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Received: April 12, 2024; Accepted: September 12, 2024

On some multiple solutions to a *p*(*x*)-Laplace equation with supercritical growth

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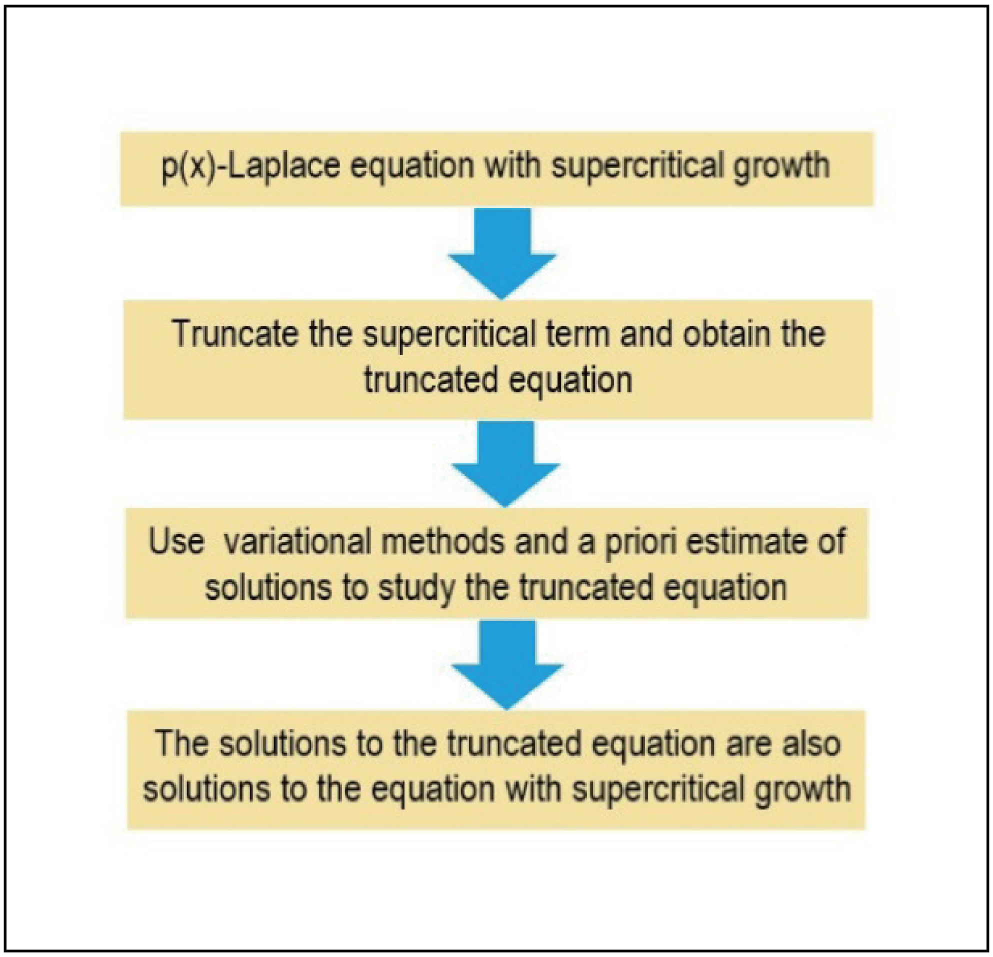
Graphical abstract



Supercritical p(x)-Laplace equation.

Public summary

- Extend the multiplicity of solutions to $p(x)$-Laplacian problem with subcritical and critical growth to supercritical growth.
- Overcome the supercritical growth via the truncated technique and De Giorgi iteration.
- The existence of solutions to the truncated equation is given by means of the variational principle of Ricceri.

Citation: Zhao L. On some multiple solutions to a $p(x)$-Laplace equation with supercritical growth. *JUSTC*, **2025**, 55(5): 0502. CSTR: 32290.14.JUSTC-2024-0056. DOI: 10.52396/JUSTC-2024-0056

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Cite This: *JUSTC*, **2025**, 55(5): 0502 (6pp)   Read Online

Abstract: We consider the multiplicity of solutions to a $p(x)$-Laplacian problem involving supercritical Sobolev growth via Ricceri's principle. By means of truncation combined with De Giorgi iteration, we can extend the results of subcritical and critical growth to supercritical growth and obtain at least three solutions to the $p(x)$-Laplacian problem.

Keywords: degenerate elliptic equations; a priori estimates; truncated technique; supercritical Sobolev exponent

CLC number: O175.25   **Document code:** A

2020 Mathematics Subject Classification: 35J70; 35B45; 35J62

1 Introduction

In this paper we study the following problem:

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda f(x,u) + \mu|u|^{q(x)-2}u, & \text{in } \Omega; \\ u(x) = 0, & \text{on } \partial\Omega; \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary, $p,q : \overline{\Omega} \to \mathbb{R}$ are continuous and satisfy

$$1 < p_- = \min_{x\in\overline{\Omega}} p(x) \le p_+ = \max_{x\in\overline{\Omega}} p(x) < N, \tag{2}$$

$$p \in L_+^\infty(\Omega) = \{p \in L^\infty(\Omega) \mid \operatorname*{ess\,inf}_{\Omega} p(x) \ge 1\}, \tag{3}$$

$$p_+ < p^*(x) = \frac{Np(x)}{N-p(x)},\ q \succ p^*; \tag{4}$$

$q \succ p^*$ means $\inf_\Omega(q(x)-p^*(x)) > 0$, $\lambda,\mu > 0$ are nonnegative parameters, and $f(x,u)$ is a Caratheodory function satisfying

$$\sup_{s\in[-M,M]} |f(x,s)| < +\infty,\ \forall M > 0. \tag{5}$$

We suppose that the nonlinearity $f(x,u)$ satisfies the following conditions:

$$\lim_{|s|\to+\infty} \frac{f(x,s)}{|s|^{p(x)-1}} = 0, \text{ uniformly in } x \in \Omega; \tag{6}$$

$$\lim_{|s|\to 0} \frac{f(x,s)}{|s|^{p(x)-1}} = 0, \text{ uniformly in } x \in \Omega; \tag{7}$$

$$\sup_{u\in W_0^{1,p(x)}} \int_\Omega \int_0^u f(x,s)\mathrm{d}s\mathrm{d}x > 0. \tag{8}$$

Several studies have investigated related problems. Fan and Deng[1] studied the $p(x)$-Laplacian equation:

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + a(x)|u|^{p(x)-2}u = \lambda f(x,u) + \mu g(x,u), \text{in } \Omega; \\ u(x) = 0, \quad \text{on } \partial\Omega; \end{cases} \tag{9}$$

where $\lambda = 1$, f and g satisfy the subcritical growth, and three solutions are obtained via the variational principle of Ricceri. Da Silva[2] addressed problem (9) with $a(x) = 0$, $\lambda = 1$, $g(x,u) = |u|^{q(x)-2}u$, $q(x) \le p^*(x)$, and $f(x,u)$ satisfying the symmetry condition; he found that problem (9) has at least m pairs of solutions for a given $m \in \mathbb{N}$ when $\mu > 0$ small by the symmetric mountain pass lemma and the concentration compactness lemma of Lions[3] for the variable exponent. Fan and Zhao[4] studied the existence of nodal radial solutions to problem (9) with $\lambda = 1$, $\mu = 0$, and $f(x,u) = |u|^{q(x)-2}u$; obtained a pair of solutions that have exactly k nodes when $a(x)$, $p(x)$, and $q(x)$ satisfy suitable conditions. Through truncation and Moser iteration[5], Zhao et al.[6] studied Eq. (1) with supercritical growth as $p(x)$ is a constant exponent, and obtained at least three solutions. In this work, we study problem (1) with supercritical growth via the truncated technique and De Giogri iteration[7–9]. If $a(x) = 0$, $g(x,u) = |u|^{q(x)-2}u$, and $q \succ p^*$, problem (9) is (1). For the case where $q \succ p^*$, we cannot directly use the variational techniques because the corresponding functional is not well defined in the Sobolev space.

Obviously, condition (8) implies

$$\theta := \sup_{\Phi(u)\ne 0} \frac{J(u)}{\Phi(u)} > 0, \tag{10}$$

where $\Phi(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x$ and $J(u) = \int_\Omega \int_0^u f(x,s)\mathrm{d}s\mathrm{d}x$.

The main results of this study are as follows:

Theorem 1.1. Let (2)–(8) hold, and θ be given by (10). Then for each compact interval $[a,b] \subset (\frac{1}{\theta},+\infty)$ there exists $\gamma > 0$ with the following property: For every $\lambda \in [a,b]$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, problem (1) has at

CSTR: 32290.14.JUSTC-2024-0056
DOI: 10.52396/JUSTC-2024-0056
JUSTC, **2025**, 55(5): 0502

least three solutions in $W_0^{1,p(x)}(\Omega)\bigcap L^\infty(\Omega)$, whose $W_0^{1,p(x)}(\Omega)$-norms are less than γ.

Remark 1.1. For the condition $q \succ p^*$ in problem (1), this case is not the natural growth condition on u. One can obtain only the Sobolev embedding $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$, $q(x)\le p^*(x)=\dfrac{Np(x)}{N-p(x)}$. In this study we are interested in the existence and multiplicity of solutions to problem (1) with $q\succ p^*$. One cannot directly use variational methods, as there does not exist an embedding $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$, where $q\succ p^*$. There is difficulty for the $p(x)$-Laplacian equation with supercritical growth. The difficulty is the lack of Sobolev embedding and compactness, and we overcome this difficulty via the truncated technique and De Giogri iteration[7–9].

2 Preliminaries

Let Ω be an open domain of $\mathbb{R}^N$, where

$$L_+^\infty(\Omega)=\{p\in L^\infty(\Omega)\mid \operatorname*{ess\,inf}_{\Omega} p(x)\ge 1\}.$$

For $p\in L_+^\infty(\Omega)$, define the space

$$L^{p(x)}(\Omega)=\{u:\Omega\to\mathbb{R}\text{ measurable}\mid \int_\Omega |u(x)|^{p(x)}\mathrm{d}x<\infty\}$$

with the norm

$$\|u\|_{L^{p(x)}}=\inf\{t>0\mid \int_\Omega |\frac{u(x)}{t}|^{p(x)}\mathrm{d}x\le 1\},$$

and Sobolev space with a variable exponent

$$W^{1,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega)\mid |\nabla u|\in L^{p(x)}\}$$

with the norm

$$\|u\|_{W^{1,p(x)}}=\|u\|_{L^{p(x)}}+\|\nabla u\|_{L^{p(x)}}.$$

We denote $W_0^{1,p(x)}(\Omega)=\overline{C_0^\infty(\Omega)}^{\|\cdot\|}$, with the norm $\|u\|=\|\nabla u\|_{L^{p(x)}}$.

Lemma 2.1[1,2,10]. Let $\rho(u)=\int_\Omega |u(x)|^{p(x)}\mathrm{d}x$. For $u,u_k\in L^{p(x)}(\Omega)$, we have

(R_1) For $u\ne 0$, $\|u\|_{L^{p(x)}}=t\Leftrightarrow \rho(\frac{u}{t})=1$;

(R_2) $\|u\|_{L^{p(x)}}<1(=1;>1)\Leftrightarrow\rho(u)<1(=1;>1)$;

(R_3) $\|u\|_{L^{p(x)}}>1\Rightarrow\|u\|_{L^{p(x)}}^{p_-}\le\rho(u)\le\|u\|_{L^{p(x)}}^{p_+}$;

(R_4) $\|u\|_{L^{p(x)}}<1\Rightarrow\|u\|_{L^{p(x)}}^{p_+}\le\rho(u)\le\|u\|_{L^{p(x)}}^{p_-}$;

(R_5) $\|u_k\|_{L^{p(x)}}=o_k(1)\Leftrightarrow\rho(u_k)=o_k(1)$;

(R_6) $\|u_k\|_{L^{p(x)}}^{-1}=o_k(1)\Leftrightarrow(\rho(u_k))^{-1}=o_k(1)$.

Lemma 2.2[10]. The spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable Banach spaces, and they are reflexive as $p_->1$.

Lemma 2.3[1,10]. Let Ω be an open bounded domain in $\mathbb{R}^N$ with the cone property, p satisfy (2), $q\in C^0(\mathbb{R}^N)$, and $p^*\succ q$, then the embedding $W^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$ is compact, where $p^*\succ q$ means $\inf_\Omega(p^*(x)-q(x))>0$.

Lemma 2.4[8]. Suppose that a sequence a_i, $i=0,1,\cdots$, of nonnegative numbers satisfies the recursion relation

$$a_{i+1}\le cb^i a_i^{1+\eta},\quad i=0,1,2,\cdots,\tag{11}$$

where c, b, and η are positive constants and $b>1$. If $a_0\le c^{-\frac{1}{\eta}}b^{-\frac{1}{\eta^2}}$, then $a_i\to 0$ as $i\to\infty$.

If X is a real Banach space, we can denote by $\mathfrak{W}_X$ the class of all functionals $\phi:X\to\mathbb{R}$ possessing the following property: If $\{u_n\}\subset X$ is a sequence converging weakly to $u\in X$ and $\liminf\limits_{n\to\infty}\phi(u_n)\le\phi(u)$, then $\{u_n\}$ has a subsequence converging strongly to u. For instance, if X is a uniformly convex Banach space and $g:[0,+\infty)\to\mathbb{R}$ is a continuous, strictly increasing function, then the functional $u\to g(\|u\|)$ belongs to the class $\mathfrak{W}_X$.

Lemma 2.5[11,12]. Let X be a separable and reflexive real Banach space; $\Phi:X\to\mathbb{R}$ a sequentially weakly lower semicontinuous C^1 functional, belonging to $\mathfrak{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*; $J:X\to\mathbb{R}$ a C^1 functional with compact derivative. Assume that Φ has a strict local minimum x_0 with $\Phi(x_0)=J(x_0)=0$. Finally, setting

$$\alpha=\max\left\{0,\limsup_{\|x\|\to\infty}\frac{J(x)}{\Phi(x)},\limsup_{x\to x_0}\frac{J(x)}{\Phi(x)}\right\},$$
$$\beta=\sup_{x\in\Phi^{-1}(0,+\infty)}\frac{J(x)}{\Phi(x)},$$

assume that $\alpha<\beta$. Then, for each compact interval $[a,b]\subseteq I=(\frac{1}{\beta},\frac{1}{\alpha})$ (with the conventions $\frac{1}{0}=+\infty$, $\frac{1}{+\infty}=0$), $\rho>0$ with the following property: For every $\lambda\in[a,b]$ and every C^1 functional $\Psi:X\to\mathbb{R}$ with compact derivative, there exists $\delta_0>0$ such that, for each $\mu\in[0,\delta_0]$, the equation

$$\Phi'(u)=\lambda J'(u)+\mu\Psi'(u)$$

has at least three solutions in X whose norms are less than γ, i.e., $\|u_i\|\le\gamma$, $i=1,2,3$.

Remark 2.1. From the proof of Lemma 2.5, Ricceri[12] shows that ρ does not depend on μ. One can refer to Ref. [12] for details.

Lemma 2.6. Let f satisfy (5) and (6). Then for every $\lambda\in(0,+\infty)$, $\Phi-\lambda J$ is sequentially weakly lower continuous and coercive on $W_0^{1,p(x)}(\Omega)$, and has a global minimizer v_λ, where $\Phi(u)=\int_\Omega\dfrac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x$ and $J(u)=\int_\Omega\int_0^u f(x,s)\mathrm{d}s\mathrm{d}x$.

Proof. Let us fix $\lambda\in(0,+\infty)$. By (6), for $\forall\varepsilon>0$, there exists $M_0>0$, such that

$$|f(x,s)|\le\varepsilon|s|^{p(x)-1}\quad\text{as }|s|\ge M_0.$$

Considering this inequality and (5), we may find a constant $C_0>0$, such that

$$|f(x,s)|\le C_0+\varepsilon|s|^{p(x)-1}\quad\text{as }s\in\mathbb{R},$$

which implies

$$|F(x,u)|\le C_0|u|+\frac{\varepsilon}{p(x)}|u|^{p(x)}.\tag{12}$$

Thus, for $u\in W_0^{1,p(x)}(\Omega)$, we obtain

CSTR: 32290.14.JUSTC-2024-0056
DOI: 10.52396/JUSTC-2024-0056
JUSTC, **2025**, 55(5): 0502

$$\begin{aligned}\Phi(u)-\lambda J(u)=\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x-\lambda\int_\Omega F(x,u)\mathrm{d}x\geq\\ \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x-\lambda\int_\Omega\left(\frac{\varepsilon}{p(x)}|u|^{p(x)}+C_0|u|\right)\mathrm{d}x\geq\\ \frac{1}{p_+}\|u\|^{p(x)}-\frac{\lambda\varepsilon}{p_-}\int_\Omega|u|^{p(x)}\mathrm{d}x-\lambda C_0\int_\Omega|u|\mathrm{d}x\geq\\ \frac{1}{p_+}\|u\|^{p(x)}-\frac{\lambda\varepsilon C_p^p}{p_-}\|u\|^{p(x)}-\lambda C_0C_1\|u\|,\end{aligned}$$

where $\|u\|_{L^{p(x)}}\leq C_p\|u\|$ and $\|u\|_{L^1}\leq C_1\|u\|$, with constants $C_1, C_p>0$. Since $p(x)>1$, $\varepsilon>0$ is small enough, we have $\Phi(u)-\lambda J(u)\to+\infty$ as $\|u\|\to\infty$. Hence $\Phi-\lambda J$ is coercive.

Since the embedding $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{p(x)}(\Omega)$ is compact and by (12), J is weakly continuous. Obviously, $\Phi(u)=\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x\in\mathfrak{W}_X$ is weakly lower semicontinuous on $W_0^{1,p(x)}(\Omega)$. We can deduce that $\Phi-\lambda J$ is sequentially weakly lower semicontinuous. Thus, $\Phi-\lambda J$ has a global minimizer v_λ.

The proof is completed.

Next, we will show that $\Phi-\lambda J$ has a strictly local, not global minimizer for some λ when f satisfies (5)–(8).

Lemma 2.7. Let f satisfy (5)–(8), then

(i) 0 is a strict local minimizer of the functional $\Phi-\lambda J$ for $\lambda\in(0,+\infty)$;

(ii) $0\neq v_\lambda$, i.e., 0 is not the global minimizer v_λ for $\lambda\in(\frac{1}{\theta},+\infty)$, where v_λ is given by Lemma 2.6.

Proof. First, we prove

$$\lim_{\|u\|\to0}\frac{J(u)}{\Phi(u)}=0.$$

In fact, by (7), for $\forall\varepsilon>0$, $\exists\delta>0$, such that

$$|f(x,u)|\leq\varepsilon|u|^{p(x)-1},\ \text{ as } |u|<\delta. \tag{13}$$

Considering the inequalities (13) and (6), we have

$$|f(x,u)|\leq\varepsilon|u|^{p(x)}+|u|^{r(x)},\ \text{ for } x\in\Omega,\ u\in\mathbb{R},$$

with fixed $r(x)\in(p(x),p^*(x))$. By continuous embedding, we have

$$\begin{aligned}|J(u)|\leq\int_\Omega|F(x,u)|\mathrm{d}x\leq\\ \frac{\varepsilon}{p_-}\int_\Omega|u|^{p(x)}\mathrm{d}x+\frac{1}{r_-}\int_\Omega|u|^{r(x)}\mathrm{d}x\leq\\ \frac{\varepsilon C_p}{p_-}\|u\|^{p(x)}+\frac{C_r}{r_-}\|u\|^{r(x)}.\end{aligned}$$

This implies

$$\lim_{\|u\|\to0}\frac{J(u)}{\Phi(u)}=0.$$

Next, we prove (i) and (ii).

(i) For $\lambda\in(0,+\infty)$, since $\lim_{\|u\|\to0}\frac{J(u)}{\Phi(u)}=0<\frac{1}{\lambda}$ and $\Phi(u)>0$ for each $u\neq0$ in some neighborhood U of 0, there exists a neighborhood $V\subseteq U$ of 0 such that $\Phi(u)-\lambda J(u)>0$ for all $u\in V\setminus\{0\}$. Hence, 0 is a strict local minimum of $\Phi-\lambda J$.

(ii) For $\lambda\in(\frac{1}{\theta},+\infty)$, from the definition of θ (Eq. (10)), there exist $u^*\in W_0^{1,p(x)}(\Omega)$, with $\min\{\Phi(u^*),\ J(u^*)\}>0$, such that $\frac{J(u^*)}{\Phi(u^*)}>\frac{1}{\lambda}$, i.e., $\Phi(u^*)-\lambda J(u^*)<0=\Phi(0)-\lambda J(0)$. Therefore, $0\neq v_\lambda$ is not a global minimum of $\Phi-\lambda J$.

The proof is completed.

3 Proof of Theorem 1.1

Our main difficulty in proving the existing solutions to problem (1) by using variational methods is that the associations with the functional are not well defined on $W_0^{1,p(x)}(\Omega)$ for $q>p^*$. We follow the idea of Refs. [6, 7]. Let the truncation of $|u|^{q(x)-2}u$ be given by

$$g_K(x,u)=\begin{cases}|u|^{q(x)-2}u, & \text{if } 0\leq|u|\leq K;\\ K^{q(x)-p_+}|u|^{p_+-2}u, & \text{if } |u|\geq K;\end{cases} \tag{14}$$

where $K\geq2$ is a real number, whose value is fixed later. Then $g_K(x,u)$ satisfies the subcritical growth

$$|g_K(x,u)|\leq K^{q(x)-p_+}|u|^{p_+-1}.$$

Setting $h_K(x,u)=\lambda f(x,u)+\mu g_K(x,u)$, we study the truncation problem associated with h_K:

$$\begin{cases}-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)=h_K(x,u)=\lambda f(x,u)+\mu g_K(x,u), & \text{in } \Omega;\\ u(x)=0, & \text{on } \partial\Omega.\end{cases} \tag{15}$$

Definition 3.1. We say that if $u\in W_0^{1,p(x)}(\Omega)$ is a weak solution to the truncated problem (15) if

$$\int_\Omega|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi\mathrm{d}x=\int_\Omega h_K(x,u)\varphi\mathrm{d}x \tag{16}$$

for every $\varphi\in W_0^{1,p(x)}(\Omega)$. We introduce

$$\Psi(u)=\int_\Omega\int_0^u g_K(x,s)\mathrm{d}s\mathrm{d}x.$$

The critical points of the functional $\Phi-\lambda J-\mu\Psi$ are the weak solutions to problem (16), where $\Phi(u)=\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x$ and $J(u)=\int_\Omega\int_0^u f(x,s)\mathrm{d}s\mathrm{d}x$. From (7), for small $\varepsilon>0$, $\exists\delta>0$ such that the function f satisfies

$$|f(x,s)|\leq\varepsilon|s|^{p(x)-1},$$

for $|s|<\delta$. For each compact interval $[a,b]\subset(\frac{1}{\theta},+\infty)$, $\lambda\in[a,b]$, considering (6) and $|g_K(x,u)|\leq K^{q(x)-p_+}|u|^{p_+-1}$, we can choose constant $C>0$ such that

$$|h_K(x,s)|\leq C|s|^{p(x)-1}+\mu K^{q(x)-p_+}|s|^{p_+-1}, \tag{17}$$

for $\forall s\in\mathbb{R}$. Hence, $h_K(x,u)$ and $g_K(u)$ satisfy the subcritical growth. $\Psi(u)$ has a compact derivative in $W_0^{1,p(x)}(\Omega)$. By Lemmas 2.2, 2.3, 2.6, and 2.7, all the hypotheses of Lemma 2.5 are satisfied. Then $\gamma>0$ exists with the following property: For every $\lambda\in[a,b]$, $\delta_0>0$ exists, such that, for $\mu\in[0,\delta_0]$, problem (15) has at least three solutions u_0, u_1, and u_2 in $W_0^{1,p(x)}(\Omega)$ whose $W_0^{1,p(x)}(\Omega)$-norms are less than γ, i.e., $\|u_i\|_{W_0^{1,p(x)}}\leq\gamma$, $i=0,1,2$, where γ is depend on λ, but not depend on μ (see (5) of Ref. [12] for details). If the three solutions u_i, $i=0,1,2$, satisfy

CSTR: 32290.14.JUSTC-2024-0056
DOI: 10.52396/JUSTC-2024-0056
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$$|u_i(x)| \le K,\ \text{a.e. } x \in \Omega,\ i = 0,1,2, \tag{18}$$

in view of the definition of g_K (Eq. (14)), we have $h_K(x,u) = \lambda f(x,u) + \mu |u|^{q(x)-2}u$, therefore u_i, $i = 0,1,2$, are also solutions to the original problem (1). Thus, to prove Theorem 1.1, it suffices to show that $\delta > 0$ exists, such that for $\mu \in [0,\delta]$, the solutions obtained by Lemma 2.5 satisfy the inequality (18).

Step 1. We first prove that $\forall$ any ball $B_R(x_0) \subset\subset \Omega$, $R \le 1$, $\forall$ real number $l \ge 1$ and $\forall$ t,s satisfying $0 \le t < s \le 1$ the solutions $u = u_i$, $i = 0,1,2$, satisfy the following inequality:

$$\int_{A_{l,t}} |\nabla u|^{p_-} \mathrm{d}x \le (C' + C''\mu K^{q_+ - p_+}) \left(\int_{A_{l,s}} \left| \frac{u-l}{s-t} \right|^{p_-^*} \mathrm{d}x + (l^{p_+} + 1)|A_{l,s}| \right), \tag{19}$$

where $p_-^* = \dfrac{Np_-}{N - p_-}$, $C', C'' > 0$ are constants, and $A_{l,s} = \{x \in B_s \mid u(x) > l\}$.

For arbitrary balls $\overline{B}_t(x') \subset B_s(x') \subset B_R(x_0)$, some $x' \in \Omega$, let ξ be a C^∞ function such that $0 \le \xi \le 1$, $\operatorname{supp}\xi \subset B_s$, $\xi = 1$ on B_t, and $|\nabla \xi| \le \dfrac{2}{s-t}$. For $l \ge 1$ taking $\varphi = \xi^{p_+} \max\{u - l, 0\} \in W_0^{1,p(x)}(\Omega)$ as a test function in problem (16), we obtain

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \mathrm{d}x = \int_\Omega [\lambda f(x,u)\varphi + \mu g_K(x,u)\varphi] \mathrm{d}x, \tag{20}$$

and

$$\begin{aligned} &\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \mathrm{d}x = \\ &\int_{A_{l,s}} |\nabla u|^{p(x)} \xi^{p_+} \mathrm{d}x + p_+ \int_{A_{l,s}} |\nabla u|^{p(x)-2} \xi^{p_+ - 1}(u-l) \nabla u \nabla \xi \mathrm{d}x. \end{aligned} \tag{21}$$

From (17), we have

$$\begin{aligned} &\int_\Omega [\lambda f(x,u)\varphi + \mu g_K(x,u)\varphi] \mathrm{d}x = \\ &\int_{A_{l,s}} [\lambda f(x,u)\varphi + \mu g_K(x,u)\varphi] \mathrm{d}x \le \\ &\lambda C \int_{A_{l,s}} |u|^{p(x)-1} \xi^{p_+}(u-l) \mathrm{d}x + \\ &\mu \int_{A_{l,s}} K^{q(x)-p_+} |u|^{p_+ - 1} \xi^{p_+}(u-l) \mathrm{d}x. \end{aligned} \tag{22}$$

By (20)–(22), it follows that

$$\begin{aligned} \int_{A_{l,s}} |\nabla u|^{p(x)} \xi^{p_+} \mathrm{d}x \le &- p_+ \int_{A_{l,s}} |\nabla u|^{p(x)-2} \xi^{p_+ - 1}(u-l) \nabla u \nabla \xi \mathrm{d}x + \\ &\lambda C \int_{A_{l,s}} |u|^{p(x)-1} \xi^{p_+}(u-l) \mathrm{d}x + \\ &\mu \int_{A_{l,s}} K^{q(x)-p_+} |u|^{p_+ - 1} \xi^{p_+}(u-l) \mathrm{d}x \le \\ &p_+ \int_{A_{l,s}} |\nabla u|^{p(x)-1} \xi^{p_+ - 1}(u-l) |\nabla \xi| \mathrm{d}x + \\ &\lambda C \int_{A_{l,s}} |u|^{p(x)-1} \xi^{p_+}(u-l) \mathrm{d}x + \\ &\mu \int_{A_{l,s}} K^{q(x)-p_+} |u|^{p_+ - 1} \xi^{p_+}(u-l) \mathrm{d}x. \end{aligned} \tag{23}$$

Let us estimate the terms of the right hand side of (23),

$$\begin{aligned} &p_+ \int_{A_{l,s}} |\nabla u|^{p(x)-1} \xi^{p_+ - 1}(u-l) |\nabla \xi| \mathrm{d}x \le \\ &p_+ \left[\int_{A_{l,s}} \frac{p(x)-1}{p(x)} \varepsilon^{\frac{p(x)}{p(x)-1}} |\nabla u|^{p(x)} \xi^{\frac{(p_+ - 1)p(x)}{p(x)-1}} \mathrm{d}x + \right. \\ &\left. \int_{A_{l,s}} \frac{1}{p(x)} \varepsilon^{-p(x)} |\nabla \xi|^{p(x)} |u-l|^{p(x)} \mathrm{d}x \right] \le \\ &(p_+ - 1)\varepsilon^{\frac{p_+}{p_+ - 1}} \int_{A_{l,s}} |\nabla u|^{p(x)} \xi^{p_+} \mathrm{d}x + \frac{p_+}{p_-} \varepsilon^{-p_-} 2^{p_+} \int_{A_{l,s}} |\frac{u-l}{s-t}|^{p(x)} \mathrm{d}x \le \\ &\frac{1}{2} \int_{A_{l,s}} |\nabla u|^{p(x)} \xi^{p_+} \mathrm{d}x + C_{p,\varepsilon}(\int_{A_{l,s}} \frac{p(x)}{p_-^*} |\frac{u-l}{s-t}|^{p_-^*} \mathrm{d}x + \frac{p_-^* - p(x)}{p_-^*} |A_{l,s}|) \le \\ &\frac{1}{2} \int_{A_{l,s}} |\nabla u|^{p(x)} \xi^{p_+} \mathrm{d}x + C_{p,\varepsilon}(\int_{A_{l,s}} |\frac{u-l}{s-t}|^{p_-^*} \mathrm{d}x + |A_{l,s}|), \end{aligned} \tag{24}$$

where we use Young's inequality with ε and take $\varepsilon \in (0,1)$ such that $(p_+ - 1)\varepsilon^{\frac{p_+}{p_+ - 1}} = \dfrac{1}{2}$ and denote $C_{p,\varepsilon} = \dfrac{p_+ 2^{p_+}}{p_-} \cdot (2p_+ - 2)^{\frac{(p_+ - 1)p_-}{p_+}}$. Using Young's inequality, we obtain

$$\begin{aligned} &\lambda C \int_{A_{l,s}} |u|^{p(x)-1} \xi^{p_+} |u-l| \mathrm{d}x \le \\ &\lambda C[\int_{A_{l,s}} \frac{1}{p(x)} |u-l|^{p(x)} \mathrm{d}x + \int_{A_{l,s}} \frac{p(x)-1}{p(x)} |u|^{p(x)} \mathrm{d}x] \le \\ &\frac{\lambda C}{p_+} \int_{A_{l,s}} |u-l|^{p(x)} \mathrm{d}x + \\ &\frac{\lambda C(p_+ - 1)2^{p_+ - 1}}{p_+} [\int_{A_{l,s}} |u-l|^{p(x)} \mathrm{d}x + l^{p_+} |A_{l,s}|] = \\ &\frac{\lambda C}{p_+} [((p_+ - 1)2^{p_+ - 1} + 1) \int_{A_{l,s}} |u-l|^{p(x)} \mathrm{d}x + (p_+ - 1)2^{p_+ - 1} l^{p_+} |A_{l,s}|] \le \\ &\frac{\lambda C}{p_+} [2^{p_+ - 1} p_+ (\int_{A_{l,s}} \frac{p(x)}{p_-^*} |\frac{u-l}{s-t}|^{p_-^*} \mathrm{d}x + \int_{A_{l,s}} \frac{p_-^* - p(x)}{p_-^*} \mathrm{d}x) + \\ &\quad 2^{p_+ - 1} p_+ l^{p_+} |A_{l,s}|] \le \\ &\lambda C 2^{p_+ - 1} [\int_{A_{l,s}} |\frac{u-l}{s-t}|^{p_-^*} \mathrm{d}x + (l^{p_+} + 1)|A_{l,s}|]. \end{aligned} \tag{25}$$

Considering $K \ge 2$ and $0 \le \xi \le 1$, we obtain

$$\begin{aligned} &\mu \int_{A_{l,s}} K^{q(x)-p_+} |u|^{p_+ - 1} \xi^{p_+} |u-l| \mathrm{d}x \le \\ &\mu K^{q_+ - p_+} \int_{A_{l,s}} |u|^{p_+ - 1} |u-l| \mathrm{d}x \le \\ &\mu K^{q_+ - p_+} (\int_{A_{l,s}} |u|^{p_+} \mathrm{d}x + \int_{A_{l,s}} |u-l|^{p_+} \mathrm{d}x) \le \\ &\mu K^{q_+ - p_+} [2^{p_+ - 1} (\int_{A_{l,s}} |u-l|^{p_+} \mathrm{d}x + l^{p_+} |A_{l,s}|) + \int_{A_{l,s}} |u-l|^{p_+} \mathrm{d}x] \le \\ &\mu K^{q_+ - p_+} 2^{p_+} (\int_{A_{l,s}} |\frac{u-l}{s-t}|^{p_+} \mathrm{d}x + l^{p_+} |A_{l,s}|) \le \\ &\mu K^{q_+ - p_+} 2^{p_+} (\int_{A_{l,s}} \frac{p_+}{p_-^*} |\frac{u-l}{s-t}|^{p_-^*} \mathrm{d}x + \frac{p_-^* - p_+}{p_-^*} |A_{l,s}| + l^{p_+} |A_{l,s}|) \le \\ &\mu K^{q_+ - p_+} 2^{p_+} (\int_{A_{l,s}} |\frac{u-l}{s-t}|^{p_-^*} \mathrm{d}x + (l^{p_+} + 1)|A_{l,s}|), \end{aligned} \tag{26}$$

where $0 < t < s \le 1$. From (23)–(26), it follows that

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$$\begin{aligned}
&\int_{A_{l,t}} |\nabla u|^{p_-}\mathrm{d}x \le \\
&\int_{A_{l,t}} \frac{p_-}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x + \frac{p(x)-p_-}{p(x)}|A_{l,t}| \le \\
&\int_{A_{l,t}} |\nabla u|^{p(x)}\mathrm{d}x + |A_{l,t}| \le \\
&\int_{A_{l,s}} |\nabla u|^{p(x)}\xi^{p_+}\mathrm{d}x + |A_{l,s}| \le \\
&2C_{p,\varepsilon}(\int_{A_{l,s}} |\frac{u-l}{s-t}|^{p_-^*}\mathrm{d}x + |A_{l,s}|)+ \\
&(C\lambda 2^{p_+} + \mu K^{q_+-p_+}2^{p_++1})[\int_{A_{l,s}} |\frac{u-l}{s-t}|^{p_-^*}\mathrm{d}x + (l^{p_+}+1)|A_{l,s}|] + |A_{l,s}| \le \\
&(2C_{p,\varepsilon} + C\lambda 2^{p_+} + \mu K^{q_+-p_+}2^{p_++1}+1)\cdot \\
&[\int_{A_{l,s}} |\frac{u-l}{s-t}|^{p_-^*}\mathrm{d}x + (l^{p_+}+1)|A_{l,s}|],
\end{aligned} \tag{27}$$

where $C_{p,\varepsilon} = \frac{p_+}{p_-}2^{p_+}(2p_+-2)^{\frac{(p_+-1)p_-}{p_+}}$. If we denote $C' = 2C_{p,\varepsilon} + C\lambda 2^{p_+} + 1$ and $C'' = 2^{p_++1}$ in (27), we can obtain inequality (19).

Step 2. We will show that $\exists\delta > 0$ such that $|u| \le K$ a.e. $x \in \Omega$ for $\mu \in [0,\delta]$.

Let us fix $B_R \subset\subset \Omega$, $K \ge 2$ in Step 1 and set

$$\rho_i = \frac{R}{2} + \frac{R}{2^{i+1}}, \quad \bar{\rho}_i = \frac{\rho_i + \rho_{i+1}}{2},$$

and

$$K_i = K(1 - \frac{1}{2^{i+1}}),$$

with $i = 0,1,2,\cdots$. Obviously,

$$\begin{gathered}
\rho_{i+1} \le \bar{\rho}_i \le \rho_i, \\
\rho_0 \ge \rho_1 \ge \cdots \ge \rho_i \cdots \to \frac{R}{2},
\end{gathered} \tag{28}$$

and

$$1 \le K_0 = \frac{K}{2} \le K_1 \le \cdots \le K_i \cdots \to K \tag{29}$$

as $i \to \infty$.

We define the sequence

$$a_i = \int_{A_{K_i\rho_i}} |u - K_i|^{p_-^*}\mathrm{d}x$$

with $p_-^* = \frac{Np_-}{N-p_-}$ and fix a function $\zeta(\cdot) \in C^1[0,+\infty)$, $0 \le \zeta(t) \le 1$,

$$\zeta(t) = \begin{cases} 1, & \text{if } t \le \frac{1}{2}; \\ 0, & \text{if } t \ge \frac{3}{4}; \end{cases}$$

and $|\zeta'(t)| \le 1$. Set

$$\zeta_i(x) = \zeta(\frac{2^{i+1}}{R}(|x| - \frac{R}{2})).$$

We have

$$\zeta_i(x) = \begin{cases} 1, & x \in B_{\rho_{i+1}}, \\ 0, & x \notin B_{\bar{\rho}_i}. \end{cases}$$

Next, we will show that a_i satisfies the inequality $a_{i+1} \le cb^i a_i^{1+\eta}$.

$$\begin{aligned}
a_{i+1} = &\int_{A_{K_{i+1},\rho_{i+1}}} |u - K_{i+1}|^{p_-^*}\mathrm{d}x \le \\
&\int_{A_{K_{i+1},\bar{\rho}_i}} |(u - K_{i+1})\zeta_i|^{p_-^*}\mathrm{d}x \le \\
&\int_{B_R} |(u - K_{i+1})\zeta_i|^{p_-^*}\mathrm{d}x \le \\
&S^{-\frac{p_-^*}{p_-}}[\int_{B_R} |\nabla((u - K_{i+1})\zeta_i)|^{p_-}\mathrm{d}x]^{\frac{p_-^*}{p_-}} \le \\
&S^{-\frac{p_-^*}{p_-}}[\int_{B_R} |\zeta_i\nabla u + (u - K_{i+1})\nabla\zeta_i|^{p_-}\mathrm{d}x]^{\frac{p_-^*}{p_-}} \le \\
&S^{-\frac{p_-^*}{p_-}}2^{p_-^*}[\int_{A_{K_{i+1},\bar{\rho}_i}} |\zeta_i\nabla u|^{p_-}\mathrm{d}x + \\
&\int_{A_{K_{i+1},\rho_i}} |(u - K_{i+1})\nabla\zeta_i|^{p_-}\mathrm{d}x]^{\frac{p_-^*}{p_-}}.
\end{aligned} \tag{30}$$

Setting $l = K_{i+1}$, $t = \bar{\rho}_i$, and $s = \rho_i$ for inequality (19) in Step 1 and from (30) we have

$$\begin{aligned}
&a_{i+1} \le \\
&S^{-\frac{p_-^*}{p_-}}2^{p_-^*}\{C_{\mu,K}[\int_{A_{K_{i+1},\rho_i}} (\frac{2^{i+3}}{R})^{p_-^*}|u - K_{i+1}|^{p_-^*}\mathrm{d}x + \\
&(K_{i+1}^{p_+}+1)|A_{K_{i+1},\rho_i}|] + \int_{A_{K_{i+1},\rho_i}} (\frac{2^{i+1}}{R})^{p_-}|u - K_{i+1}|^{p_-}\mathrm{d}x\}^{\frac{p_-^*}{p_-}} \le \\
&S^{-\frac{p_-^*}{p_-}}2^{p_-^*}\{C_{\mu,K}[\int_{A_{K_{i+1},\rho_i}} (\frac{2^{i+3}}{R})^{p_-^*}|u - K_{i+1}|^{p_-^*}\mathrm{d}x + (K_{i+1}^{p_+}+1)|A_{K_{i+1},\rho_i}|] + \\
&(\frac{2^{i+1}}{R})^{p_-}(\int_{A_{K_{i+1},\rho_i}} \frac{p_-}{p_-^*}|u - K_{i+1}|^{p_-^*}\mathrm{d}x + \frac{p_-^*-p_-}{p_-^*}|A_{K_{i+1}},\rho_i|)\}^{\frac{p_-^*}{p_-}} \le \\
&S^{-\frac{p_-^*}{p_-}}2^{p_-^*}\{C_{\mu,K}[(\frac{2^{i+3}}{R})^{p_-^*}\int_{A_{K_{i+1},\rho_i}} |u - K_{i+1}|^{p_-^*}\mathrm{d}x + (K_{i+1}^{p_+}+1)|A_{K_{i+1},\rho_i}|] + \\
&(\frac{2^{i+1}}{R})^{p_-}(\int_{A_{K_{i+1},\rho_i}} |u - K_{i+1}|^{p_-^*}\mathrm{d}x + |A_{K_{i+1}},\rho_i|)\}^{\frac{p_-^*}{p_-}},
\end{aligned} \tag{31}$$

where

$$C_{\mu,K} = (C' + C''\mu K^{q_+-p_+}) = (2C_{p,\varepsilon} + C\lambda 2^{p_+} + \mu K^{q_+-p_+}2^{p_++1}+1).$$

Since $u > K_{i+1} > K_i \ge \cdots \ge K_0 \ge 1$ on A_{K_{i+1},ρ_i} and the definition of K_i, we have

$$(K_{i+1} - K_i)^{p_-^*}|A_{K_{i+1},\rho_i}| \le \int_{A_{K_{i+1},\rho_i}} |u - K_{i+1}|^{p_-^*}\mathrm{d}x \le a_i, \tag{32}$$

$$|A_{K_{i+1},\rho_i}| \le (\frac{2^{i+2}}{K})^{p_-^*}a_i \le \frac{2^{(i+2)p_-^*}}{K_0^{p_-^*}}a_i \le 2^{p_-^*(i+2)}a_i, \tag{33}$$

and

$$K_{i+1}^{p_+}|A_{K_{i+1},\rho_i}| \le \frac{2^{p_-^*(i+2)}}{K_{i+1}^{p_-^*-p_+}}a_i \le 2^{p_-^*(i+2)}a_i. \tag{34}$$

By inequalities (31)–(34), we have

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$$
\begin{aligned}
a_{i+1} \le & S^{-\frac{p_-^*}{p_-}} 2^{p_-^*}\{C_{\mu,K}[(\frac{2^{i+3}}{R})^{p_-^*}a_i + 2^{p_-^*(i+2)+1}a_i] + \\
& \frac{2^{(i+1)p_-}}{R^{p_-}}(a_i + 2^{p_-^*(i+2)}a_i)\}^{\frac{p_-^*}{p_-}} \le \\
& S^{-\frac{p_-^*}{p_-}} 2^{p_-^*}[C_{\mu,K}(\frac{2^{3p_-^*}}{R^{p_-^*}} + 2^{2p_-^*+1}) + \frac{2^{p_-}}{R^{p_-}} + \frac{2^{p_-+2p_-^*}}{R^{p_-}}]^{\frac{p_-^*}{p_-}} 2^{\frac{2p_-^{*2} i}{p_-}} a_i^{\frac{p_-^*}{p_-}} = \\
& c(2^{\frac{2p_-^{*2}}{p_-}})^i a_i^{1+\eta} = cb^i a_i^{1+\eta},
\end{aligned} \tag{35}
$$

where $c = S^{-\frac{p_-^*}{p_-}} 2^{p_-^*}\left[C_{\mu,K}\left(\frac{2^{3p_-^*}}{R^{p_-^*}} + 2^{2p_-^*+1}\right) + \frac{2^{p_-}}{R^{p_-}} + \frac{2^{p_-+2p_-^*}}{R^{p_-}}\right]^{\frac{p_-^*}{p_-}}$, $C_{\mu,K} = 2C_{p,\varepsilon} + C\lambda 2^{p_+} + \mu K^{q_+-p_-}2^{p_+} + 1$, $b = 2^{\frac{2p_-^{*2}}{p_-}} > 1$, and $\eta = \frac{p_-}{N-p_-}$. From $u \in W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega) \subset L^{p_-^*}(\Omega)$, we obtain

$$
\int_{A_{\frac{K}{2},R}} |u - \frac{K}{2}|^{p_-^*}\mathrm{d}x \le \int_{A_{\frac{K}{2},R}} |u|^{p_-^*}\mathrm{d}x < +\infty,
$$

which implies that $|u(x)| < +\infty$ a.e. in $A_{\frac{K}{2},R}$. That is $|A_{\frac{K}{2}}, R| \to 0$ as $K \to \infty$. Obviously,

$$
a_0 = \int_{A_{\frac{K}{2},R}} \left|u - \frac{K}{2}\right|^{p_-^*}\mathrm{d}x \le \int_{A_{\frac{K}{2},R}} |u|^{p_-^*}\mathrm{d}x \to 0, \tag{36}
$$

as $K \to \infty$. Next, we find some suitable values of K and μ such that the inequality

$$
a_0 \le c^{-\frac{1}{\eta}} b^{-\frac{1}{\eta^2}} \tag{37}
$$

holds in Lemma 2.4, where b, c, and η are given by (35), and $a_0 = \int_{A_{\frac{K}{2},R}} \left|u - \frac{K}{2}\right|^{p_-^*}\mathrm{d}x$. This implies

$$
C_{\mu,K}\left(\frac{2^{3p_-^*}}{R^{p_-^*}} + 2^{2p_-^*+1}\right) + \frac{2^{p_-}}{R^{p_-}} + \frac{2^{p_-+2p_-^*}}{R^{p_-}} \le \frac{S\,2^{-p_- - 2N}}{\left(\int_{A_{\frac{K}{2},R}}\left|u - \frac{K}{2}\right|^{p_-^*}\mathrm{d}x\right)^{\frac{p_-}{N}}}.
$$

From (36) choose a suitable K to satisfy the following inequality

$$
\begin{aligned}
& (\frac{2^{3p_-^*}}{R^{p_-^*}} + 2^{2p_-^*+1})^{-1}[\frac{S\,2^{-p_- \cdot 2N}}{(\int_{A_{\frac{K}{2},R}}|u - \frac{K}{2}|^{p_-^*}\mathrm{d}x)^{\frac{p_-}{N}}} - \frac{2^{p_-}}{R^{p_-}} - \frac{2^{p_-+2p_-^*}}{R^{p_-}}] - \\
& 2C_{\varepsilon,p} - \lambda C2^{p_+} - 1 > 0,
\end{aligned}
$$

and fix δ_1 such that

$$
\begin{aligned}
\mu \le \delta_1 = & \frac{1}{K^{q_+-p_+}2^{p_++1}}\left\{\left(\frac{2^{3p_-^*}}{R^{p_-^*}} + 2^{2p_-^*+1}\right)^{-1}\cdot\right. \\
& \left[\frac{S\,2^{-p_- - 2N}}{\left(\int_{A_{\frac{K}{2},R}}|u - \frac{K}{2}|^{p_-^*}\mathrm{d}x\right)^{\frac{p_-}{N}}} - \frac{2^{p_-}}{R^{p_-}} - \frac{2^{p_-+2p_-^*}}{R^{p_-}}\right] - \\
& \left. 2C_{\varepsilon,p} - \lambda C2^{p_+} - 1\right\},
\end{aligned}
$$

where $C_{p,\varepsilon} = \frac{p_+}{p_-}2^{p_+}(2p_+ - 2)^{\frac{(p_+-1)p_-}{p_+}}$. Thus we obtain $a_0 \le c^{-\frac{1}{\eta}} b^{-\frac{1}{\eta^2}}$ for $\mu \in [0, \delta_1]$. All the hypotheses of Lemma 2.4 are satisfied when (35) and (37), and we obtain $a_i \to 0$ as $i \to \infty$, which implies

$$
u \le K \text{ on } B_{\frac{R}{2}}, \text{ for } \mu \in [0, \delta_1],
$$

by (28) and (29). Set $\delta = \min\{\delta_0, \delta_1\}$, where $\delta_0 > 0$ is given by Lemma 2.5. Similarly, we can prove that $-u$ is also bounded on $B_{\frac{R}{2}}$. Since $u(x)|_{\partial\Omega} = 0$ and Ω is a bounded smooth domain, we can obtain $|u| \le K$ a.e. Ω for $\mu \in [0, \delta]$. We obtain inequality (18), and the proof is completed.

Acknowledgements

This work was supported by the Fundamental Research Funds for the Central Universities (2024KYJD2006).

Conflict of interest

The author declares that he has no conflict of interest.

Biography

Lin Zhao is a Lecturer at China University of Mining and Technology. He received his Ph.D. degree in Mathematics Science from Lanzhou University in 2013. His research mainly focuses on nonlinear functional analysis and elliptic equations.

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CSTR: 32290.14.JUSTC-2024-0056
DOI: 10.52396/JUSTC-2024-0056
JUSTC, **2025**, 55(5): 0502