are associated with any complex simple Lie algebra and feature cone singularities on compact Riemann surfaces (Definition 1). In the next three sub-sections, we aim to highlight the principal advancements that are pertinent to our thematic focus within this field. To lay the foundation for our discussion, we initially explore the basic correspondence between solutions to the $\mathrm{SU}(n+1)$ Toda system with cone singularities and their associated unitary curves in $\mathbb{P}^n$ with regular singularities (Definition 2) in the first two subsections.

1.1 A differential-geometric framework for solutions to Toda systems

Building on the seminal contributions of Refs. [9, 10], this manuscript establishes on Riemann surfaces a new conceptual framework for solutions to $\mathrm{SU}(n+1)$ Toda systems with cone singularities (Definition 1). As elucidated in Ref. [11, Theorem 2. (ii)], our analytical approach scrutinizes the solutions as n-tuples of Kähler metrics with cone singularities on Riemann surfaces. Importantly, our methodology not only aligns with but also translates into the ones employed by a spectrum of researchers including Refs. [4–6, 8, 12–17], thereby establishing a conceptual equivalence across these various approaches. This framework is versatile, allowing for an extension to Toda systems associated with any complex simple Lie algebra on Kähler manifolds.

Consider a Riemann surface $\mathfrak{X}$, which is not necessarily compact, alongside a closed discrete subset $\mathfrak{S}$ within $\mathfrak{X}$, where notably, $\mathfrak{S}$ is at most countable and possibly empty. We consider a vector $\vec{\omega}$ of Kähler metrics on the punctured Riemann surface $\mathfrak{X}\setminus\mathfrak{S}$, defined as

$$\vec{\omega}:=\left(\omega_1=\frac{\mathrm{i}}{2}\mathrm{e}^{u_1}\mathrm{d}z\wedge\mathrm{d}\bar{z},\cdots,\omega_n=\frac{\mathrm{i}}{2}\mathrm{e}^{u_n}\mathrm{d}z\wedge\mathrm{d}\bar{z}\right),\qquad(1)$$

together with the corresponding vector $\mathrm{Ric}(\vec{\omega})$ of Ricci $(1,1)$-forms derived from the metric components of $\vec{\omega}$,

$$\mathrm{Ric}(\vec{\omega}):=\left(\mathrm{Ric}(\omega_1)=-\mathrm{i}\partial\bar{\partial}u_1,\cdots,\mathrm{Ric}(\omega_n)=-\mathrm{i}\partial\bar{\partial}u_n\right).$$

We define $\vec{\omega}$ as a solution to the $\mathrm{SU}(n+1)$ Toda system on $\mathfrak{X}\setminus\mathfrak{S}$ if it satisfies the equation

$$\mathrm{Ric}(\vec{\omega})=\vec{\omega}\cdot(2a_{ij})_{n\times n}\qquad(2)$$

on $\mathfrak{X}\setminus\mathfrak{S}$, where (a_{ij}) is the Cartan matrix for $\mathfrak{su}(n+1)$, represented by

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$$(a_{ij})_{n\times n}=\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0\\ -1 & 2 & -1 & 0 & \cdots & 0\\ 0 & -1 & 2 & -1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & 0 & -1 & 2 & -1\\ 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix}_{n\times n}.$$

This system, $\mathrm{Ric}(\vec{\omega})=\vec{\omega}\cdot(2a_{ij})_{n\times n}$, is identified as the $\mathrm{SU}(n+1)$ Toda system on the Riemann surface, introducing a factor of 2 for consistency with the notation in Ref. [11]. It is important to note that a solution $\vec{\omega}$ to Eq. (2) can exhibit wild behavior in the vicinity of punctures within $\mathfrak{S}$. For instance, the integral of ω_j in a neighborhood around a point in $\mathfrak{S}$ may diverge. Since Ref. [18, Proposition 3.1], there has been an ongoing investigation into solutions of Toda systems with cone singularities. We will soon provide a precise definition for these solutions. Specifically, we assign to each point $P\in\mathfrak{S}$ a vector $\vec{\gamma}_P=(\gamma_{P,1},\cdots,\gamma_{P,n})$ of real numbers that are greater than -1 and do not vanish simultaneously, encapsulated into the $\mathbb{R}$-divisor vector $\vec{\mathfrak{D}}:=(\mathfrak{D}_1,\cdots,\mathfrak{D}_n)$ with $\mathfrak{D}_1=\sum_{P\in\mathfrak{S}}\gamma_{P,1}[P],\cdots,\mathfrak{D}_n=\sum_{P\in\mathfrak{S}}\gamma_{P,n}[P]$. Denoting by $\delta_{\vec{\mathfrak{D}}}$ the following vector of $(1,1)$-currents[2, Chapter 3],

$$\delta_{\vec{\mathfrak{D}}}:=2\pi\left(\sum_{P\in\mathfrak{S}}\gamma_{P,1}\delta_P,\cdots,\sum_{P\in\mathfrak{S}}\gamma_{P,n}\delta_P\right), \tag{3}$$

we give the following:

Definition 1. We consider $\vec{\omega}$ a solution to the $\mathrm{SU}(n+1)$ Toda system on $\mathfrak{X}$ with cone singularities $\vec{\mathfrak{D}}$ (i.e., representing $\vec{\mathfrak{D}}$) if it has finite area over each compact subset K of $\mathfrak{X}$, i.e., $\int_K\omega_j<\infty$ for all $1\le j\le n$, and satisfies the system

$$\mathrm{Ric}(\vec{\omega})=-\delta_{\vec{\mathfrak{D}}}+\vec{\omega}\cdot(2a_{ij})_{n\times n} \tag{4}$$

in the sense of $(1,1)$-current on $\mathfrak{X}$. In particular, this system, when restricted to a sufficiently small chart (U,z) around each $P\in\mathfrak{S}$, takes the following form:

$$\frac{\mathrm{i}}{2}\left(\partial\bar{\partial}u_i+\sum_{j=1}^{n}a_{ij}\mathrm{e}^{u_j}\mathrm{d}z\wedge\mathrm{d}\bar{z}\right)=\pi\gamma_{P,i}\delta_P,\quad 1\le i\le n.$$

By Ref. [11, Theorem 2. (ii)], for each $1\le i\le n$, the component $\omega_i=\frac{\mathrm{i}}{2}\mathrm{e}^{u_i}\mathrm{d}z\wedge\mathrm{d}\bar{z}$ of a solution $\vec{\omega}$ to Eq. (4) forms a cone Kähler metric on X with cone angle $2\pi(1+\gamma_{P,i})$ at $P\in\mathfrak{S}$. In this context, we say that $\vec{\omega}$ is a solution with cone singularities $\vec{\mathfrak{D}}$ (i.e., representing $\vec{\mathfrak{D}}$).

Notably, when $n=1$, the term $\vec{\omega}=\omega_1$ denotes a cone spherical metric that represents the divisor $\mathfrak{D}_1$ on $\mathfrak{X}$. For further insights into cone spherical metrics, see Ref. [19] and the references contained therein. Unlike the conventional definition of the Toda system with singularities, our intrinsic approach does not rely on a predetermined background Kähler metric and situates the problem within the context of Kähler geometry. Specifically, our ongoing project is dedicated to investigating vectors of Kähler metrics with cone singularities along divisors[20] as potential solutions to the Toda systems associated with any complex simple Lie algebra on Kähler manifolds. This approach naturally extends both the Toda system (4) on Riemann surfaces and the Monge–Ampère equation concerning cone Kähler–Einstein metrics with positive scalar curvatures on Kähler manifolds of dimension ≥ 2[20].

1.2 A basic correspondence

In this subsection, drawing upon Refs. [1, Section 2.2], [12, Section 3], and [11, Section 2], we present a concise overview of the basic correspondence between solutions to the $\mathrm{SU}(n+1)$ Toda systems with cone singularities $\vec{\mathfrak{D}}$ on $\mathfrak{X}$, and totally unramified unitary curves on $\mathfrak{X}\setminus\mathfrak{S}$ and with regular singularities $\vec{\mathfrak{D}}$. This overview lays a robust foundation for articulating the main results of our manuscript and situating them within the context of classical findings such as Refs. [1, 12, 13, 21].

A totally unramified unitary curve f on $\mathfrak{X}\setminus\mathfrak{S}$ is a multivalued holomorphic map $f:\mathfrak{X}\setminus\mathfrak{S}\to\mathbb{P}^n$, characterized by the following properties:

- The monodromy of f resides within $\mathrm{PSU}(n+1)$, the group of holomorphic isometries preserving the Fubini–Study metric ω_{FS} on $\mathbb{P}^n$.
- At each point $z\in\mathfrak{X}\setminus\mathfrak{S}$, any germ $\mathfrak{f}$ of f at z is totally unramified and, notably, nondegenerate near z.

For a more detailed exposition of this concept, please refer to Ref. [11, Definition 2.1].

Definition 2. We define a totally unramified unitary curve $f:\mathfrak{X}\setminus\mathfrak{S}\to\mathbb{P}^n$ to have regular singularities

$$\vec{\mathfrak{D}}=\left(\sum_{P\in\mathfrak{S}}\gamma_{P,1}[P],\cdots,\sum_{P\in\mathfrak{S}}\gamma_{P,n}[P]\right),$$

i.e., representing $\vec{\mathfrak{D}}$, if, for each point $P\in\mathfrak{S}$, there exists an element $\varphi\in\mathrm{PSU}(n+1)$ such that the composition $\varphi\circ f$ when restricted to a punctured disk $\{0<|z|<1\}\subset\mathfrak{X}\setminus\mathfrak{S}$ around P can be expressed as

$$\varphi\circ f(z)=[z^{\beta_{P,0}}g_0(z):z^{\beta_{P,1}}g_1(z):\cdots:z^{\beta_{P,n}}g_n(z)], \tag{5}$$

where the functions $g_0(z),\cdots,g_n(z)$ are holomorphic and nonvanishing at $z=0$, i.e., P. The exponents $\beta_{P,0},\cdots,\beta_{P,n}$ are real numbers satisfying

$$\beta_{P,j}-\beta_{P,j-1}=\gamma_{P,j}+1\quad\text{for all}\quad 1\le j\le n. \tag{6}$$

In this context, given a point P in $\mathfrak{S}$, if all $\gamma_{P,j}$'s are nonnegative integers and do not vanish simultaneously, then $P\in\mathfrak{S}$ is referred to as a ramification point of f[2, pp. 266–268]; if at least one of $\{\gamma_{P,j}\}_{j=1}^n$ is a noninteger, then P is called a branch point of f. Both ramification points and branch points are called regular singularities of f. Note that f is totally unramified at a point $P\in X$ if and only if $\gamma_{P,j}=0$ for all $1\le j\le n$.

From a totally unramified unitary curve $f:\mathfrak{X}\setminus\mathfrak{S}\to\mathbb{P}^n$, one can derive a solution to the $\mathrm{SU}(n+1)$ Toda system (2) on $\mathfrak{X}\setminus\mathfrak{S}$. In fact, for $0\le j\le n-1$, the jth associated curve $f_j:\mathfrak{X}\setminus\mathfrak{S}\to G(j+1,n+1)\subset\mathbb{P}(\Lambda^{j+1}\mathbb{C}^{n+1})$ of f is also a unitary curve (see Refs. [2, pp. 263–264] and [11, Definition 2.1]). For simplicity, we uniformly adopt the symbol ω_{FS} to represent the Fubini–Study metrics on $\mathbb{P}(\Lambda^{j+1}\mathbb{C}^{n+1})$ for all $0\le j\le n-1$. By employing the infinitesimal Plücker formula, $f:\mathfrak{X}\setminus\mathfrak{S}\to\mathbb{P}^n$ produces a vector of Kähler metrics

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$$\vec{\omega}=(f_0^*\omega_{\rm FS},\cdots,f_{n-1}^*\omega_{\rm FS}), \tag{7}$$

which serves as a solution to Eq. (2) on $\mathfrak{X}\setminus\mathfrak{S}$, as delineated in Ref. [11, Lemma 2.2]. Conversely, each solution $\vec{\omega}$ to the $\mathrm{SU}(n+1)$ Toda system (2) on $\mathfrak{X}\setminus\mathfrak{S}$ gives rise to a series of totally unramified unitary curves $\mathfrak{X}\setminus\mathfrak{S}\to\mathbb{P}^n$, where any two are distinguishable by a post-composition of an element in $\mathrm{PSU}(n+1)$. Furthermore, these curves reconstruct the solution ω as defined in Eq. (7), and they are termed curves associated with ω. The intricacies will be expounded in Lemma 1. Therefore, we have finalized the exposition detailing the correspondence between solutions to the $\mathrm{SU}(n+1)$ Toda system, as described in Eq. (2), and totally unramified unitary curves on $\mathfrak{X}\setminus\mathfrak{S}$. In Theorem 3, we further elucidate a refined correspondence between solutions to the $\mathrm{SU}(n+1)$ Toda system (4) with cone singularities $\vec{\mathfrak{D}}$ on $\mathfrak{X}$, and totally unramified unitary curves on $\mathfrak{X}\setminus\mathfrak{S}$ with regular singularities $\vec{\mathfrak{D}}$.

1.3 Toric curves and toric solutions

We introduce the following innovative concept associated with the $\mathrm{SU}(n+1)$ Toda system, inspired by its $n=1$ scenario—specifically, the reducible cone spherical metric[21–24].

Definition 3. A toric curve $f:\mathfrak{X}\setminus\mathfrak{S}$ is defined as a totally unramified unitary curve whose monodromy resides within a maximal torus of $\mathrm{PSU}(n+1)$. A solution ω to the $\mathrm{SU}(n+1)$ Toda system on $\mathfrak{X}\setminus\mathfrak{S}$ is termed toric if it yields toric associated curves. This leads to the concepts of a toric curve with regular singularities $\vec{\mathfrak{D}}$ and a toric solution with cone singularities $\vec{\mathfrak{D}}$, respectively.

Having established the foundation for our methodology in addressing Toda systems, it is now an opportune moment to review the classifications of solutions and the significant findings regarding their existence. The pioneering effort by Jost and Wang[12] in 2002 revealed that curves associated with finite-area solutions to the $\mathrm{SU}(n+1)$ Toda system on the complex plane $\mathbb{C}$ extend to rational normal curves from $\mathbb{P}^1$ to $\mathbb{P}^n$. In this way, they obtained a thorough classification of solutions on $\mathbb{P}^1$. In 2007, Eremenko[26, Theorem 2] further classified curves linked to solutions that expand polynomially in the area at $\mathbb{C}$. Continuing this trajectory, Lin et al.[13], five years on, managed to classify all solutions to the $\mathrm{SU}(n+1)$ system with two cone singularities on $\mathbb{P}^1$. Simultaneously, they also characterized the corresponding cone singularities. Recently, Karmakar et al.[7] broadened the scope to include any Toda system tied to a complex simple Lie algebra. The preceding solutions are all toric since the fundamental groups of the underlying surfaces are either trivial or isomorphic to $\mathbb{Z}$. In 2018, Lin et al.[21] obtained some existence results about solutions to the $\mathrm{SU}(n+1)$ system with three cone singularities on the Riemann sphere. Between 2015 and 2016, Battaglia[4–6] laid out extensive theorems on the existence and absence of solutions for the SU(3) Toda system with cone singularities on compact Riemann surfaces. Following suit, Lin et al.[8] made significant strides by proving existence theorems for Toda systems related to Lie algebras of types A_n, B_n, C_n, and G_2, featuring cone singularities on compact Riemann surfaces with positive genera. More recently, Chen and Lin[14–16] reported numerous findings regarding the SU(3) Toda system with cone singularities on tori. In their most recent collaboration, Sun et al.[11] meticulously categorized the solutions to the $\mathrm{SU}(n+1)$ Toda system, delineated on the disk $|z|<1$. These solutions are characterized by a cone singularity at $z=0$, possess finite area, and inherently exhibit the toric property.

In the remainder part of this section, we shall focus on toric solutions to the $\mathrm{SU}(n+1)$ Toda system with cone singularities

$$\vec{D}=\left(D_1=\sum_{i=1}^k\gamma_{i,1}[P_i],\cdots,D_n=\sum_{i=1}^k\gamma_{i,n}[P_i]\right),$$

where $P_1,\cdots,P_k$ are distinct points on a compact Riemann surface X, and $\left(\gamma_{i,j}\right)_{\substack{1\le i\le k\\1\le j\le n}}$ is a $k\times n$ matrix of real numbers greater than -1 such that for each $1\le i\le k$, at least one of $\{\gamma_{ij}\}_{1\le j\le n}$ does not vanish. We call $\{P_1,\cdots,P_k\}$ the support, and $\left(\gamma_{i,j}\right)_{\substack{1\le i\le k\\1\le j\le n}}$ the coefficient matrix of $\vec{D}$. Chen et al.[23, Theorems 1.4. and 1.5] established on X a correspondence between toric solutions to the SU(2) Toda system with cone singularities, namely reducible cone spherical metrics, and meromorphic one-forms with simple poles and purely imaginary periods on X. This correspondence is particularly noteworthy because it allows for the explicit characterization of the singularity information of a reducible metric in terms of the corresponding one-form, which was referred to as the character one-form of the metric therein. Moreover, meromorphic one-forms with simple poles and purely imaginary periods are plentiful on Riemann surfaces, cf. Refs. [27, §15], [28, §8-1], and [29, §Ⅱ.4-5] for precise statements and their proofs. In particular, the explicit formulation of such one-forms was meticulously documented in Ref. [23, Example 4.7] on the Riemann sphere. To generalize this correspondence for toric solutions to $\mathrm{SU}(n+1)$ Toda systems with cone singularities on X, we introduce the following:

Definition 4. A character n-ensemble on X is defined as a vector $\vec{\Omega}=(\Omega_1,\cdots,\Omega_n)$, composed of n meromorphic one-forms with simple poles and purely imaginary periods on X such that there exist finitely many points $P_1,\cdots,P_k$ on X and

$$f_{\vec{\Omega},\vec{\rho}}(z):=\left[1:\rho_1\cdot\exp\left(\int^z\Omega_1\right):\cdots:\rho_n\cdot\exp\left(\int^z\Omega_n\right)\right] \tag{8}$$

generates a family of totally unramified unitary curves mapping from $X\setminus\{P_1,\cdots,P_k\}$ to $\mathbb{P}^n$, where $\vec{\rho}=(\rho_1,\cdots,\rho_n)$ varies over all vectors of n positive real numbers. Importantly, these curves exhibit monodromy within the diagonal maximal torus $\mathbb{T}^n$ of $\mathrm{PSU}(n+1)$, specified by

$$\mathbb{T}^n:=\left\{\varphi([z_0:\cdots:z_n])=[\mathrm{e}^{\mathrm{i}\theta_0}z_0:\cdots:\mathrm{e}^{\mathrm{i}\theta_n}z_n]\,\Big|\,(\theta_0,\cdots,\theta_n)\in\mathbb{R}^{n+1}\right\}.$$

Hence, they are inherently toric curves from $X\setminus\{P_1,\cdots,P_k\}$ to $\mathbb{P}^n$. Within the given context, it can be demonstrated that the preceding curves described inherently possess the same regular singularities, denoted by $\vec{D}$. These singularities are characterized by a support $\{P_1,\cdots,P_k\}$ and their coefficient matrix $\left(\gamma_{i,j}\right)_{\substack{1\le i\le k\\1\le j\le n}}$, both of which are implicitly determined by the n-ensemble $\vec{\Omega}$ (Proposition 1 and Lemma 2).

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1.4 Main results

We establish on X the following general correspondence for the $\mathrm{SU}(n+1)$ Toda system which puts the one in Ref. [23] as its $n=1$ scenario.

Theorem 1. There exists a correspondence between character n-ensembles and toric solutions ω to the $\mathrm{SU}(n+1)$ Toda system with cone singularities on a compact Riemann surface X. Specifically, the following two statements hold:

(i) Given a character n-ensemble $\vec{\Omega}$ on X, it induces a family of toric curves via (8) with regular singularities $\vec{D}$ on X, which in turn defines a corresponding family of toric solutions representing $\vec{D}$, via (7).

(ii) Given a toric solution $\vec{\omega}$ with cone singularities $\vec{D}$ on X, it has an associated toric curve $f=[f_0,\cdots,f_n]$ with monodromy in $\mathbb{T}^n$ and with regular singularities $\vec{D}$ on X such that

$$\Omega:=\left(\mathrm{d}\left[\log\frac{f_1}{f_0}\right],\cdots,\mathrm{d}\left[\log\frac{f_n}{f_0}\right]\right)$$

forms a character n-ensemble on X.

A natural question arises regarding the characterization of regular singularities on curves generated by a character ensemble, as defined in (8). The following theorem offers a partial response to this inquiry.

Theorem 2. Let $\vec{\Omega}$ be a character n-ensemble and f one in the family of curves generated by $\vec{\Omega}$ in terms of (8) on a compact Riemann surface X. Denote by $\mathcal{P}$ the set of poles of all Ω_j's. The following statements hold:

(i) A point P on X is a branch point of f if and only if it is a pole of some component Ω_j of $\vec{\Omega}$ such that the residue of Ω_j at P is a noninteger.

(ii) A zero of $\vec{\Omega}$, at which all components of $\vec{\Omega}$ vanish, is a ramification point of f.

(iii) An algorithm is presented that identifies the regular singularities $\vec{D}$ of f in finitely many steps. Its details will be provided within the proof.

Below, we present two examples of toric curves with regular singularities that yield novel toric solutions to $\mathrm{SU}(n+1)$ Toda systems with cone singularities. These examples extend beyond the scope of recent existence results presented in Ref. [8, Theorems 1.8-9].

Example 1. Consider a compact Riemann surface X and let Ω be a nontrivial meromorphic one-form on X that has simple poles and purely imaginary periods. Define the vector of scaled one-forms

$$\vec{\Omega}:=(\lambda_1\Omega,\cdots,\lambda_n\Omega),$$

where $\lambda_1,\cdots,\lambda_n$ are distinct nonzero real numbers. This vector $\vec{\Omega}$ constitutes a character n-ensemble on X and generates a family of toric solutions that are parametrized by $(\mathbb{R}_{>0})^n$ as per Eq. (8).

Furthermore, the singularities of these toric solutions can be fully characterized by $\vec{\Omega}$. In particular, all the cone singularities of the solutions are confined to the set of zeroes and poles of Ω. Specifically, at a zero $\mathfrak{q}$ of Ω, each metric component in the toric solution exhibits a cone angle of $2\pi(1+\mathrm{ord}_{\mathfrak{q}}\,\Omega)$.

Example 2. Identify $\mathbb{P}^1$ with the Riemann sphere $\mathbb{C}\cup\{\infty\}$. Consider the vector $(\gamma_1,\cdots,\gamma_n)\in(-1,\infty)^n\setminus\mathbb{Z}^n$ and a positive integer m. Given m distinct points $z_1,\cdots,z_m$ on $\mathbb{C}\setminus\{0\}$ and a matrix of nonnegative integers $\left(\gamma_{i,j}\right)_{\substack{1\le i\le m\\1\le j\le n}}$, there exists a family of toric curves from $\mathbb{P}^1\setminus\{0,\infty,z_1,\cdots,z_m\}$ to $\mathbb{P}^n$ parametrized by $(\mathbb{R}_{>0})^n$ satisfying the following properties:

- All the curves have the same regular singularities. In particular, they have exactly two branch points at 0 and ∞, with $\gamma_{[0],j}=\gamma_j$ for all $1\le j\le n$;
- The points $z_1,\cdots,z_m$ are all ramification points of them, where $\gamma_{[z_i],j}=\gamma_{i,j}$ for all $1\le i\le m$ and $1\le j\le n$.

Moreover, the vector $\left(\gamma_{[\infty],j}\right)_{j=1}^n$ has finitely many choices, bounded from above by

$$\begin{pmatrix}\displaystyle\sum_{i=1}^m(n\gamma_{i,1}+(n-1)\gamma_{i,2}+\cdots+\gamma_{i,n})+n+1\\ n\end{pmatrix}.$$

Outline. The introduction concludes with an outline of the structure for the remainder of this manuscript. Section 2 focuses on establishing the relationship between curves with regular singularities and solutions with cone singularities, as detailed in Theorem 3. The proof of Theorem 2 is presented in Section 3, followed by a detailed substantiation of Theorem 1 in Section 4. Additionally, Section 3 introduces a new concept of a nondegenerate n-tuple of one-forms. This concept is simpler yet equivalent to the character n-ensemble, as demonstrated in Proposition 1. Section 5 provides a comprehensive discussion of the two preceding examples. In the final section, we propose three open questions to guide further investigation.

2 Correspondence between curves and solutions

In this section, we prove a lemma and a theorem that sequentially facilitates the establishment of the basic correspondence between solutions to the $\mathrm{SU}(n+1)$ Toda systems with cone singularities and unitary curves with regular singularities. This correspondence was initially outlined in subsection 1.2. Throughout, we will consistently use the notations previously introduced.

Lemma 1. (i) From curve to solution: A totally unramified unitary curve $f:\mathfrak{X}\setminus\mathfrak{S}\to\mathbb{P}^n$ induces a solution $\vec{\omega}:=(f_0^*\omega_{\mathrm{FS}},\cdots,f_{n-1}^*\omega_{\mathrm{FS}})$ to the $\mathrm{SU}(n+1)$ Toda system on $\mathfrak{X}\setminus\mathfrak{S}$. Moreover, any curve $\varphi\circ f$, with φ in $\mathrm{PSU}(n+1)$, produces the same solution as f.

(ii) From solution to curve: Every solution $\vec{\omega}$ to the $\mathrm{SU}(n+1)$ Toda system on $\mathfrak{X}\setminus\mathfrak{S}$ corresponds to at least one totally unramified unitary curve $f:\mathfrak{X}\setminus\mathfrak{S}\to\mathbb{P}^n$ that generates $\vec{\omega}$. Furthermore, any other curve that corresponds to $\vec{\omega}$ will be in the form of $\varphi\circ f$ for some φ in $\mathrm{PSU}(n+1)$.

Proof. (i) Consider a totally unramified unitary curve $f:\mathfrak{X}\setminus\mathfrak{S}\to\mathbb{P}^n$. For the definition of the jth associated function f_j where $0\le j\le n$, we refer to Refs. [2, pp. 263–264] and [11, Definition 2.1]. The proof of the first statement closely mirrors that of Lemma 2.2 in Ref. [11], which specifically addresses the plane domain scenario. The second statement follows from the fact that each element φ of $\mathrm{PSU}(n+1)$ preserves the Fubini–Study metric on $\mathbb{P}^n$.

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(ii) Consider a solution $\vec{\omega}=(\omega_1,\cdots,\omega_n)$ to the $\mathrm{SU}(n+1)$ Toda system on $\mathfrak{X}\setminus\mathfrak{S}$. Fix a point $\mathfrak{p}$ on $\mathfrak{X}\setminus\mathfrak{S}$. Restricting $\vec{\omega}$ to a disc chart $(D,|z|<1)$ around $\mathfrak{p}$, represented as $\vec{\omega}|_D$, results in a solution over D. Employing Ref. [11, Lemma 2.3], this restriction leads to the formation of a totally unramified holomorphic curve $\mathfrak{f}_{\mathfrak{p}}:D\to\mathbb{P}^n$, which specifically induces $\vec{\omega}|_D$. Notably, the Fubini–Study form pullback via $\mathfrak{f}_{\mathfrak{p}}$, denoted $\mathfrak{f}_{\mathfrak{p}}^*\omega_{\mathrm{FS}}$, coincides with the restricted ω_1 on D. According to the local rigidity theorem posited by Calabi (refer to Refs. [30, Theorem 9] and [31, (4.12)]), the curve $\mathfrak{f}_{\mathfrak{p}}$ is uniquely determined by $\vec{\omega}$, up to a transformation in $\mathrm{PSU}(n+1)$. Furthermore, leveraging the developing map concept outlined in Ref. [32, §3.4], a unique, totally unramified unitary curve $f:\mathfrak{X}\setminus\mathfrak{S}\to\mathbb{P}^n$ can be realized through analytic continuations of $\mathfrak{f}_{\mathfrak{p}}$ along curves on $\mathfrak{X}\setminus\mathfrak{S}$. Its monodromy results in a group homomorphism

$$\mathcal{M}_f:\pi_1(\mathfrak{X}\setminus\mathfrak{S},\mathfrak{p})\to\mathrm{PSU}(n+1).$$

This curve f not only induces $\vec{\omega}$ as initially described in (i) but is also uniquely characterized by the same local rigidity theorem within the confines of $\mathrm{PSU}(n+1)$.

The lemma above can be generalized to include cases with singularities as follows:

Theorem 3. We use the notations from Definitions 1 and 2.

(i) From curve with regular singularities to solution with cone singularities: A totally unramified unitary curve $f:\mathfrak{X}\setminus\mathfrak{S}\to\mathbb{P}^n$ with regular singularities $\vec{\mathfrak{D}}$ induces a solution $\vec{\omega}=(f_0^*\omega_{\mathrm{FS}},\cdots,f_{n-1}^*\omega_{\mathrm{FS}})$ to the $\mathrm{SU}(n+1)$ Toda system on $\mathfrak{X}$ with cone singularities $\vec{\mathfrak{D}}$. Moreover, any curve $\varphi\circ f$, where φ is in $\mathrm{PSU}(n+1)$, produces the same solution as f.

(ii) From solution with cone singularities to curve with regular singularities: Every solution $\vec{\omega}$ to the $\mathrm{SU}(n+1)$ Toda system on $\mathfrak{X}$ with cone singularities $\vec{\mathfrak{D}}$ corresponds to at least one totally unramified unitary curve $f:\mathfrak{X}\setminus\mathfrak{S}\to\mathbb{P}^n$ with regular singularities $\vec{\mathfrak{D}}$ that generates $\vec{\omega}$. We call f a curve associated with $\vec{\omega}$. Furthermore, any other curve that corresponds to $\vec{\omega}$ will be in the form of $\varphi\circ f$ for some φ in $\mathrm{PSU}(n+1)$.

Proof. By Lemma 1, it is sufficient to verify the properties concerning the singularities $\vec{D}$.

(i) Consider a totally unramified unitary curve $f:\mathfrak{X}\setminus\mathfrak{S}\to\mathbb{P}^n$ with regular singularities $\vec{\mathfrak{D}}$. According to Lemma 1(i), this curve induces a solution $\vec{\omega}$ on $\mathfrak{X}\setminus\mathfrak{S}$. Furthermore, based on Ref. [11, Formula (3)] and the infinitesimal Plücker formula, $\vec{\omega}$ exhibits cone singularities $\vec{\mathfrak{D}}$.

(ii) Consider a solution $\vec{\omega}=(\omega_1,\cdots,\omega_n)$ on $\mathfrak{X}$ with cone singularities $\vec{D}$. Lemma 1(ii) states that this solution corresponds to a totally unramified unitary curve $f:\mathfrak{X}\setminus\mathfrak{S}\to\mathbb{P}^n$. Moreover, according to Ref. [11, Theorem 1.2(i)], curve f has regular singularities $\vec{D}$.

3 Character ensembles and toric curves with regular singularities

In this section, we prove Theorem 2. To this end, we demonstrate in Proposition 1 that character n-ensembles correspond to n-tuples of meromorphic one-forms with simple poles and purely imaginary periods. These n-tuples are nondegenerate in the sense that they define nondegenerate curves under Eq. (8), as established in Definition 5. This explanation renders the concept of a character ensemble much more accessible and relatable.

3.1 Nondegenerate *n*-tuple of one-forms

Definition 5. Let $\vec{\Omega}=(\Omega_1,\cdots,\Omega_n)$ be an n-tuple of meromorphic one-forms on a compact Riemann surface X, characterized by simple poles and purely imaginary periods. We define $\vec{\Omega}$ to be nondegenerate if there exists a point $\mathfrak{p}$ in X such that:

- All components of $\vec{\Omega}$ are holomorphic at $\mathfrak{p}$.
- In a disc chart $(D,|z|<1)$ around $\mathfrak{p}$, i.e., $z=0$, the map

$$\mathfrak{f}_{\mathfrak{p}}(z):=\left[1:\exp\left(\int_0^z\Omega_1\right):\cdots:\exp\left(\int_0^z\Omega_n\right)\right]\tag{9}$$

defines a holomorphic curve from D to $\mathbb{P}^n$ that is nondegenerate, meaning that its image is not contained within any hyperplane of $\mathbb{P}^n$.

It is noteworthy that in this context, the curve $\mathfrak{f}_{\mathfrak{p}}(z)$ remains nondegenerate near every point $\mathfrak{p}$, where all components of $\vec{\Omega}$ are holomorphic. Denote by $\mathcal{P}$ the set of poles of all Ω_j's. Similar to Eq. (8), $\vec{\Omega}$ induces a family of nondegenerate, multivalued holomorphic curves

$$\{f_{\vec{\Omega},\vec{\rho}}:\vec{\rho}\in(\mathbb{R}_{>0})^n\}\tag{10}$$

mapping from $X\setminus\mathcal{P}$ to $\mathbb{P}^n$. These curves share identical monodromy in $\mathbb{T}^n$ and possess the same regular singularities on X.

Lemma 2. Adopting the context of Definition 5 and considering a fixed vector $\vec{\rho}$ consisting of n positive numbers, we have the following properties for the curve $f_{\vec{\Omega},\vec{\rho}}$:

(I) A point $\mathfrak{p}$ on X is identified as a branch point of the curve if and only if $\mathfrak{p}$ is a pole of some Ω_j where the residue $\mathrm{Res}_{\mathfrak{p}}\Omega_j$ is not an integer.

(II) The curve is totally unramified on X, except at a finite number of regular singularities.

Proof. (I) Take a disc chart $(D,|z|<1)$ around $\mathfrak{p}$ and denote by $\vec{\mathfrak{r}}=(\mathfrak{r}_g,\cdots,\mathfrak{r}_n)\in\mathbb{R}^n$ the vector of residues of $\vec{\Omega}=(\Omega_1,\cdots,\Omega_n)$ at $\mathfrak{p}$. We choose a germ $\mathfrak{f}_{\mathfrak{p}}$ of $f_{\vec{\Omega},\vec{\rho}}$ in $D^*:=\{0<|z|<1\}$ with form

$$[1:z^{\mathfrak{r}_1}h_1(z):\cdots:z^{\mathfrak{r}_n}h_n(z)]=[z^0:z^{\mathfrak{r}_1}h_1(z):\cdots:z^{\mathfrak{r}_n}h_n(z)],$$

where each h_j is holomorphic in D and does not vanish at $z=0$.

Assume that the residues $\mathfrak{r}_j$ are all integers. In this case, the curve $f_{\vec{\Omega},\vec{\rho}}$ extends holomorphically to point $\mathfrak{p}$. Therefore, either the curve is totally unramified at $\mathfrak{p}$, or $\mathfrak{p}$ is a ramification point of the curve. Specifically, $\mathfrak{p}$ is not a branch point. Furthermore, the ramification indices $\gamma_{\mathfrak{p},j}$ for $1\le j\le n$ can be calculated using the algorithm described in Ref. [2, pp. 266–268].

Suppose that at least one of the residues $\mathfrak{r}_j$ is not an integer. Consequently, set $\{0,\mathfrak{r}_1,\cdots,\mathfrak{r}_n\}$ is partitioned into at least two equivalence classes under the modulo $\mathbb{Z}$ equivalence relation. According to the argument used in the proof in Ref. [11, Theorem 3.1], there exists a transformation $\varphi\in\mathrm{PSU}(n+1)$ such that $\varphi(\mathfrak{f}_{\mathfrak{p}})$ is represented by

$$[z^{b_0}:z^{b_1}H_1(z):\cdots:z^{b_n}H_n(z)],\tag{11}$$

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where $b_0<\cdots<b_n$, and each function $H_j(z)$ is holomorphic in D and nonvanishing at $z=0$. Additionally, there is at least one index j (where $0\le j\le n-1$) for which $b_0\equiv\cdots\equiv b_j \pmod{\mathbb{Z}}$, and $\gamma_{\mathfrak{p},j+1}=b_{j+1}-b_j-1\notin\mathbb{Z}$. Therefore, $\mathfrak{p}$ is identified as a branch point of the curve $f_{\vec{\Omega},\vec{\beta}}$. Notably, we could read all $\gamma_{\mathfrak{p},\cdot}$ indices as

$$\gamma_{\mathfrak{p},j}=b_{j+1}-b_j-1 \quad \text{for all} \quad 1\le j\le n$$

from (11), called the quasi-canonical form of f.

(Ⅱ) We aim to show that the function $f_{\vec{\Omega},\vec{\beta}}$ has finitely many ramification points on a compact Riemann surface X. Assuming the contrary, let $\{q_n\}$ be a sequence of distinct ramification points converging to some point $\mathfrak{p}$ on X. We consider two cases based on the location of $\mathfrak{p}$:

(ⅰ) Case $\mathfrak{p}\notin\mathcal{P}$: Select a disc chart $(D,|z|<1)$ around $\mathfrak{p}$, which includes $\{q_n\}$ and excludes $\mathcal{P}$. Consider a germ $\mathfrak{f}_p$ of f in D and define the nth associated curve $\Lambda_n(\mathfrak{f}_{\mathfrak{p}},z)=\mathfrak{f}_{\mathfrak{p}}(z)\wedge\mathfrak{f}_{\mathfrak{p}}{}'(z)\wedge\cdots\wedge\mathfrak{f}_{\mathfrak{p}}{}^{(n)}(z)$, mapping D to $\Lambda^{n+1}(\mathbb{C}^{n+1})$. According to the computations in Ref. [2], $\Lambda_n(\mathfrak{f}_{\mathfrak{p}},z)$ vanishes at $\{q_n\}\cup\{\mathfrak{p}\}$, and consequently, it vanishes identically on D. This implies that $\Lambda_n(f)$ and f are degenerate on $X\setminus\mathcal{P}$. This is a contradiction since both $\vec{\Omega}$ and f are nondegenerate.

(ⅱ) Case $\mathfrak{p}\in\mathcal{P}$: Using a similar setup with a disc chart $(D,|z|<1)$ around $\mathfrak{p}$ as in (1), we consider the germ $\mathfrak{f}_{\mathfrak{p}}=[z^{b_0}H_0(z):\cdots:z^{b_n}H_n(z)]$ where $b_0<\cdots<b_n$ and each H_j is holomorphic in D and nonvanishing at $z=0$. By Lemma 4.1 in Ref. [11], we have

$$\Lambda_n(\mathfrak{f}_{\mathfrak{p}},z)=z^{\sum_{i=0}^n b_i-\frac{n(n+1)}{2}}\cdot G_n(b_0,\cdots,b_n;H_0(z),\cdots,H_n(z);z)\cdot e_0\wedge\cdots\wedge e_n,$$

where G_n is holomorphic in D and nonvanishing at $z=0$. The vanishing of $\Lambda_n(\mathfrak{f}_{\mathfrak{p}},z)$ at each q_n and thus identically in D implies that G_n vanishes identically, which is another contradiction.

Now we present the key proposition of this subsection.

Proposition 1. On a compact Riemann surface X, the concept of a character n-ensemble, as outlined in Definition 4, is equivalent to that of a nondegenerate n-tuple of one-forms, as described in Definition 5.

Proof. By definition, a character n-ensemble is inherently nondegenerate when considered as an n-tuple of one-forms. This nondegeneracy is essential for defining character n-ensembles. According to the second statement of Lemma 2, any nondegenerate n-tuple of one-forms automatically constitutes a character n-ensemble. Therefore, the properties required for an n-tuple to be a character n-ensemble are exactly those that prevent degeneracy among the one-forms in the tuple.

In the analysis that follows, we categorize three specified n-tuples of one-forms on a compact Riemann surface. We identify the nature of each tuple, highlighting whether it is degenerate, i.e., whether it forms a character n-ensemble.

Example 3. Let $\mathfrak{C}=\mathfrak{C}_X$ denote the infinite-dimensional real linear space of meromorphic one-forms with simple poles and purely imaginary periods on a compact Riemann surface X (see Refs. [27, §15], [28, §8-1], and [29, §Ⅱ.4-5]). Denote by $\mathfrak{C}^*=\mathfrak{C}_X\setminus\{0\}$ the nonzero elements of this space.

(ⅰ) It is possible to find two linear independent one-forms Ω_1 and Ω_2 in $\mathfrak{C}_X^*$ such that the pair $\vec{\Omega}=(\Omega_1,\Omega_2)$ is degenerate, meaning that $\vec{\Omega}$ does not form a character 2-ensemble. An example of such a pair on the Riemann sphere is given by $\Omega_1=\frac{\mathrm{d}z}{z}$ and $\Omega_2=\frac{\mathrm{d}z}{z+1}$. By using (10) and setting $\mathfrak{p}=1$, we can see that (Ω_1,Ω_2) generates a line in $\mathbb{P}^2$.

(ⅱ) Consider a one-form $\Omega\in\mathfrak{C}^*$ and n nonzero real numbers $\lambda_1,\cdots,\lambda_n$. The tuple $\vec{\Omega}=(\lambda_1\Omega,\cdots,\lambda_n\Omega)$ is nondegenerate if and only if all coefficients λ_j are mutually distinct. We can argue as follows. Define $y=\exp\left(\int_0^z\Omega\right)$. In a small disc chart $(D,|z|<1)$, where Ω is holomorphic and nonvanishing, we consider the nth associated curve $\Lambda_n(f)$ of the curve

$$f(z)=\left[1:\exp\left(\int_0^z\Omega_1\right):\cdots:\exp\left(\int_0^z\Omega_n\right)\right]=[1:y^{\lambda_1}:\cdots:y^{\lambda_n}].$$

This curve is nondegenerate in D, thereby confirming that $\vec{\Omega}$ is nondegenerate, as shown by the following computation:

$$\begin{aligned}\Lambda_n(f)&=f\wedge f'\wedge\cdots\wedge f^{(n)}=f\wedge\frac{\partial f}{\partial y}\wedge\cdots\wedge\frac{\partial^n f}{\partial y^n}\cdot\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^{\frac{n(n+1)}{2}}=\\&\begin{vmatrix}1 & y^{\lambda_1} & \cdots & y^{\lambda_n}\\ 0 & \lambda_1 y^{\lambda_1-1} & \cdots & \lambda_n y^{\lambda_n-1}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & \lambda_1(\lambda_1-1)\cdots(\lambda_1-n+1)y^{\lambda_1-n} & \cdots & \lambda_n(\lambda_n-1)\cdots(\lambda_n-n+1)y^{\lambda_n-n}\end{vmatrix}\cdot(y')^{\frac{n(n+1)}{2}}e_0\wedge e_1\wedge\cdots\wedge e_n=\\&\prod_{i=1}^n\lambda_i\prod_{1\le j<i\le n}(\lambda_i-\lambda_j)\cdot y^{\sum_{i=0}^n\lambda_i-\frac{n(n+1)}{2}}\cdot(y')^{\frac{n(n+1)}{2}}e_0\wedge e_1\wedge\cdots\wedge e_n.\end{aligned}\tag{12}$$

(iii) This example can be generalized as follows: Let $\Omega_1,\cdots,\Omega_n$ be n distinct one-forms in $\mathfrak{C}^*$, and assume there is a point $\mathfrak{p}$ on X where the residues of each Ω_j at $\mathfrak{p}$ are distinct nonzero numbers. Under these conditions, $\vec{\Omega}=(\Omega_1,\cdots,\Omega_n)$ forms a nondegenerate n-tuple. The proof is similar to (ⅱ).

3.2 Proof of Theorem 2

The first statement of the theorem aligns with Lemma 2(Ⅰ). We proceed to demonstrate the second statement as follows:

Consider a disc chart $(D,|z|<1)$ centered at a zero $\mathfrak{p}$ of $\vec{\Omega}$. Let $(k_1,\cdots,k_n)\in\mathbb{Z}_{>0}^n$ represent the vector of multiplicities of $\vec{\Omega}=(\Omega_1,\cdots,\Omega_n)$ at $\mathfrak{p}$. We choose a germ $\mathfrak{f}_{\mathfrak{p}}$ of $f_{\vec{\Omega},\vec{\beta}}$ in D as follows:

$$[1:C_1+z^{k_1+1}h_1(z):\cdots:C_n+z^{k_n+1}h_n(z)],$$

where each $h_j(z)$ is a holomorphic function in D that does not vanish at $z=0$, and each C_j is a nonzero complex number. Given that each k_i+1 is an integer greater than 1, and using the calculations presented in Ref. [2, pp. 266–268], we determine that the ramification index of $f=f_0$ at $\mathfrak{p}$ is $\min(k_1,\cdots,k_n)>0$. Consequently, $\mathfrak{p}$ is confirmed as a ramification point of f.

Finally, we present an algorithm to identify the regular singularities of the curve $f(z)=\left[1:\exp\left(\int^z\Omega_1\right):\cdots:\exp\left(\int^z\Omega_n\right)\right]$ generated by a character n-ensemble $\vec{\Omega}$. The procedure is as follows:

(i) Cover the manifold X using a finite collection of disk charts $(D_i,|z_i|<1)$ for $i=1,\cdots,N$. Ensure that each pole in the set $\mathcal{P}$ is contained within exactly one chart.

(ii) Within these charts, and excluding the poles in $\mathcal{P}$, solve the equation $\Lambda_n(f)=0$ to identify $\mathcal{Z}$, the set of all ramification points of f not in $\mathcal{P}$.

(iii) For each point $\mathfrak{p}\in\mathcal{P}\cup\mathcal{Z}$, the appropriate chart $(D,|z|<1)$ around it is selected. The process outlined in the proof of Lemma 2(I) is used to determine the quasi-canonical form of f. This step allows us to ascertain all $\gamma_{\mathfrak{p},\cdot}$ indices.

4 Correspondence between character ensembles and toric solutions

In this section, we prove Theorem 1 and establish a correspondence between character ensembles and toric solutions with cone singularities on a compact Riemann surface X.

Proof of Theorem 1. (i) The first statement of Theorem 1 follows from Proposition 1 and Lemma 2.

(ii) Consider a toric solution $\vec{\omega}$ with cone singularities at $\vec{D}$ on a compact Riemann surface X. Let $f:X\setminus\{P_1,\cdots,P_k\}\to\mathbb{CP}^n$ be an associated curve derived from this solution. This curve is characterized as a totally unramified unitary curve, and its monodromy is constrained within a maximal torus of the group $\mathrm{PSU}(n+1)$. To specifically align the monodromy of the curve within $\mathbb{T}^n$, a subgroup of the maximal torus, we select an appropriate element $\varphi\in\mathrm{PSU}(n+1)$. Applying φ, the transformed curve $\varphi\circ f:X\setminus\{P_1,\cdots,P_k\}\to\mathbb{CP}^n$ indeed has its monodromy contained within $\mathbb{T}^n$. We will proceed under the assumption that this simplification applies to the curve f.

Choose a sufficiently small disc chart $(D,|z|<1)$ around $\mathfrak{p}\in\{P_1,\cdots,P_k\}$. Using the argument in the proof of Ref. [28, Theorem 3.1], the restriction of $f=[f_0:\cdots:f_n]$ to D can be expressed as

$$f(z)=[z^{b_0}\phi_0(z):\cdots:z^{b_n}\phi_n(z)] \tag{13}$$

where $b_i\in\mathbb{R}$ and ϕ_i is a holomorphic function that does not vanish on D for all $1\le i\le n$.

We define an n-tuple $\vec{\Omega}=(\Omega_1,\cdots,\Omega_n)$ by

$$\Omega_k=\mathrm{d}\left(\log\frac{f_k}{f_0}\right),\quad k=1,\cdots,n. \tag{14}$$

By computation, we have

$$\Omega_k=\left((b_k-b_0)\frac{1}{z}+\frac{\phi_k'(z)}{\phi_k(z)}-\frac{\phi_0'(z)}{\phi_0(z)}\right)\mathrm{d}z. \tag{15}$$

Hence, each Ω_j has at most simple poles. Additionally, it can be verified that

$$2\Re\,\Omega_k=\mathrm{d}\left(\log\left|\frac{f_k}{f_0}\right|^2\right), \tag{16}$$

i.e., the real part of each Ω_k is exact outside of its poles. Consequently, each Ω_k is a meromorphic one-form with simple poles and purely imaginary periods. Since $\vec{\Omega}$ is nondegenerate (as f is nondegenerate), it is a character n-ensemble on X by Proposition 1.

By Theorem 1.5 in Ref. [23], the regular singularities of a toric curve to $\mathbb{P}^1$, which is derived from a character one-form, are confined to the union of the zeros and poles of the one-form. However, the following example demonstrates that a toric curve to $\mathbb{P}^n$, generated from a character n-ensemble for $n>1$, can possess ramification points that do not coincide with the zeros or the poles of any ensemble component.

Example 4. Consider the toric curve defined by $f=[1:z:z^2:\cdots:z^{n-1},z^{n+1}]$, where $n>1$. $z=0$ is a ramification point. The associated character n-ensemble, $\vec{\Omega}=(\Omega_1,\cdots,\Omega_n)$, is given by:

$$\Omega_k=\frac{k}{z}\mathrm{d}z,\quad \text{for } k=1,2,\cdots,n-1;\quad \Omega_n=\frac{n+1}{z}\mathrm{d}z.$$

Here, $z=0$ is a common pole of all components Ω_j. However, applying a nondegenerate linear transformation to f results in

$$\tilde{f}=[1:1+z:1+z+z^2:\cdots:1+z+z^{n-1}:1+z+z^{n+1}],$$

with the modified character n-ensemble $\vec{\widetilde{\Omega}}=(\widetilde{\Omega}_1,\cdots,\widetilde{\Omega}_n)$ represented as:

$$\widetilde{\Omega}_1=\frac{\mathrm{d}z}{1+z},\quad \widetilde{\Omega}_2=\frac{1+2z}{1+z+z^2}\mathrm{d}z,\ \cdots,$$
$$\widetilde{\Omega}_{n-1}=\frac{1+(n-1)z^{n-2}}{1+z+z^{n-1}}\mathrm{d}z,\quad \widetilde{\Omega}_n=\frac{1+(n+1)z^n}{1+z+z^{n+1}}\mathrm{d}z.$$

Although $z=0$ remains a ramification point for $\tilde{f}$, it is notably no longer a zero or pole of any new one-forms.

5 Two examples

In this section, we explore the practical application of correspondence between character ensembles and toric solutions, as discussed in the previous section. We conduct a thorough analysis of Examples 1 and 2. These cases introduce innovative toric solutions to the $\mathrm{SU}(n+1)$ Toda system with cone singularities. Notably, these examples expand upon the recent findings detailed in Ref. [8, Theorems 1.8-9], extending the known boundaries of this research area.

5.1 Example 1

This subsection details Example 1. Utilizing Example 3(ii) and Proposition 1, we can see that Ω constitutes a character n-ensemble on X. This ensemble generates a family of toric curves parametrized by $(\mathbb{R}_{>0})^n$ as per Eq. (8), all sharing the same regular singularities, denoted by $\vec{D}$. According to Theorem 3, these curves are associated with a family of toric solutions that exhibit the same cone singularities $\vec{D}$. Notations from Example 3(ii) are used herein.

By Eq. (12), the singularities of the curve

$$f(z)=\left[1:\exp\left(\int_0^z\lambda_1\Omega\right):\cdots:\exp\left(\int_0^z\lambda_n\Omega\right)\right]=[1:y^{\lambda_1},\cdots,y^{\lambda_n}]$$

are limited to the set of zeros and poles of Ω. Considering a point $\mathfrak{p}$ within this set, we categorize the analysis into the following two cases:

(i) $\mathfrak{p}$ is a zero of Ω with order k: We establish that $\mathfrak{p}$ is a ramification point of the curve f with all ramification indices equal to k. Assuming without loss of generality that $y(\mathfrak{p})=1$, we express $y(z)=1+z^{k+1}\phi(z)$ in a small disc chart $(D,|z|<1)$ around $\mathfrak{p}$, where $\phi(z)$ is holomorphic and nonvanishing in D.

Expanding each y^{λ_i} around $\mathfrak{p}$ into a power series, we obtain:

$$y^{\lambda_i}=1+\lambda_i z^{k+1}\varphi(z)+\frac{\lambda_i(\lambda_i-1)}{2}z^{2k+2}\varphi(z)^2+\cdots,\quad i=1,2,\cdots,n.$$

Consequently, f can be expressed as:

$$\begin{aligned}f=&(1,y^{\lambda_1},y^{\lambda_2},\cdots,y^{\lambda_n})=\\&(1,z^{k+1}\varphi(z),z^{2k+2}\varphi(z)^2,\cdots)\cdot\\&\begin{pmatrix}1&1&\cdots&1\\0&\lambda_1&\cdots&\lambda_n\\0&\dfrac{\lambda_1(\lambda_1-1)}{2}&\cdots&\dfrac{\lambda_n(\lambda_n-1)}{2}\\\vdots&\vdots&\ddots&\vdots\end{pmatrix}.\end{aligned}$$

Right-multiplying f by an $(n+1)\times(n+1)$ invertible matrix corresponds to applying column transformations to the $\infty\times(n+1)$ infinite matrix on the right-hand side of the preceding equation. The first $n+1$ rows of this matrix

$$\begin{pmatrix}1&1&\cdots&1\\0&\lambda_1&\cdots&\lambda_n\\0&\dfrac{\lambda_1(\lambda_1-1)}{2}&\cdots&\dfrac{\lambda_n(\lambda_n-1)}{2}\\\vdots&\vdots&\ddots&\vdots\\0&\dfrac{\lambda_1(\lambda_1-1)\cdots(\lambda_1-n+1)}{n!}&\cdots&\dfrac{\lambda_n(\lambda_n-1)\cdots(\lambda_n-n+1)}{n!}\end{pmatrix}$$

forms an $(n+1)\times(n+1)$ invertible matrix, with a determinant given by:

$$\prod_{i=1}^n i!\prod_{i=1}^n\lambda_i\prod_{1\le j<i\le n}(\lambda_i-\lambda_j)\neq 0.$$

Thus, it can be transformed into a lower triangular matrix by applying a finite number of column transformations, such as those used in the Gaussian elimination process. Correspondingly, f has been transformed into the quasi-canonical form:

$$(1,z^{k+1}\varphi(z)+\cdots,z^{2k+2}\varphi(z)^2+\cdots,\cdots\cdots)$$

under finitely many transforms in $\mathrm{PGL}(n+1,\mathbb{C})$. By the definition of ramification indices[2, pp. 266–268] for a holomorphic curve, applying such transformations to the curve does not alter its ramification indices. Hence, all the n ramification indices of f at $\mathfrak{p}$ coincide with those of the preceding quasi-canonical curve, each equal to k.

(ii) $\mathfrak{p}$ is a pole of Ω: Assume $\mathrm{RES}_{\mathfrak{p}}(\Omega)=a\in\mathbb{R}\setminus\{0\}$. In a small disk chart $(D,|z|<1)$ around point $\mathfrak{p}$, the function $y=\exp\left(\int_0^z\Omega\right)$ is represented as $z^a\phi(z)$, where $\phi(z)$ is holomorphic and nonvanishing within D. Accordingly, we define $f(z)=[1:z^{\lambda_1 a}\phi(z)^{\lambda_1}:\cdots:z^{\lambda_n a}\phi(z)^{\lambda_n}]$, representing a totally unramified unitary curve in D^* with a potential regular singularity at $z=0$ by Eq. (12).

By Theorem 3, f induces a solution $\vec{\omega}$ to the $\mathrm{SU}(n+1)$ Toda system on D, characterized by a possible cone singularity at $z=0$. Theorem 3 assures that the singularities of both types at $z=0$ are equivalent. Given the distinct real numbers $0,\lambda_1 a,\lambda_2 a,\cdots,\lambda_n a$, we reorder them in ascending order to achieve $\mu_0<\mu_1<\cdots<\mu_n$. Concurrently, the components of f are rearranged in ascending powers of z to attain the quasi-canonical form:

$$\left[z^{\mu_0}\phi(z)^{\frac{\mu_0}{a}}:z^{\mu_1}\phi(z)^{\frac{\mu_1}{a}}:\cdots:z^{\mu_n}\phi(z)^{\frac{\mu_n}{a}}\right].$$

The solution $\vec{\omega}$, and consequently its cone singularity at $z=0$, remains unaltered, as this constitutes merely a transformation within $\mathrm{PSU}(n+1)$. It is established that $\gamma_{\mathfrak{p},i}=\mu_i-\mu_{i-1}-1$ for all $1\le i\le n$. Depending on the specific λ values, point $\mathfrak{p}$ may be classified as a branch point, a ramification point, or a totally unramified point of f.

5.2 New solutions with cone singularities on a compact Riemann surface

The curve in this example corresponds to a novel class of solutions to the $\mathrm{SU}(n+1)$ Toda system with cone singularities on a compact Riemann surface. Ref. [8, Theorems 1.8-9] presented various sufficient conditions for the existence of such solutions on a compact Riemann surface X of genus $g_X>0$. Specifically, by transforming Eq. (4), it can be reformulated into the equivalent form:

$$\Delta_g\tilde{u}_i+\sum_{j=1}^n a_{ij}\mathrm{e}^{\tilde{u}_j}-K_0=4\pi\sum_{k=1}^m\gamma_{k,i}\delta_{p_k}\quad i=1,2,\cdots,n,$$

where K_0 is the Gaussian curvature function of the background conformal metric g on X and Δ_g is its Laplacian. The function $\tilde{u}_i$ represents a global transformation of u_i from Eq. (1), adjusted by the background metric in each coordinate chart. The singularity coefficients $\{\gamma_{i,j}\}$ remain consistent with those described in Eq. (4). Ref. [8] defined ρ_i for all $1\le i\le n$ as follows:

$$\rho_i:=4\pi\sum_{k=1}^m\sum_{j=1}^n a^{ij}\gamma_{k,j}+\sum_{j=1}^n a^{ij}\int_X K_0.$$

Here, $(a^{ij})=\left(\dfrac{j(n+1-i)}{n+1}\right)$ represents the inverse of the Cartan matrix (a_{ij}) for $\mathfrak{su}(n+1)$. According to Theorem 1.9 in Ref. [8], if $\rho_i\notin\Gamma_i$ for any $i=1,2,\cdots,n$, a solution to the Toda system is feasible. The definition of Γ_i $(1\le i\le n)$ is detailed in Ref. [8, pp. 340–341], noting that each includes $4\pi\mathbb{N}$.

In this example, Ω is a meromorphic one-form on a compact Riemann surface with positive genus, featuring two simple poles (denoted p_1 and p_2). Assuming $0<\lambda_1<\cdots<\lambda_n$, with $\mathrm{Res}_{p_1}(\omega)=a\neq 0$, the corresponding $\gamma_{1,j}=(\lambda_j-\lambda_{j-1})a-1$ for $j=1,2,\cdots,n$ (assuming $\lambda_0=0$ for consistency). According to the residue theorem, $\mathrm{Res}_{p_2}(\omega)=-a$, and as previously noted, $\gamma_{2,j}=(\lambda_{n-j+1}-\lambda_{n-j})a-1$. Additionally, the other $m-2$ singular points $p_3,\cdots,p_m$ are identified as ramification points,

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with $\gamma_{i,j} \equiv k_i$ for $i = 3, \cdots, m$ and $j = 1, 2, \cdots, n$, where k_i denotes the order of zero of Ω at p_i. After calculations, we derive:

$$\begin{aligned}\rho_i =& 4\pi \sum_{k=1}^{m}\sum_{j=1}^{n} a^{ij}\gamma_{k,j} + \sum_{j=1}^{n} a^{ij}\int_X K_0 = \\ & 4\pi\left(\sum_{j=1}^{n}\frac{j(n+1-i)}{n+1}((\lambda_j - \lambda_{j-1})a - 1) + \right.\\ & \left.\sum_{j=1}^{n}\frac{j(n+1-i)}{n+1}((\lambda_{n-j+1} - \lambda_{n-j})a - 1)\right) + \\ & 4\pi\sum_{k=3}^{m}\sum_{j=1}^{n}\frac{j(n+1-i)}{n+1}k_i + \sum_{j=1}^{n}\frac{j(n+1-i)}{n+1}\cdot 2\pi(2-2g_X) = \\ & 4\pi(n+1-i)a\lambda_n + 4\pi(k_3 + \cdots + k_n - 1 - g_X)\cdot n(n+1-i).\end{aligned}$$

Choosing $a\lambda_n$ to be a positive integer, by this calculation, we can see that all ρ_i's lie within $4\pi\mathbb{N} \subset \Gamma_i$ for all $1 \le i \le n$, violating the criteria of Theorem 1.9 in Ref. [8]. Furthermore, Theorem 1.8 in Ref. [8] also outlines sufficient conditions for the existence of solutions to the SU$(n+1)$ Toda system, requiring all $\{\gamma_{i,j}\}$ to be nonnegative integers. By choosing appropriate $\lambda_1, \cdots, \lambda_n$ such that p_1 and p_2 act as branch points, these conditions are not met. Both theorems stipulate that the genus of X must be positive.

In summary, the solutions of the Toda system associated with the curves in Example 1 on a general compact Riemann surface and Example 2 on the Riemann sphere introduce new possibilities beyond the scope outlined in Ref. [8].

5.3 Example 2

We establish the existence of the necessary curves and provide an approximate enumeration of all possible singularity data at ∞ for these curves in this example.

Existence. The following lemma is useful for establishing existence.

Lemma 3. Let $\alpha > 0$. For any polynomial $P(z)$, there exists a unique polynomial $Q(z)$, having the same degree as $P(z)$, such that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^\alpha Q(z)\right) = z^{\alpha-1}P(z). \tag{17}$$

Proof. First, consider the monomial $Q(z) = \dfrac{z^j}{\alpha + j}$ and observe the derivative

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^\alpha Q(z)\right) = z^{\alpha-1}(\alpha Q(z) + zQ'(z)) = z^{\alpha-1}\cdot z^j.$$

This calculation allows us to solve the linear ODE (17) for any polynomial $P(z)$. The uniqueness of the solution is straightforward to establish.

Proposition 2. We adopt the notations from Example 2 and assume $0 = \beta_0 < \beta_1 < \cdots < \beta_n$ with the increments $\beta_i - \beta_{i-1} = 1 + \gamma_i$ for all $1 \le i \le n$. Consequently, there exist n polynomials $\varphi_1(z), \varphi_2(z), \cdots, \varphi_n(z)$, each nonvanishing at $z = 0$, such that the family of curves

$$f_{\vec{\rho}}(z) = [1 : \rho_1 z^{\beta_1}\varphi_1(z) : \cdots : \rho_n z^{\beta_n}\varphi_n(z)],\ \vec{\rho} = (\rho_1, \cdots, \rho_n) \in (\mathbb{R}_{>0})^n,$$

meets the requirements specified in Example 2.

Proof. We construct these n polynomials $\varphi_1(z), \varphi_2(z), \cdots, \varphi_n(z)$ by using the iteration argument.

Step 1. Take

$$\psi_1(z) := \prod_{i=1}^{m}(z - z_i)^{\gamma_{i,1}},$$
$$\psi_k(z) := \prod_{i=1}^{m}(z - z_i)^{\gamma_{i,1}}\cdot\varphi_k^{(1)}(z), \quad 2 \le k \le n,$$

where $\varphi_2^{(1)}(z), \cdots, \varphi_n^{(1)}(z)$ are $(n-1)$ polynomials that will be chosen later. Given that $0 < \beta_1 < \cdots < \beta_n$, Lemma 3 ensures the existence of n polynomials $\varphi_1(z), \cdots, \varphi_n(z)$ such that the derivative of $\hat{f}(z) := (1, z^{\beta_1}\varphi_1(z), z^{\beta_2}\varphi_2(z), \cdots, z^{\beta_n}\varphi_n(z))$ is given by

$$\frac{\mathrm{d}\hat{f}(z)}{\mathrm{d}z} = (0, z^{\beta_1 - 1}\psi_1(z), \cdots, z^{\beta_n - 1}\psi_n(z)) =: z^{\beta_1 - 1}\prod_{i=1}^{m}(z - z_i)^{\gamma_{i,1}}\vec{g}_1(z), \tag{18}$$

where $\vec{g}_1(z) = \left(0, 1, z^{\beta_2 - \beta_1}\varphi_2^{(1)}(z), \cdots, z^{\beta_n - \beta_1}\varphi_n^{(1)}(z)\right)$. Moreover, we could determine the polynomial $\varphi_1(z)$ of degree $\sum_{\ell=1}^{m}\gamma_{\ell,1}$, which does not vanish at $z = 0$.

Step 2. Differentiating $\dfrac{\mathrm{d}\hat{f}(z)}{\mathrm{d}z}$ again, we obtain

$$\frac{\mathrm{d}^2\hat{f}(z)}{\mathrm{d}z^2} = \frac{\mathrm{d}}{\mathrm{d}z}\left(z^{\beta_1 - 1}\prod_{i=1}^{m}(z - z_i)^{\gamma_{i,1}}\right)\cdot\vec{g}_1(z) + z^{\beta_1 - 1}\prod_{i=1}^{m}(z - z_i)^{\gamma_{i,1}}\cdot\frac{\mathrm{d}\vec{g}_1(z)}{\mathrm{d}z}. \tag{19}$$

For any $(n-2)$ polynomials $\varphi_3^{(2)}(z), \cdots, \varphi_n^{(2)}(z)$ that will be chosen later, Lemma 3 ensures that $(n-1)$ polynomials $\varphi_2^{(1)}(z), \cdots, \varphi_n^{(1)}(z)$ exist such that

$$\begin{aligned}\frac{\mathrm{d}\vec{g}_1}{\mathrm{d}z} =& \frac{\mathrm{d}}{\mathrm{d}z}\left(0, 1, z^{\beta_2 - \beta_1}\varphi_2^{(1)}(z), \cdots, z^{\beta_n - \beta_1}\varphi_n^{(1)}(z)\right) = \\ & z^{\beta_2 - \beta_1 - 1}\prod_{i=1}^{m}(z - z_i)^{\gamma_{i,2}}\cdot\left(0, 0, 1, z^{\beta_3 - \beta_2}\varphi_3^{(2)}(z), \cdots, z^{\beta_n - \beta_2}\varphi_n^{(2)}(z)\right) := \\ & z^{\beta_2 - \beta_1 - 1}\prod_{i=1}^{m}(z - z_i)^{\gamma_{i,2}}\cdot\vec{g}_2(z),\end{aligned} \tag{20}$$

where $\vec{g}_2(z) = \left(0, 0, 1, z^{\beta_3 - \beta_2}\varphi_3^{(2)}(z), \cdots, z^{\beta_n - \beta_2}\varphi_n^{(2)}(z)\right)$. Moreover, by Eqs. (18) and (20), we could determine these three polynomials $\varphi_2^{(1)}(z)$, $\psi_2(z)$, and $\varphi_2(z)$ such that

$$\deg\varphi_2 = \sum_{\ell=1}^{m}(\gamma_{\ell,1} + \gamma_{\ell,2})$$

and all of them do not vanish at $z = 0$. Substituting this equation into Eq. (19) yields

$$\frac{\mathrm{d}^2\hat{f}(z)}{\mathrm{d}z^2} = \left(z^{\beta_1 - 1}\prod_{i=1}^{m}(z - z_i)^{\gamma_{i,1}}\right)'\cdot\vec{g}_1(z) + z^{\beta_2 - 2}\prod_{i=1}^{m}(z - z_i)^{\gamma_{i,1} + \gamma_{i,2}}\cdot\vec{g}_2(z). \tag{21}$$

Repeating the above process, we continue to differentiate $\dfrac{\mathrm{d}^2\hat{f}(z)}{\mathrm{d}z^2}$ and $\vec{g}_2$ and obtain

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$$\vec{g}_3 = \left(0,0,0,1,z^{\beta_4-\beta_3}\varphi_4^{(3)}(z),\cdots,z^{\beta_n-\beta_3}\varphi_n^{(3)}(z)\right),$$
$$\vdots$$
$$\vec{g}_n \equiv \left(0,0,\cdots,0,1\right).$$

In summary, for each increment in the subscript i of $\vec{g}_i$, the vector-valued function $\vec{g}_i$ exhibits one additional vanishing component in its earlier entries. After carrying out this procedure n times, we find all the n polynomials $\varphi_1(z),\cdots,\varphi_n(z)$ such that

$$\deg \varphi_k = \sum_{j=1}^{k}\sum_{\ell=1}^{m}\gamma_{\ell,j}, \quad 1\le k\le n, \tag{22}$$

and all of them do not vanish at $z=0$.

Step 3. In the final step, we show that the curve

$$f(z) := [\hat{f}(z)] = [1 : z^{\beta_1}\varphi_1(z) : \cdots : z^{\beta_n}\varphi_n(z)]$$

satisfies the requirements specified in Example 2. Since $\hat{f}(z)$ itself is of quasi-canonical form near $z=0$, it has the desired singularity information there, i.e., $\gamma_{[0],j}=\gamma_j=\beta_j-\beta_{j-1}-1$ for all $1\le j\le n$. By the induction argument, we obtain

$$\begin{aligned}&\hat{f}^{(k)}(z) - z^{\beta_k-k}\prod_{i=1}^{m}(z-z_i)^{\gamma_{i,1}+\cdots+\gamma_{i,k}}\cdot\vec{g}_k(z) = \\ &a_0^{(k)}\hat{f}(z)+a_1^{(k)}\hat{f}'(z)+\cdots+a_{k-1}^{(k)}\hat{f}^{(k-1)}(z), \quad 1\le k\le n,\end{aligned} \tag{23}$$

where all $a_i^{(j)}$ are multivalued holomorphic functions. Then we obtain the following equation for each $1\le k\le n$,

$$\begin{aligned}&\Lambda_k(\hat{f}) = \\ &\hat{f}\wedge\hat{f}'\wedge\cdots\wedge\hat{f}^{(k)} = \\ &\hat{f}\wedge z^{\beta_1-1}\prod_{i=1}^{m}(z-z_i)^{\gamma_{i,1}}\cdot\vec{g}_1\wedge\cdots\wedge z^{\beta_k-k}\prod_{i=1}^{m}(z-z_i)^{\gamma_{i,1}+\cdots+\gamma_{i,k}}\cdot\vec{g}_k = \\ &z^{\sum_{i=1}^{k}\beta_i-\frac{k(k+1)}{2}}\prod_{i=1}^{m}(z-z_i)^{k\gamma_{i,1}+(k-1)\gamma_{i,2}+\cdots+\gamma_{i,k}}\hat{f}\wedge\vec{g}_1\wedge\cdots\wedge\vec{g}_k = \\ &z^{\sum_{i=1}^{k}\beta_i-\frac{k(k+1)}{2}}\prod_{i=1}^{m}(z-z_i)^{k\gamma_{i,1}+(k-1)\gamma_{i,2}+\cdots+\gamma_{i,k}}\cdot\\ &\begin{pmatrix}1\\ z^{\beta_1}\varphi_1(z)\\ z^{\beta_2}\varphi_2(z)\\ \vdots\\ z^{\beta_n}\varphi_n(z)\end{pmatrix}^{\mathrm{T}}\wedge\begin{pmatrix}0\\ 1\\ z^{\beta_2-\beta_1}\varphi_2^{(1)}(z)\\ \vdots\\ z^{\beta_n-\beta_1}\varphi_n^{(1)}(z)\end{pmatrix}^{\mathrm{T}}\wedge\cdots\wedge\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ z^{\beta_{k+1}-\beta_k}\varphi_{k+1}^{(k)}(z)\\ \vdots\\ z^{\beta_n-\beta_k}\varphi_n^{(k)}(z)\end{pmatrix}^{\mathrm{T}}.\end{aligned} \tag{24}$$

Notably, the first term of $\hat{f}\wedge\vec{g}_1\wedge\cdots\wedge\vec{g}_k$ equals $e_0\wedge\cdots\wedge e_k$. From this relationship, $\Lambda_n(\hat{f})$ remains nonzero outside the set $\{0,\infty,z_1,\cdots,z_m\}$, where the curve f is totally unramified. Moreover, for the nondegenerate unitary curve $f:\mathbb{C}\setminus\{0\}\to\mathbb{P}^n$, $\Lambda_n(\hat{f})$ vanishes at $z_1,\cdots,z_m$, which constitute all the ramification points of f.

For each $1\le\ell\le m$, by applying a linear transformation in $\mathrm{GL}(n+1,\mathbb{C})$ to $\hat{f}$, we can achieve the quasi-canonical form of $\hat{f}$ near its ramification point z_k. This form is given by

$$\begin{aligned}\hat{f}(z) = \Big(&1,(z-z_\ell)^{\gamma_{[z_\ell],1}+1}g_1(z),(z-z_\ell)^{\gamma_{[z_\ell],1}+\gamma_{[z_\ell],2}+2}g_2(z),\cdots,\\ &(z-z_\ell)^{\gamma_{[z_\ell],1}+\cdots+\gamma_{[z_\ell],n}+n}g_n(z)\Big),\end{aligned}$$

where $g_1,\cdots,g_n$ are multivalued holomorphic functions that do not vanish at z_ℓ. By straightforward computation, for each $1\le k\le n$, the kth associated curve $\Lambda_k(\hat{f})$ is expressed as

$$\Lambda_k(\hat{f}) = \prod_{i=1}^{m}(z-z_\ell)^{k\gamma_{[z_\ell],1}+(k-1)\gamma_{[z_\ell],2}+\cdots+\gamma_{[z_\ell],k}}\cdot\vec{\lambda}_k(z), \tag{25}$$

where $\vec{\lambda}_k(z)$ is a multivalued holomorphic curve valued in $\Lambda^{k+1}(\mathbb{C}^{n+1})$ and does not vanish at z_ℓ. By using Eq. (25) and Ref. [11, Lemma 4.1], we obtain

$$\begin{aligned}&\Lambda_n(\hat{f}) = \\ &\mathrm{Const}\cdot z^{\sum_{i=1}^{n}\beta_i-\frac{n(n+1)}{2}}\prod_{\ell=1}^{m}(z-z_\ell)^{n\gamma_{[z_\ell],1}+(n-1)\gamma_{[z_\ell],2}+\cdots+\gamma_{[z_\ell],n}}\cdot e_0\wedge\cdots\wedge e_n.\end{aligned} \tag{26}$$

Comparing this with Eq. (24) yields the following system of linear equations:

$$\begin{aligned}&k\gamma_{[z_\ell],1}+(k-1)\gamma_{[z_\ell],2}+\cdots+\gamma_{[z_\ell],k} = k\gamma_{\ell,1}+(k-1)\gamma_{\ell,2}+\cdots+\gamma_{\ell,k},\\ &1\le k\le n.\end{aligned}$$

Consequently, we find that $\gamma_{[z_\ell],k}=\gamma_{\ell,k}$ for all $1\le\ell\le m$ and $1\le k\le n$.

Remark 1. The singularity information at ∞ of the curve

$$\begin{aligned}&\hat{f}(z) = (z^{\beta_0}\varphi_0(z),z^{\beta_1}\varphi_1(z),\cdots,z^{\beta_n}\varphi_n(z))\\ &\text{with } \beta_0 = 1 \text{ and } \varphi_0(z)\equiv 1,\end{aligned}$$

constructed in the proof is given by

$$\begin{aligned}(\gamma_{[\infty],i})_{i=1}^{n} = \Bigg(&\beta_n-\beta_{n-1}-1+\sum_{i=1}^{m}\gamma_{i,n},\cdots,\\ &\beta_2-\beta_1-1+\sum_{i=1}^{m}\gamma_{i,2},\ \beta_1-1+\sum_{i=1}^{m}\gamma_{i,1}\Bigg).\end{aligned}$$

Indeed, substituting $z=\dfrac{1}{w}$ into $\hat{f}(z)$, we can obtain its local expression near ∞ as

$$\hat{f}(w) = (w^{-\beta_0-\deg\varphi_0}\Phi_0(w),w^{-\beta_1-\deg\varphi_1}\Phi_1(w),\cdots,w^{-\beta_n-\deg\varphi_n}\Phi_n(w)), \tag{27}$$

where $\Phi_0(w),\cdots,\Phi_n(w)$ are polynomials nonvanishing at $w=0$. Since $\beta_0<\cdots<\beta_n$ and $\deg\varphi_0<\cdots<\deg\varphi_n$, we obtain the quasi-canonical form of $\hat{f}(w)$ near $w=0$ as

$$(w^{-\beta_n-\deg\varphi_n}\Phi_n(w),\cdots,w^{-\beta_1-\deg\varphi_1}\Phi_1(w),\cdots,w^{-\beta_0-\deg\varphi_0}\Phi_0(w)).$$

The statement follows from Eq. (22).

Counting. We shall enumerate the possibilities of $(\gamma_{[\infty],i})_{i=1}^{\infty}$ for the curves in Example 2. To this end, we need the following lemma.

Lemma 4. Consider $\beta_0<\cdots<\beta_n$, $(n+1)$ polynomials $\varphi_0(z)$, $\varphi_1,\cdots,\varphi_n(z)$ and the following unitary curve on $\mathbb{C}\setminus\{0\}$:

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$$\hat{f}(z)=(z^{\beta_0}\varphi_0(z),z^{\beta_1}\varphi_1(z),\cdots,z^{\beta_n}\varphi_n(z)).$$

Then by applying a suitable nondegenerate linear transformation to $\hat{f}(z)$, we can obtain

$$\hat{g}(z)=(z^{\beta_0}\psi_0(z),z^{\beta_1}\psi_1(z),\cdots,z^{\beta_n}\psi_n(z)), \tag{28}$$

where $\psi_0(z),\cdots,\psi_n(z)$ are polynomials such that $\beta_0+\deg\psi_0$, $\beta_1+\deg\psi_1,\cdots,\beta_n+\deg\psi_n$ are mutually distinct. Moreover, these two curves share the same regular singularities on the Riemann sphere $\mathbb{C}\cup\infty$.

Proof. For simplicity, we define the term "quasi-degree" for the expressions $\beta_0+\deg\varphi_0$, $\beta_1+\deg\varphi_1$, $\cdots$, $\beta_n+\deg\varphi_n$. These quasi-degrees correspond to the quasi-polynomials $z^{\beta_0}\varphi_0(z)$, $z^{\beta_1}\varphi_1(z)$, $\cdots$, $z^{\beta_n}\varphi_n(z)$, where each component can include non-integer powers of z.

Should any two quasi-degrees be equal, such that $\beta_i+\deg\varphi_i=\beta_j+\deg\varphi_j$ for $i<j$, we can alter $z^{\beta_i}\varphi_i(z)$ by adding a suitable multiple of $z^{\beta_j}\varphi_j(z)$. This operation transforms $z^{\beta_i}\varphi_i(z)$ into $z^{\beta_i}\tilde{\varphi}_i(z)$, where $\deg\tilde{\varphi}_i(z)<\deg\varphi_i(z)$ and $\tilde{\varphi}_i(0)\neq 0$. Importantly, the terms z^{β_i} and z^{β_j} remain unchanged.

Consider the scenario where $z^{\beta_m}\varphi_m(z)$ has the highest quasi-degree among all components, and it has the largest index among those with this quasi-degree. Beginning with this component, we use the aforementioned transformation to adjust other components with the same quasi-degree, reducing their degrees. This ensures that $\beta_m+\deg\varphi_m$ stands out as the unique highest quasi-degree. Subsequently, we identify the next highest quasi-degree among the remaining components and apply similar transformations. After a finite series of these steps, the quasi-degrees of all components become distinct. This results in the transformed curve maintaining its form as $\hat{g}(z)=(z^{\beta_0}\psi_0(z),z^{\beta_1}\psi_1(z),\cdots,z^{\beta_n}\psi_n(z))$.

Since curve $\hat{g}$ differs from curve $\hat{f}$ only by a nondegenerate linear transformation, they share the same regular singularities on the Riemann sphere.

By Lemma 4, we assume that for each curve

$$\hat{f}(z)=(z^{\beta_0}\varphi_0(z),z^{\beta_1}\varphi_1(z),\cdots,z^{\beta_n}\varphi_n(z))$$

in Example 2, the $(n+1)$ positive numbers $\beta_0+\deg\varphi_0,\cdots$, $\beta_n+\deg\varphi_n$ are mutually distinct. Substituting $z=\dfrac{1}{w}$ into $\hat{f}$, we obtain its local expression near $z=\infty$, i.e., $w=0$, as

$$\hat{f}(w)=(w^{-\beta_0-\deg\varphi_0}\Phi_0(w),w^{-\beta_1-\deg\varphi_1}\Phi_1(w),\cdots,w^{-\beta_n-\deg\varphi_n}\Phi_n(w)), \tag{29}$$

where $\Phi_0,\Phi_1,\cdots,\Phi_n$ are polynomials nonvanishing at $w=0$. Since $\beta_0+\deg\varphi_0$, $\beta_1+\deg\varphi_1$, $\cdots$, $\beta_n+\deg\varphi_n$ are pairwise distinct, we can rearrange the components in Eq. (29) according to the ascending order of the powers of w. This yields the quasi-canonical form of $\hat{f}(w)$ near $w=0$, which determines the sequence $(\gamma_{[\infty],i})_{i=1}^{\infty}$. The challenge then reduces to determining how many distinct degree vectors $\vec{d}=(\deg\varphi_0$, $\deg\varphi_1,\cdots,\deg\varphi_n)$ are possible. The subsequent lemma is critical for this counting.

Lemma 5. $\displaystyle\sum_{j=0}^{n}\deg\varphi_j=\sum_{i=1}^{m}\left(n\gamma_{i,1}+(n-1)\gamma_{i,2}+\cdots+\gamma_{i,n}\right).$

Proof. Denote by A the sum

$$\sum_{i=1}^{m}(n\gamma_{i,1}+(n-1)\gamma_{i,2}+\cdots+\gamma_{i,n}).$$

Consider $\Lambda_n(\hat{f}(z))$ as the Wronskian $W_{n+1}(z^{\beta_0}\varphi_0(z),\cdots,z^{\beta_n}\varphi_n(z))$ of the terms $z^{\beta_0}\varphi_0(z),\cdots,z^{\beta_n}\varphi_n(z)$. This can be expanded by columns into a linear combination of finitely many quasi-monomials of the form $W_{n+1}(z^{\beta_0+k_0},z^{\beta_1+k_1},\cdots,z^{\beta_n+k_n})$. Among these, the quasi-monomial summand

$$W_{n+1}(z^{\beta_0+\deg\varphi_0},z^{\beta_1+\deg\varphi_1},\cdots,z^{\beta_n+\deg\varphi_n})$$

in $W_{n+1}(z^{\beta_0}\varphi_0(z),\cdots,z^{\beta_n}\varphi_n(z))$ has the highest quasi-degree, calculated as $\displaystyle\sum_{i=0}^{n}(\beta_i+\deg\varphi_i)-\frac{n(n+1)}{2}$. The statement follows from Eq. (26).

By this lemma, the degree vector $\vec{d}=(\deg\varphi_0,\deg\varphi_1,\cdots$, $\deg\varphi_n)$ could achieve a finite number of possible values that do not exceed $\dbinom{A+n+1}{n}$. Therefore, we complete the counting process.

In summary, we conclude the detailed discussion of Example 2.

6 Three open questions

We conclude this manuscript by posing the following three open questions.

Question 1. Using the notations from subsection 1.3, we inquire about the characterization of the coefficient matrices $\Gamma:=(\gamma_{i,j})_{\substack{1\le i\le k\\1\le j\le n}}$ associated with the regular singularities $\vec{D}$ represented by toric solutions to the $\mathrm{SU}(n+1)$ Toda system on the Riemann sphere. Remarkably, Lin et al.[13] addressed the case when $k=2$, and Eremenko[25] resolved the case when $n=1$ for this inquiry.

Question 2. A question akin to the one previously discussed emerges for compact Riemann surfaces of positive genera. In particular, for a specified genus g, we aim to characterize the $k\times n$ matrices Γ that correspond to regular singularities $\vec{D}$ manifested by toric solutions on compact Riemann surfaces of genus g. It is noteworthy that Gendron and Tahar[33] addressed this issue for the scenario when $n=1$.

Question 3. Consider a $k\times n$ matrix Γ and a nonnegative integer g. The task is to determine the dimension of the moduli space of toric solutions on compact Riemann surfaces of genus g, where the regular singularities $\vec{D}$ are characterized by the coefficient matrix Γ. Notably, Lu and Xu[19] explored this scenario for the case $n=1$.

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Conflict of interest

The authors declare that they have no conflict of interest.

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References

[1] Gervais J-L, Matsuo Y. Classical A_n-W-geometry. *Comm. Math. Phys.*, **1993**, *152* (2): 317–368.
[2] Griffiths P, Harris J. Principles of Algebraic Geometry. Hoboken, NJ, USA: Wiley, **1994**.
[3] Doliwa A. Holomorphic curves and Toda systems. *Lett. Math. Phys.*, **1997**, *39* (1): 21–32.
[4] Battaglia L. Existence and multiplicity result for the singular Toda system. *J. Math. Anal. Appl.*, **2015**, *424* (1): 49–85.
[5] Battaglia L, Jevnikar A, Malchiodi A, et al. A general existence result for the Toda system on compact surfaces. *Advances in Mathematics*, **2015**, *285*: 937–979.
[6] Battaglia L, Malchiodi A. Existence and non-existence results for the *SU*(3) singular Toda system on compact surfaces. *J. Funct. Anal.*, **2016**, *270* (10): 3750–3807.
[7] Karmakar D, Lin C-S, Nie Z, et al. Total masses of solutions to general Toda systems with singular sources. *Advances in Mathematics*, **2024**, *447*: 109702.
[8] Lin C-S, Yang W, Zhong X. A priori estimates of Toda systems, I: the Lie algebras of $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{G}_2$. *J. Differential Geom.*, **2020**, *114* (2): 337–391.
[9] Givental' A B. Plücker formulas and Cartan matrices. *Russian Math. Surveys*, **1989**, *44* (3): 193–194.
[10] Positselskii L E. Local Plücker formulas for a semisimple Lie group. *Funktsional'nyi Analiz i Ego Prilozheniya*, **1991**, *25* (4): 74–76.
[11] Mu J, Shi Y, Sun T, et al. Classifying solutions of SU(n + 1) Toda system around a singular source. *Proc. Amer. Math. Soc.*, **2024**, *152* (6): 2585–2600.
[12] Jost J, Wang G. Classification of solutions of a Toda system in R^2. *International Mathematics Research Notices*, **2002**, (6): 277–290.
[13] Lin C-S, Wei J, Ye D. Classification and nondegeneracy of *SU*(n+1) Toda system with singular sources. *Invent. Math.*, **2012**, *190* (1): 169–207.
[14] Chen Z, Lin C-S. Sharp existence, symmetry and asymptotic result for the singular *SU*(3) Toda systems with critical parameters. *J. Differential Geom.*, **2024**, *127* (3): 899–943.
[15] Chen Z, Lin C-S. Monodromy of a generalized Lamé equation of third order. *Math. Ann.*, **2025**, *392* (1): 263–312.
[16] Chen Z J, Lin C-S. On number and evenness of solutions of the *SU*(3) Toda system on flat tori with non-critical parameters. *J. Differential Geom.*, **2023**, *125* (1): 85–120.
[17] Chen Z J. Singular SU(3) Toda system on the flat tori. *Sci Sin Math*, **2023**, *53* (8): 1067–1084. (in Chinese)
[18] Jost J, Lin C, Wang G. Analytic aspects of the Toda system. II. Bubbling behavior and existence of solutions. *Comm. Pure Appl. Math.*, **2006**, *59* (4): 526–558.
[19] Lu S, Xu B. Moduli space of dihedral spherical surfaces and measured foliations. arXiv: 2312.01807, **2024**.
[20] Donaldson S K. Kähler metrics with cone singularities along a divisor. In: Essays in Mathematics and Its Applications. Berlin, Heidelberg: Springer, **2012**: 49–79.
[21] Lin C-S, Nie Z, Wei J. Toda systems and hypergeometric equations. *Trans. Amer. Math. Soc.*, **2018**, *370* (11): 7605–7626.
[22] Umehara M, Yamada K. Metrics of constant curvature 1 with three conical singularities on the 2-sphere. *Illinois J. Math.*, **2000**, *44* (1): 72–94.
[23] Chen Q, Wang W, Wu Y, et al. Conformal metrics with constant curvature one and finitely many conical singularities on compact Riemann surfaces. *Pacific J. Math.*, **2015**, *273* (1): 75–100.
[24] Mondello G, Panov D. Spherical metrics with conical singularities on a 2-sphere: angle constraints. *International Mathematics Research Notices*, **2016**, *2016* (16): 4937–4995.
[25] Eremenko A. Co-axial monodromy. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, **2020**, *20* (2): 619–634.
[26] Eremenko A. A Toda lattice in dimension 2 and Nevanlinna theory. *Zh. Mat. Fiz. Anal. Geom.*, **2007**, *3* (1): 39–46.
[27] Weyl H. The Concept of a Riemann Surface. Reading, MA, USA: Addison-Wesley, **1964**.
[28] Springer G. Introduction to Riemann Surfaces. New York: Chelsea, **1981**.
[29] Farkas H M, Kra I. Riemann Surfaces. New York: Springer-Verlag, **1992**.
[30] Calabi E. Isometric imbedding of complex manifolds. *Annals of Mathematics*, **1953**, *58* (1): 1–23.
[31] Griffiths P. On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry. *Duke Math. J.*, **1974**, *41*: 775–814.
[32] Thurston W P. Three-Dimensional Geometry and Topology: Volume 1. Princeton, NJ, USA: Princeton University Press, **1997**.
[33] Gendron Q, Tahar G. Dihedral monodromy of cone spherical metrics. *Illinois J. Math.*, **2023**, *67* (3): 457–483.

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