JUSTC

Article

http://justc.ustc.edu.cn

Received: January 25, 2024; Accepted: September 12, 2024

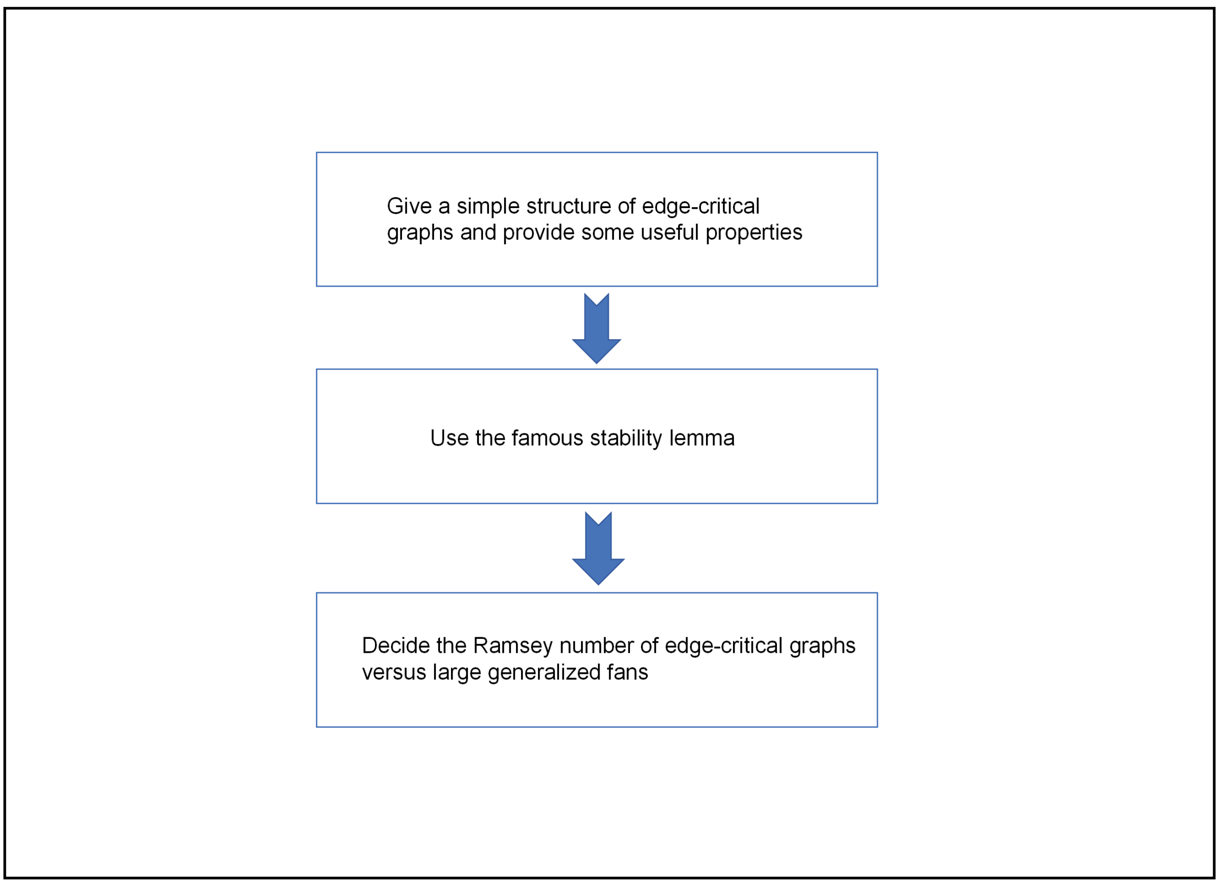
Ramsey numbers of edge-critical graphs versus large generalized fans

Taiping Jiang[1], and Xinmin Hou[1,2,3] ✉

[1]School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China;
[2]CAS Key Laboratory of Wu Wen-Tsun Mathematics, University of Science and Technology of China, Hefei 230026, China;
[3]Hefei National Laboratory, University of Science and Technology of China, Hefei 230088, China

✉Correspondence: Xinmin Hou, E-mail: xmhou@ustc.edu.cn

Graphical abstract



Research framework.

Public summary

- The Ramsey number $R(G, H)$ is the smallest positive integer N such that every 2-coloring of the edges of K_N contains either a red G or a blue H.
- Ramsey numbers of odd cycles and complete graphs versus large generalized fans have a few excellent results.
- Extend odd cycles or complete graphs to edge-critical graphs, the relevant result has the natural promotion.

Citation: Jiang T P, Hou X M. Ramsey numbers of edge-critical graphs versus large generalized fans. *JUSTC*, **2025**, 55(5): 0506. CSTR: 32290.14.JUSTC-2024-0008. DOI: 10.52396/JUSTC-2024-0008

JUSTC Article
http://justc.ustc.edu.cn Received: January 25, 2024; Accepted: September 12, 2024

Ramsey numbers of edge-critical graphs versus large generalized fans

Taiping Jiang[1], and Xinmin Hou[1,2,3] ✉

[1]School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China;
[2]CAS Key Laboratory of Wu Wen-Tsun Mathematics, University of Science and Technology of China, Hefei 230026, China;
[3]Hefei National Laboratory, University of Science and Technology of China, Hefei 230088, China

✉Correspondence: Xinmin Hou, E-mail: xmhou@ustc.edu.cn

Cite This: *JUSTC*, **2025**, 55(5): 0506 (5pp) Read Online

Abstract: Given two graphs G and H, the Ramsey number $R(G,H)$ is the smallest positive integer N such that every 2-coloring of the edges of K_N contains either a red G or a blue H. Let $K_{N-1}\sqcup K_{1,k}$ be the graph obtained from K_{N-1} by adding a new vertex v connecting k vertices of K_{N-1}. A graph G with $\chi(G)=k+1$ is called edge-critical if G contains an edge e such that $\chi(G-e)=k$. A considerable amount of research has been conducted by previous scholars on Ramsey numbers of graphs. In this study, we show that for an edge-critical graph G with $\chi(G)=k+1$, when $k\ge 2$, $t\ge 2$, and n is sufficiently large, $R(G,K_1+nK_t)=knt+1$ and $r_*(G,K_1+nK_t)=(k-1)nt+t$.

Keywords: Ramsey number; color critical graph; generalized fan

CLC number: O157.5 **Document code:** A

2020 Mathematics Subject Classification: Primary 05D10; Secondary 05C55

1 Introduction

Given two graphs G and H, the Ramsey number $R(G,H)$ is the smallest positive integer N such that every 2-coloring of the edges of K_N contains either a red G or a blue H. Let $\chi(G)$ be the chromatic number of G, and $s(G)$ be the minimum size of a color class over all proper $\chi(G)$-coloring of G. Burr[1] gave the following fundamental result about the lower bound of the Ramsey number $R(G,H)$.

Theorem 1.1.[1] Let G and H be graphs such that H is connected and $|V(H)|\ge s(G)$. Then,

$$R(G,H)\ge(\chi(G)-1)(|V(H)|-1)+s(G). \tag{1}$$

We call H G-good if the lower bound in (1) is tight for G and H. There are fruitful results on Ramsey numbers $R(G,H)$; one can refer to an excellent survey[2]. In this note, we are mainly concerned with the pairs (G,H) with H being G-good (we also call (G,H) a good pair). Before we introduce more results for good (G,H)-pairs, we give some notations. Write K_k for a complete graph on k vertices, and T_n for a tree on n vertices. For disjoint graphs G and H, let $G+H$ be the joint graph of G and H, i.e., the graph obtained by connecting $V(G)$ and $V(H)$ completely, and write nG for the disjoint union of n copies of G. For example, a fan $F_n=K_1+nK_2$. Write C_k for a cycle on k vertices. There are some good pairs (G,H) that we have known, here is a list of them.

(a)[3] Let $k,n\ge 1$ be integers. Then $R(K_k,T_n)=(k-1)\cdot(n-1)+1$.

(b)[4] If $k\ge 2$ and $t\ge 2$ are fixed, then $R(K_{k+1},K_1+nK_t)=knt+1$ for sufficiently large n.

(c)[5] If $k\ge 2$ and $t\ge 3$ are fixed, then $R(K_{k+1},K_t+nK_1)=k(n+t-1)+1$ for sufficiently large n.

(d)[6] If integers $k,t\ge 1$ are fixed, then $R(C_{2k+1},K_1+nK_t)=2nt+1$ for sufficiently large n.

An interesting problem proposed by Hook and Isaak[7] is the following: If we know the Ramsey number $R(G,H)=r$, what is the largest number of edges we can delete from K_r such that every 2-coloring of the edges of the resulting graph still contains either a red G or a blue H. More precisely, let $K_{r-1}\sqcup K_{1,k}$ be the graph obtained from K_{r-1} by adding a new vertex v connecting k vertices of K_{r-1}. Hook and Isaak[7] defined the star-critical Ramsey number $r_*(G,H)$ as the smallest integer k such that every 2-coloring of the edges of $K_{r-1}\sqcup K_{1,k}$ contains either a red G or a blue H. There are a few known results of the star-critical Ramsey number, and we list some of them.

(e)[8] For any fixed integers $k\ge 2$ and $t\ge 2$, $r_*(K_{k+1},K_1+nK_t)=(k-1)tn+t$ for all sufficiently large n.

(f)[8] For any fixed integers $k\ge 2$ and $t\ge 2$, $r_*(K_{k+1},K_t+nK_1)=(k-1+o(1))n$ as $n\to\infty$. They also asked the question of the exact value of $r_*(K_{k+1},K_t+nK_1)$.

(g)[9] For a fixed odd cycle C_{2m+1}, if n is large, then $r_*(C_{2m+1},K_1+nK_t)=nt+t$.

A graph H with $\chi(H)=k+1$ is called an edge-critical graph if H contains an edge e such that $\chi(H-e)=k$. Note that C_{2m+1} and K_{k+1} are edge-critical graphs. Recall that a fan $F_n=K_1+nK_2$. We call K_1+nK_t a generalized fan. In this study, we give the Ramsey number and star-critical Ramsey number of an edge-critical graph versus a generalized fan.

The main results of this work are as follows.

Theorem 1.2. Let $k \geq 2$ and $t \geq 2$ be integers and G be a fixed edge-critical graph with $\chi(G) = k+1$. If n is sufficiently large, then $R(G, K_1 + nK_t) = knt + 1$.

Theorem 1.3. Let $k \geq 2$ and $t \geq 2$ be integers and G be a fixed edge-critical graph with $\chi(G) = k+1$. If n is sufficiently large, then $r_*(G, K_1 + nK_t) = (k-1)nt + t$.

Clearly, Theorems 1.2 and 1.3 extend the results of (b), (d), (e), and (g).

The rest of this article is organized as follows. In Section 2, we provide some preliminaries. Next, we prove Theorem 1.2 and Theorem 1.3 in Section 3 and Section 4, respectively. Finally, We present some conclusions in Section 5.

2 Preliminaries

In this section we present some useful lemmas and results needed in our proofs. The first lemma is an upper bound of $R(G, nH)$ due to Ref. [10].

Lemma 2.1.[10] Let G and H be graphs. Then, $R(G, nH) \leq (n-1)|V(H)| + R(G, H)$.

The following is the classical stability lemma due to Refs. [11–14]. Let G be a graph and $V_i, V_j \subseteq V(G)$. Write $E_G(V_i, V_j)$ for the set of edges with one end in V_i and the other in V_j, and write $E_G(V_i)$ for $E_G(V_i, V_i)$. For a vertex $v \in V(G)$, write $d_G(v, V_i) = |E_G(\{v\}, V_i)|$. Let $[k] = \{1, 2, \cdots, k\}$ for positive integer k.

Lemma 2.2.[13] Let G be a given graph with $\chi(G) = p+1$. For every $\xi > 0$ there exist $\delta = \delta(\xi)$ and $n_0 = n_0(\delta) > 0$ such that if H is a graph of order $N > n_0$ and $e(H) > \frac{1}{2p}(p-1)N^2 - \delta N^2$ that does not contain G, then there exists a partition of $V(H)$ into classes $V_1, V_2, \cdots, V_p$ such that

- $\sum_{1 \leq i < j \leq p} (|V_i||V_j| - |E_H(V_i, V_j)|) < \xi N^2$;
- $\sum_{i=1}^{p} |E_H(V_i)| < \xi N^2$ for $i \in [p]$;
- for every $v \in V_i$ and $i, j \in [p]$, $d_G(v, V_i) \leq d_G(v, V_j)$;
- $\frac{N}{p} - \xi N < |V_i| < \frac{N}{p} + \xi N$ for each $i \in [p]$.

The following lemma obtained by Hao and Lin[8] gives a general lower bound for the star-critical Ramsey number $r_*(G, H)$. Denote a proper k-coloring of G with color classes $U_1, U_2, \cdots, U_k$ by $(U_1, U_2, \cdots, U_k)$. Let

$$\tau(G) = \min_{v \in U_\chi, 1 \leq i < \chi} \{d_G(v, U_i) : (U_1, U_2, \cdots, U_\chi) \text{ is a proper } \chi\text{-coloring of } G \text{ with } |U_\chi| = s(G)\}.$$

Lemma 2.3.[8] Let G be a graph with $\chi(G) \geq 2$, and H be a connected graph of order $n \geq s(G)+1$ with minimum degree $\delta(H)$. We have

$$r_*(G, H) \geq (\chi(G) - 2)(n-1) + \min\{n, \delta(H) + \tau(G) - 1\}.$$

Furthermore, if H contains no cut vertex or $\delta(H) = 1$, then

$$r_*(G, H) \geq (\chi(G) - 2)(n-1) + \min\{n, \delta(H) + \tau(G) - 1\} + s(G) + 1.$$

We also call (G, H) a good pair if the lower bound in Lemma 2.3 is tight for $r_*(G, H)$. Finally, we give a structural property of the edge-critical graph.

Lemma 2.4. Let G be an edge-critical graph with $\chi(G) = k$. Then there exists a proper k-coloring $(V_1, V_2, \cdots, V_k)$ of G such that $|V_k| = 1$ and $|E(V_k, V_i)| = 1$ for some $i \in [k-1]$.

Proof. Choose an edge $e = uv \in E(G)$ with $\chi(G - e) = k-1$. Then $G - e$ has a proper $(k-1)$-coloring π' with $\pi'(u) = \pi'(v)$, otherwise, we obtain a proper $(k-1)$-coloring of G, a contradiction. Now we recolor the vertices by setting $\pi : V(G) \to [k]$ as follows:

$$\pi(x) = \begin{cases} \pi'(x), & x \neq v; \\ k, & x = v. \end{cases}$$

Then π is a proper k-coloring of G, and vertex v is what we want to find.

As a direct corollary of Lemma 2.4, we have

Corollary 2.1. Let G be an edge-critical graph. Then $s(G) = 1$ and $\tau(G) = 1$.

3 Proof of Theorem 1.2

Write $K_{n_1, n_2, \cdots, n_k}$ for a k-partite graph with color classes of sizes $n_1, n_2, \cdots, n_k$ and let $K^+_{n_1, n_2, \cdots, n_k}$ be the graph obtained by adding an extra edge in one class of $K_{n_1, n_2, \cdots, n_k}$.

Proof of Theorem 1.2. Let G be an edge-critical graph with $\chi(G) = k+1$. By Corollary 2.1, $s(G) = 1$. By Theorem 1.1, $R(G, K_1 + nK_t) \geq knt + 1$. Let $N = knt + 1$. Then it suffices to show $R(G, K_1 + nK_t) \leq knt + 1$, i.e., we prove that any 2-coloring of the edges of K_N contains either a red G or a blue $K_1 + nK_t$ when n is sufficiently large.

In contrast, suppose that there exists a 2-coloring of the edges of K_N that contains neither a red G nor a blue $K_1 + nK_t$. Let R and B denote the subgraphs consisting of red and blue edges in K_N, respectively. The subgraph of K_N induced by $N_B(v)$ contains neither a red G nor a blue nK_t. By Lemma 2.1, we have

$$d_B(v) < R(G, nK_t) \leq (n-1)t + R(G, K_t).$$

Thus $d_R(v) = N - 1 - d_B(v) \geq (k-1)tn - O(1)$, which implies that

$$e(R) = \frac{1}{2}\sum_v d_R(v) \geq \frac{k-1}{2k}N^2 - O(N).$$

Set $\xi > 0$, a very small real number. Applying Lemma 2.2 to graph G and the G-free graph R, we have a partition of $V(R) = V(K_N)$ into k classes $V_1, V_2, \cdots, V_k$ such that

(i) $\sum_{1 \leq i < j \leq k} (|V_i||V_j| - |E_R(V_i, V_j)|) < \xi N^2$;

(ii) $|E_R(V_i)| < \xi N^2$ for every $i \in [k]$;

(iii) for every $v \in V_i$ and $i, j \in [k]$, $d_R(v, V_i) \leq d_R(v, V_j)$;

(iv) $\frac{N}{k} - \xi N < |V_i| < \frac{N}{k} + \xi N$ for every $i \in [k]$.

With loss of generality, we may assume that $|V_1| \geq |V_i|$ for $i = 2, \cdots, k$. Thus $|V_1| \geq tn + 1 > \frac{N}{k}$. Define a subset $V'_i \subset V_i$ as follows:

$$V'_i = \{x \in V_i \mid d_R(x, V_j) \geq (1 - 2\sqrt{\xi})|V_j|, \text{ for all } j \in [k] \text{ and } j \neq i\}.$$

CSTR: 32290.14.JUSTC-2024-0008
DOI: 10.52396/JUSTC-2024-0008
JUSTC, **2025**, 55(5): 0506

Claim 1. $|V_i'| \geq (1-2k\sqrt{\xi})|V_i|$ for $i \in [k]$.

Proof. Suppose to the contrary that there is an $i \in [k]$ with

$$|V_i \setminus V_i'| > 2k\sqrt{\xi}|V_i| > 2k\sqrt{\xi}\left(\frac{1}{k}-\xi\right)N.$$

By the definition of V_i', for any vertex $v \in V_i \setminus V_i'$, we have

$$d_B(v,V_j) = |V_j| - d_R(x,V_j) \geq 2\sqrt{\xi}|V_j| \geq 2\sqrt{\xi}\left(\frac{1}{k}-\xi\right)N,$$

$$\text{for all } j \in [k] \text{ and } j \neq i.$$

Thus

$$\sum_{\substack{j=1\\ j\neq i}}^{k} |E_B(V_i,V_j)| > (k-1)\cdot 2k\sqrt{\xi}\left(\frac{1}{k}-\xi\right)N \cdot 2\sqrt{\xi}\left(\frac{1}{k}-\xi\right)N = 4k(k-1)\xi\left(\frac{1}{k}-\xi\right)^2 N^2 > \xi N^2.$$

However,

$$\sum_{j=1,j\neq i}^{k} |E_B(V_i,V_j)| \leq \sum_{1\leq i<j\leq k} |E_B(V_i,V_j)| = \sum_{1\leq i<j\leq k} (|V_i||V_j| - |E_R(V_i,V_j)|) < \xi N^2,$$

a contradiction.

Let m be the maximum size of a color class over all proper $(k+1)$-coloring of G. The construction of a copy of red G in R will be used throughout the remainder of the proof.

Claim 2. If there is a red edge $e = u_{i1}u_{i2}$ in some V_i such that $d_R(u_{i1},V_j') + d_R(u_{i2},V_j') \geq (1+\alpha)|V_j|$ for some constant $\alpha > 0$ and $j \neq i$, then R contains a copy of G.

Proof. Without loss of generality, we assume that $e = u_{11}u_{12}$ is a red edge in V_1. Note that, for any vertex $v \in V_i'$, we have

$$d_R(v,V_j') \geq (1-2\sqrt{\xi})|V_j| - 2k\sqrt{\xi}|V_j| = (1-2(k+1)\sqrt{\xi})|V_j|.$$

We will construct a copy of G in R with $e = u_{11}u_{12} \in E(G)$. In the following proof of this claim, the host graph we considered is R. First, we choose $m-1$ vertices $u_{13},\cdots,u_{1(m+1)}$ from V_1' and let $U_1 = \{u_{11},u_{12},u_{13},\cdots,u_{1(m+1)}\}$. Then, when n is large enough, the common neighbors of U_1 in V_j' with $j \geq 2$ are

$$\left|\bigcap_{i=1}^{m+1} N_R(u_{1i},V_j')\right| \geq (1+\alpha)|V_j| + \sum_{i=3}^{m+1} d_R(u_{1i},V_j') - m|V_j'| \geq \left(\alpha - 2(k+1)(m-1)\sqrt{\xi}\right)|V_j| \geq m.$$

For ease of notations, we denote $\bigcap_{i=1}^{m+1} N_R(u_{1i},V_j') = W_j^{(1)}$ for $j = 2,3,\cdots,k$. Second, we take m vertices $U_2 = \{u_{21},u_{22},\cdots,u_{2m}\}$ from $W_2^{(1)}$. When n is large enough, the common neighbors of $U_1 \cup U_2$ in V_j' for $j = 3,\cdots,k$ is

$$\left|\left(\bigcap_{i=1}^{m+1} N_R(u_{1i},V_j')\right) \bigcap \left(\bigcap_{i=1}^{m} N_R(u_{2i},V_j')\right)\right| \geq \left(\alpha - 2(k+1)(2m-1)\sqrt{\xi}\right)|V_j| \geq m.$$

Similarly, denote $\left(\bigcap_{i=1}^{m+1} N_R(u_{1i},V_j')\right) \bigcap \left(\bigcap_{i=1}^{m} N_R(u_{2i},V_j')\right) = W_j^{(2)}$, for $j = 3,\cdots,k$. Note that $W_3^{(2)} \subseteq W_3^{(1)}$. We continue to choose m vertices $U_3 = \{u_{31},u_{32},\cdots,u_{3m}\}$ from $W_3^{(2)}$ and compute the common neighbors of $U_1 \cup U_2 \cup U_3$ in $V_4',\cdots,V_k'$. Repeat the above process, until the last step we get $k-1$ disjoint vertex sets $U_1,U_2,\cdots,U_{k-1}$, consider their common neighbors in V_k'. Let

$$\left(\bigcap_{i=1}^{m+1} N_R(u_{1i},V_k')\right) \bigcap \left(\bigcap_{i=1}^{m} N_R(u_{2i},V_k')\right) \bigcap \cdots \bigcap \left(\bigcap_{i=1}^{m} N_R(u_{(k-1)i},V_k')\right) = W_k^{(k-1)}.$$

When n is large enough, we still have

$$|W_k^{(k-1)}| \geq \left(\alpha - 2(k+1)((k-1)m-1)\sqrt{\xi}\right)|V_k| \geq m.$$

Thus we can still choose m vertices $U_k = \{u_{k1},u_{k2},\cdots,u_{km}\}$ from $W_k^{(k-1)}$. Note that $W_i^{(j)} \subseteq W_i^{(j')}$ for $2 \leq i \leq k$ and $1 \leq j' \leq j \leq k-1$. Thus $R[U_1 \cup U_2 \cup \cdots \cup U_k]$ contains a copy of $K^+_{(m+1),m,\cdots,m}$ with an extra edge $e = u_{11}u_{12}$ in $R[U_1]$. By the definition of m and Lemma 2.4, we know that there is a copy of G in $K^+_{(m+1),m,\cdots,m} \subseteq R$.

Claim 3. The induced subgraph $B[V_i']$ is a complete graph for $i \in [k]$.

Proof. By symmetry, it is sufficient to show that $B[V_1']$ is a complete graph. Suppose not, then there exists an edge $e = \{u_{11},u_{12}\}$ in $R[V_1']$. Note that, for any vertex $v \in V_i'$, we have

$$d_R(v,V_j') \geq (1-2\sqrt{\xi})|V_j| - 2k\sqrt{\xi}|V_j| = (1-2(k+1)\sqrt{\xi})|V_j|.$$

Thus $d_R(u_{11},V_j') + d_R(u_{12},V_j') \geq (2-4(k+1)\sqrt{\xi})|V_j| \geq (1+\alpha)|V_j|$ for some constant $\alpha > 0$. By Claim 2, we have a red copy of G in R, a contradiction.

Claim 4. $E(V_i', V_i \backslash V_i') \subseteq E(B)$ for every $i \in [k]$.

Proof. By symmetry, it suffices to prove that $E(V_1', V_1 \backslash V_1') \subseteq E(B)$. Suppose that there exists a red edge $e = u_{11}u_{12}$ with $u_{11} \in V_1 \backslash V_1'$ and $u_{12} \in V_1'$. Again note that, for $j = 2,3,\cdots,k$,

$$d_R(u_{12},V_j') \geq (1-2\sqrt{\xi})|V_j| - 2k\sqrt{\xi}|V_j| = (1-2(k+1)\sqrt{\xi})|V_j|.$$

By Lemma 2.2, $d_R(u_{11},V_j) \geq d_R(u_{11},V_1)$ for $u_{11} \in V_1 \setminus V_1'$. Thus, when ξ is very small and n is large enough, we have

$$\begin{aligned} d_R(u_{11},V_j') &\geq \frac{1}{2}(d_R(u_{11},V_1) + d_R(u_{11},V_j)) - |V_j \setminus V_j'| = \\ &\frac{1}{2}\left(d_R(u_{11}) - \sum_{\substack{i=2\\ i\neq j}}^{k} |V_i|\right) - |V_j \setminus V_j'| \geq \\ &\frac{1}{2}\left((k-1)nt - O(1) - (k-2)\left(\frac{N}{k}+\xi N\right)\right) - |V_j \setminus V_j'| \geq \\ &\frac{1}{2}\left((k-1)\frac{N-1}{k} - O(1) - (k-2)\left(\frac{N}{k}+\xi N\right)\right) - 2k\sqrt{\xi}|V_j| > \\ &\left(\frac{1}{3} - 2k\sqrt{\xi}\right)|V_j|. \end{aligned}$$

Therefore,

CSTR: 32290.14.JUSTC-2024-0008
DOI: 10.52396/JUSTC-2024-0008
JUSTC, **2025**, 55(5): 0506

$$d_R(u_{11}, V_j') + d_R(u_{12}, V_j') \geq \left(\frac{4}{3} - 2(2k+1)\sqrt{\xi}\right)|V_j| \geq (1+\alpha)|V_j|$$

for some constant $\alpha > 0$. Again by Lemma 2.2, we have a red copy of G in R, a contradiction again.

Notice any vertex $v \in V_1 \setminus V_1'$ and $t-1$ vertices in V_1' induces a complete graph K_t in B. Since $|V_1 \setminus V_1'| < |V_1'|/(t-1)$, we can repeat the above process until the vertices of $V_1 \setminus V_1'$ have been used up; now we have $|V_1 \setminus V_1'|$ copies of blue K_t in $B[V_1]$. Note that $|V_1| \geq nt+1$. By Claim 3, we can continue to find copies of blue K_t in the remaining vertices of $B[V_1]$ until we find n disjoint copies of blue K_t in $B[V_1]$. Note that if $|V_1| \geq nt+1$, at least one vertex left in V_1' that is completely connected with the n vertex disjoint copies of blue K_t, thus we construct a blue $K_1 + nK_t$, which leads to a contradiction.

4 Proof of Theorem 1.3

Proof of Theorem 1.3. By Corollary 2.1, we have $\tau(G) = 1$. Note that $\delta(K_1 + nK_t) = t$. Thus when n is large, we have $r_*(G, K_1 + nK_t) \geq (k-1)nt + t$ by Lemma 2.3. In the following, we will show that $r_*(G, K_1 + nK_t) \leq (k-1)nt + t$.

Let $N = knt$. Then $R(G, K_1 + nK_t) = N + 1$ by Theorem 1.2. Thus there is a $\{R, B\}$-coloring of the edges of K_N such that it contains neither a red G nor a blue $K_1 + nK_t$. By Lemma 2.1, we have

$$d_B(v) < R(G, nK_t) \leq t(n-1) + R(G, K_t).$$

Therefore, $d_R(v) = N - 1 - d_B(v) \geq (k-1)tn - O(1)$ which implies that

$$e(R) = \frac{1}{2}\sum_{v \in V(R)} d_R(v) \geq \frac{k-1}{2k}N^2 - O(N).$$

Apply the stability Lemma 2.2 to R with the forbidden graph G, for a sufficiently small real number $\xi > 0$, there is a partition of $V(R) = V(K_N)$ into k classes $V_1, V_2, \cdots, V_k$ such that

(i) $\sum_{1 \leq i < j \leq k} (|V_i||V_j| - |E_R(V_i, V_j)|) < \xi N^2$;

(ii) $|E_R(V_i)| < \xi N^2$ for every $i \in [k]$;

(iii) for every $v \in V_i$ and $i \in [k]$, $d_R(v, V_i) \leq d_R(v, V_j)$ for every $j \in [k]$;

(iv) $\frac{N}{k} - \xi N < |V_i| < \frac{N}{k} + \xi N$ for every $i \in [k]$.

Similar to the proof of Theorem 1.2, define $V_i' \subset V_i$ as follows:

$$V_i' = \{x \in V_i | d_R(x, V_j) \geq (1 - 2\sqrt{\xi})|V_j|, \text{ for all } j \in [k] \text{ and } j \neq i\}.$$

With the same discussion as in the proof of Theorem 1.2, we have the following three claims.

Claim 5. $|V_i'| \geq (1 - 2k\sqrt{\xi})|V_i|$ for every $i \in [k]$.

Claim 6. $B[V_i']$ is a complete graph for every $i \in [k]$.

Claim 7. $E(V_i', V_i) \subseteq E(B)$ for every $i \in [k]$.

By Claims 6 and 7, we have $|V_i| \leq nt$, otherwise, we can construct a copy of $K_1 + nK_t$ in B with the same discussion as in the last paragraph of the proof of Theorem 1.2. Note that $V_1, V_2, \cdots, V_k$ is a partition of $V(K_N)$ and $N = knt$. We have the following claim.

Claim 8. For every $i \in [k]$, $|V_i| = nt$ and there are n disjoint copies of blue K_t in $B[V_i]$.

We further claim that there are no red edges in $V_i \setminus V_i'$ for $i \in [k]$. Recall that m denotes the maximum size of a color class over all proper $(k+1)$-coloring of G.

Claim 9. $B[V_i \setminus V_i']$ is a complete graph for every $i \in [k]$.

Proof. By symmetry, we will show that $B[V_1 \setminus V_1']$ is a complete graph. Suppose to the contrary that there exists a red edge $v_{11}v_{12}$ with $v_{11}, v_{12} \in V_1 \setminus V_1'$. Then, $d_B(v_{11}, V_j') \leq t-1$ for $j = 2, \cdots, k$, since, otherwise, $\{v_{11}\} \cup V_j$ contains a copy of blue $K_1 + nK_t$. Similarly, we have $d_B(v_{12}, V_j') \leq t-1$. Thus, we have

$$d_R(v_{1i}, V_j') \geq |V_j'| - t + 1 \geq (1 - 2k\sqrt{\xi})|V_j| - t + 1 \text{ for } i = 1, 2.$$

Therefore, when n is sufficiently large and ξ is very small,

$$d_R(v_{11}, V_j') + d_R(v_{12}, V_j') \geq (2 - 4k\sqrt{\xi})|V_j| - 2t + 2 \geq (1+\alpha)|V_j|$$

for some constant $\alpha > 0$. By Claim 2, R contains a copy of G, a contradiction.

By Claims 6, 7, and 9, the graph induced by V_i is a blue K_{tn} for $i \in [k]$.

Now, assume v_0 is a new vertex outside $V(K_N)$. We claim that if $d(v_0, V(K_N)) \geq (k-1)nt + t$, then there exists either a red G or a blue $K_1 + nK_t$.

Without loss of generality, suppose $d(v_0, V_k) \leq d(v_0, V_{k-1}) \leq \cdots \leq d(v_0, V_1)$. Then we have $t \leq d(v_0, V_k) \leq (k-1)nt/k + t/k$. By Claim 8, $d(v_0, V_i) \leq |V_i| = nt$. We can easily prove that $d(v_0, V_{k-1}) \geq nt/2$, since, otherwise,

$$\sum_{i=1}^{k-2} d(v_0, V_i) + d(v_0, V_{k-1}) + d(v_0, V_k) < (k-2)nt + nt/2 + nt/2 = (k-1)nt < d(v_0, V(K_N)),$$

a contradiction. Thus we have $d(v_0, V_i) \geq d(v_0, V_{k-1}) \geq nt/2$ for $i \in [k-1]$. Note that V_i induces a blue K_{nt}. Then for each $i \in [k]$, $d_B(v_0, V_i) \leq t-1$, otherwise, $\{v_0\} \cup V_i$ contains a blue $K_1 + nK_t$. For the same reason, we know that, for $u \in V_i$, $d_B(u, V_j) \leq t-1$ for every $j \in [k]$ and $j \neq i$. Therefore,

$$d_R(v_0, V_k) \geq d(v_0, V_k) - (t-1) \geq t - (t-1) = 1$$

and for $u \in V_i$,

$$d_R(u, V_j') \geq |V_j'| - (t-1) \geq (1 - 2k\sqrt{\xi})|V_j| - t + 1, \quad j \in [k] \text{ and } j \neq i.$$

Thus there exists a vertex $v_{11} \in V_k$ such that $v_0 v_{11}$ is a red edge. Note that

$$d_R(v_0, V_j') \geq \frac{nt}{2} - |V_j \setminus V_j'| - (t-1) \geq \left(\frac{1}{2} - 2k\sqrt{\xi}\right)|V_j| - t + 1, \quad \text{for } j \in [k-1].$$

Therefore, when n is sufficiently large and ξ is very small, we have

CSTR: 32290.14.JUSTC-2024-0008
DOI: 10.52396/JUSTC-2024-0008
JUSTC, **2025**, 55(5): 0506

$$d_R(v_0,V_j')+d_R(v_{11},V_j')\geq\left(\frac{3}{2}-4k\sqrt{\xi}\right)|V_j|-2t+2\geq(1+\alpha)|V_j|,$$

for some constant $\alpha>0$ and $j\in[k-1]$. We can view v_0 as a vertex in V_k and v_0v_{11} as a red edge in V_k. Then, by Claim 2, we have a copy of red G in R, a contradiction. This completes the proof of Theorem 1.3.

5 Conclusions

In this study, we show that for an edge-critical graph G with $\chi(G)=k+1$, when $k\geq 2$, $t\geq 2$, and n is sufficiently large, $R(G,K_1+nK_t)=knt+1$ and $r_*(G,K_1+nK_t)=(k-1)nt+t$. We extend some results as mentioned in the intrduction. We ask the question of what the exact value of $r_*(K_{k+1},K_t+nK_1)$ is. One may naturally ask the question of what the exact values of (G,K_t+nK_1) and $r_*(G,K_t+nK_1)$ are. More generally, determining the exact values of (G,K_t+nK_r) and $r_*(G,K_t+nK_r)$ deserves more further discussion.

Acknowledgements

This work was supported by the National Key Research and Development Program of China (2023YFA1010200, 2020YFA0713100), the National Natural Science Foundation of China (12071453), and the Innovation Program for Quantum Science and Technology (2021ZD0302902).

Conflict of interest

The authors declare that they have no conflicts of interest.

Biographies

Taiping Jiang received his M.D. degree from the University of Science and Technology of China in 2024. His research mainly focuses on graph and conbinatorics.

Xinmin Hou is a Professor at the University of Science and Technology of China. He received his Ph.D. degree in Mathematics from Dalian University of Technology in 2002. His research mainly focuses on extreme graph theory, structural graph theory, and combinatorial optimization theory.

References

[1] Burr S. Ramsey numbers involving graphs with long suspended paths. *J. London Math. Soc.*, **1981**, *24*: 405–413.
[2] Radziszowski S P. Small Ramsey numbers. *Electron. J. Comb.*, **2021**: DS1.16.
[3] Chvátal V. Tree-complete graph ramsey numbers. *J. Graph Theory*, **1977**, *1* (1): 93.
[4] Li Y, Rousseau C. Fan-complete graph Ramsey numbers. *J. Graph Theory*, **1996**, *23*: 413–420.
[5] Nikiforov V, Rousseau C. Large generalized books are *p*-good. *J. Combin. Theory, Ser. B*, **2004**, *92*: 85–97.
[6] Liu M, Li Y. Ramsey numbers of a fixed odd-cycle and generalized books and fans. *Discrete Mathematics*, **2016**, *339*: 2481–2489.
[7] Hook J, Issak G. Star-critical Ramsey numbers. *Discrete Applied Mathematics*, **2011**, *159*: 328–334.
[8] Hao Y, Lin Q. Star-critical Ramsey numbers for large generalized fans and books. *Discrete Mathematics*, **2018**, *341*: 3385–3393.
[9] Li Y, Li Y, Wang Y. Minimal Ramsey graphs on deleting stars for generalized fans and books. *Applied Mathematics and Computation*, **2020**, *372*: 125006.
[10] Burr S, Erdös P, Spencer J. Ramsey theorems for multiple copies of graphs. *Trans. Amer. Math. Soc.*, **1975**, *209*: 87–99.
[11] Erdös P. Some recent results on extremal problems in graph theory. In: Theory of Graphs, International Symposium in Rome. New York: Gordon and Breach, **1966**: 117–130.
[12] Erdös P. On some new inequalities concerning extremal properties of graphs. In: Theory of Graphs: Proceedings of the Colloquium Held at Tihany, Hungary, September, 1966. New York: Academic Press, **1968**: 77–81.
[13] Erdös P, Simonovits M. A limit theorem in graph theory. *Studia Sci. Math. Hung.*, **1966**, *1*: 51–57.
[14] Simonovits M. A method for solving extremal problems in graph theory, stability problems. In: Theory of Graphs: Proceedings of the Colloquium Held at Tihany, Hungary, September, 1966. New York: Academic Press, **1968**: 279–319.

CSTR: 32290.14.JUSTC-2024-0008
DOI: 10.52396/JUSTC-2024-0008
JUSTC, **2025**, 55(5): 0506