JUSTC
Article
http://justc.ustc.edu.cn
Received: March 07, 2023; Accepted: June 05, 2023

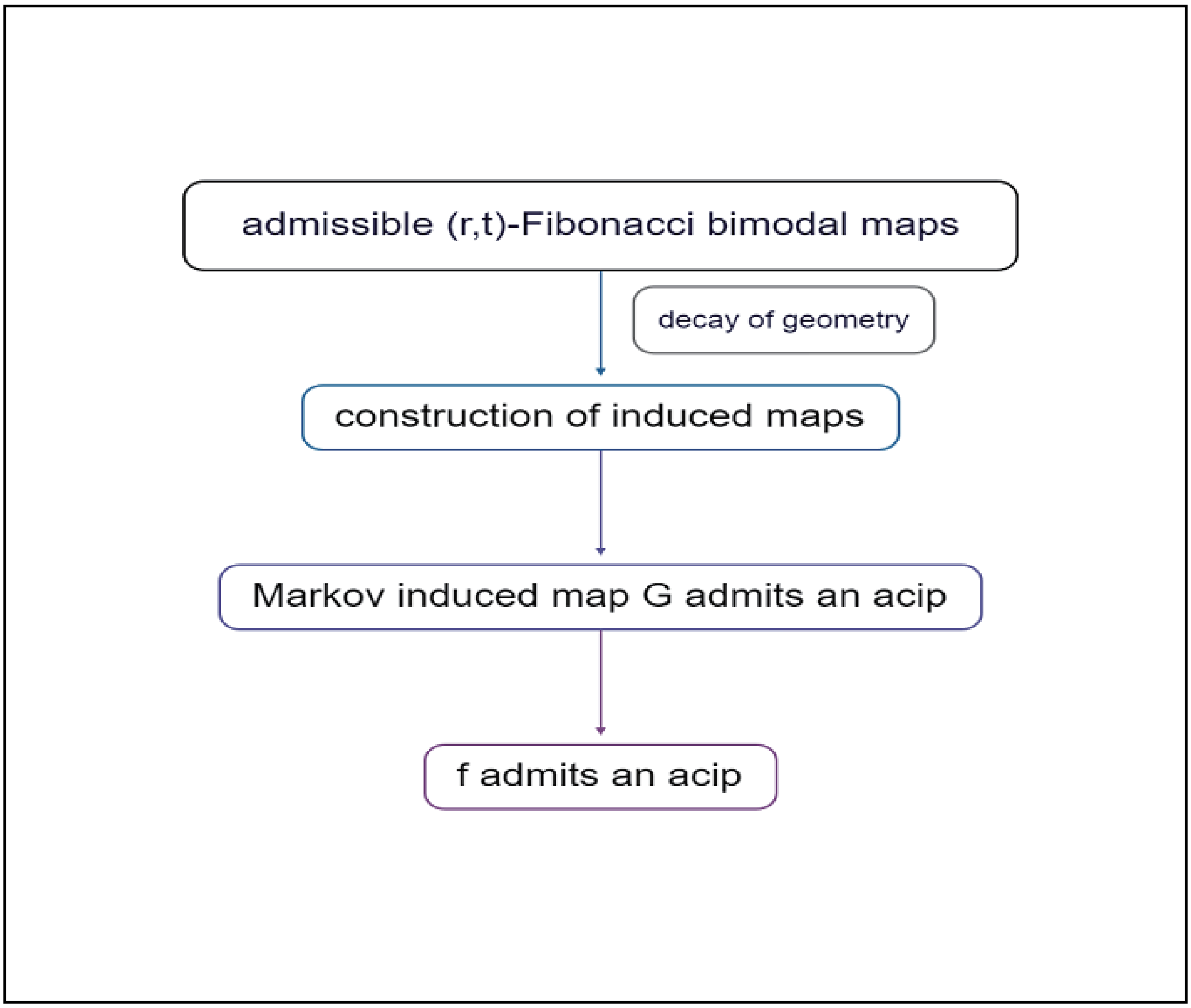
Invariant measure for cubic Fibonacci-like polynomials

Wenxiu Ma ✉

School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China

✉Correspondence: Wenxiu Ma, E-mail: mwx@mail.ustc.edu.cn

Graphical abstract



Public summary

■ We study the combinatorial properties of (r, t)-Fibonacci bimodal maps.

■ We construct an induced map G and show that G admits an acip.

■ We prove that for any $f \in B$, f has an acip.

Citation: Ma W X. Invariant measure for cubic Fibonacci-like polynomials. *JUSTC*, **2024**, 54(8): 0802. DOI: 10.52396/JUSTC-2023-0036

JUSTC

Article

http://justc.ustc.edu.cn

Received: March 07, 2023; Accepted: June 05, 2023

Invariant measure for cubic Fibonacci-like polynomials

Wenxiu Ma ✉

School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China

✉Correspondence: Wenxiu Ma, E-mail: mwx@mail.ustc.edu.cn

Cite This: *JUSTC*, **2024**, 54(8): 0802 (6pp)

Read Online

Abstract: A special class of cubic polynomials possessing decay of geometry property is studied. This class of cubic bimodal maps has generalized Fibonacci combinatorics. For maps with bounded combinatorics, we show that they have an absolutely continuous invariant measure.

Keywords: Fibonacci combinatorics; cubic polynomial; decay of geometry; invariant measure

CLC number: O193 **Document code:** A

2020 Mathematics Subject Classification: 37E05; 37A10

1 Introduction

The dynamical properties of unimodal interval maps have been extensively studied. The 'decay of geometry' property plays an essential role in the study of quadratic dynamics. Several results, including density of hyperbolicity and Milnor’s attractor problem, rely on this phenomenon.

For the unimodal case, let $I^0 \supset I^1 \supset I^2 \supset \dots$ be the principal nest of *f*. The scaling factor of *f* is defined as $\mu_n := \dfrac{|I^{n+1}|}{|I^n|}$. Decay of geometry means that μ_{n_i} decreases to 0 exponentially fast for a subsequence n_i. This concept first appeared in the work of Jakobson and Światiek[3] for non-renormalizable maps with negative Schwarzian derivatives. Lyubich[9], Graczyk and Światiek[2] solved it independently using complex techniques. These proofs make elaborate use of complex methods and do not seem to work for critical orders smaller than 2. More recently, Shen[12] used real analysis techniques to prove the decay of geometry property for smooth unimodal maps with a critical order of no more than 2, thus solving the Milnor attractor problem in this case.

However, the decay of geometry loses its universality for unimodal maps with a larger critical order, or even for multimodal maps with quadratic critical points. For the unimodal case, the typical example is the Fibonacci unimodal maps. It is well-known that Fibonacci unimodal maps possess decay of geometry and admit an absolutely continuous (with respect to the Lebesgue measure) invariant probability measure for critical order $\ell \leqslant 2$, and have bounded geometry for critical order $\ell > 2$, see Refs. [8, 6]. For multimodal maps, the real cubic polynomial with two nondegenerate critical points does not have uniform decay of geometry property either. A preliminary investigation of this phenomenon was carried out by Światiek and Vargas[14], who constructed two cubic polynomials, one with bounded geometry and the other with decay of geometry.

To date, multimodal maps are rarely known. The principal nest is a useful tool when studying the geometric properties of interval maps, but it seems inonvenient for treating metric problems in multimodal cases. Unlike the unimodal case, the scaling factors fail to give distortion control of the first return map of the principal nest. However, in Ref. [15], Vargas constructed the Fibonacci bimodal map using a new tool, named 'twin principal nest', which we will explain later. Our recent work[5] showed that a wide class of cubic maps have the 'decay of geometry' property in the sense that the ratio of twin principal nests decreases at least exponentially fast. In this paper, we concentrate on the metric properties of generalized Fibonacci bimodal maps with uniformly bounded combinatorics.

1.1 Preliminaries

For convenience, (a,b) denotes the interval with endpoints a and b, even though $a > b$. For example, let $(2,1)$ refer to $(1,2)$. If J and J' are two intervals on the real line, by $J < J' (J \leqslant J')$, we mean that $y < y' (y \leqslant y')$ for every $y \in J$ and $y' \in J'$; analogously, we define $a < J$ and $a \leqslant J'$ for real number a.

Denote $I = [0,1]$. A continuous map $f: I \to I$ is called *bimodal* if:

1. $f(\{0,1\}) = \{0,1\}$;
2. there exist exactly two points $c < d$ (called *turning points*) that are the local extreme of *f*;
3. *f* is strictly monotone on subintervals determined by these points.

If the points $\{0,1\}$ are fixed, then we say that the bimodal map *f* is *positive*, and in the case that these points are permuted, we say that *f* is *negative*. Examples of bimodal maps are parameterized families of real cubic polynomials P^+_{ab} and P^-_{ab} given by $P^+_{ab}(x) = ax^3 + bx^2 + (1-a-b)x$ and $P^-_{ab}(x) = 1 - ax^3 - bx^2 - (1-a-b)x$. We are mainly interested in bimodal maps that have neither periodic attractors nor wandering intervals.

For $T \subset I$, let $D(T) = \{x \in I : f^k(x) \in T \text{ for some } k \geqslant 1\}$. The *first entry map* $R_T : D(T) \to T$ is defined as $x \to f^{k(x)}(x)$, where $k(x)$ is the *entry time* of x into T, i.e., the minimal positive integer such that $f^{k(x)}(x) \in T$. The map $R_T|(D(T)\cap T)$ is called the *first return map* of T. A component of $D(T)$ is called an *entry domain* of T and a component of $D(T)\cap T$ is called a *return domain*.

An open set $J \subset I$ is called *nice* if $f^n(\partial J)\cap J = \emptyset$ for all $n \geqslant 0$. Let $T \subset I$ be a nice interval. Let $\mathcal{L}_x(T)$ denote the entry domain of T containing x.

A point $x \in I$ is called *recurrent* provided $x \in \omega(x)$.

Let $\mathcal{B}$ denote the collection of C^3 bimodal maps $f: I \to I$, which have no wandering intervals and all periodic cycles of hyperbolic repelling. Let Crit(f) denote the set of critical points of f, i.e., the set of points where Df vanishes. Note that $\{c,d\}\subset$ Crif(f). Let $\mathcal{B}^+$ and $\mathcal{B}^-$ denote the subset of positive and negative bimodal maps, respectively, from class $\mathcal{B}$. If $f \in \mathcal{B}^+$, then there exists a fixed point p between c and d; otherwise, ∂I contains an attracting fixed point. Let $p_1 < p_2$ be such that $f(p_1)=f(p_2)=p$. Define $I^0=(p_1,p)$, $J^0=(p,p_2)$. If $f\in\mathcal{B}^-$, we discuss three cases:

1. f has three fixed point in $(0,1)$. In this case, there exists a fixed point p in (c,d), then define I^0 and J^0 as above.

2. f has one fixed point p in $(0,1)$ with three preimages $\{p,p_1,p_2\}$ specified by $p_1<p_2$. If $p<p_1<p_2$, define $I^0=(p,p_1)\ni c$ and $J^0=(p_1,p_2)\ni d$; if $p_1<p_2<p$, define $I^0=(p_1,p_2)$ and $J^0=(p_2,p)$.

3. f has one fixed point p in $(0,1)$ with only one preimage, that is, $f^{-1}(p)=\{p\}$. This case can be reduced to the positive case since f^2 restricted on $[p,1]$ is always a positive bimodal map.

Assume that both c and d are recurrent. For every $n\geqslant 1$, define $I^n := \mathcal{L}_c(I^{n-1}\cup J^{n-1})$ and $J^n := \mathcal{L}_d(I^{n-1}\cup J^{n-1})$ inductively. The two sequences of nested intervals

$$I^0 \supset I^1 \supset I^2 \supset \ldots \supset \{c\} \text{ and } J^0 \supset J^1 \supset J^2 \supset \ldots \supset \{d\}$$

are called the twin principal nest of f. The scaling factor of f is defined as

$$\lambda_n := \max\left\{\frac{|I^n|}{|I^{n-1}|}, \frac{|J^n|}{|J^{n-1}|}\right\}.$$

Let g_n denote the first return map to $I^{n-1}\cup J^{n-1}$. The restriction of g_n on I^n and J^n are unimodal, while its restriction on any other branches are monotone and onto I^{n-1} or J^{n-1}. The first return map g_n is called a central return if $g_n(c)\in I^n\cup J^n$ or $g_n(d)\in I^n\cup J^n$; otherwise, g_n is called non-central return. In the case when g_n is non-central, let I_1^n and J_1^n (possibly coincide) denote the first return domains intersecting $\{g_n(c), g_n(d)\}$.

Given $f\in\mathcal{B}$, f is called combinatorially symmetric if there exists an orientation-reversing homeomorphism $h: I\to I$ such that $h\circ f = f\circ h$.

Note that any combinatorially symmetric maps can be quasisymmetrically conjugated to an odd function. Finally, let $\mathcal{B}_*$ denote the collection of combinatorially symmetric bimodal maps from $\mathcal{B}$ with recurrent turning points satisfying $\omega(c)=\omega(d)$. Class $\mathcal{B}_*$ is nonempty since it contains infinitely renormalizable maps and Fibonacci bimodal maps.

1.2 Statement of results

Definition 1.1. A bimodal map $f\in\mathcal{B}_*$ is called (r,t)-Fibonacci if:

(1) $f(c), f(d)\notin I^0\cup J^0$ while $f^i(c), f^i(d)\in I^0\cup J^0$ for $i=2,3$;

(2) I_1^n and J_1^n are defined and disjoint for all $n\geqslant 1$;

(3) for each $n\geqslant 1$, $I^n\cup J^n\subset g_n(I^n\cup J^n)$;

(4) for each $n\geqslant 1$, $(\omega(c)\cup\omega(d))\cap(I^{n-1}\cup J^{n-1})\subset I^n\cup I_1^n\cup J^n\cup J_1^n$;

(5) for each $n\geqslant 1$, $g_n|(I^n\cup J^n) = g_{n-1}^r|(I^n\cup J^n)$ for some integer $r\geqslant 2$;

(6) for each $n\geqslant 1$, $g_n|(I_1^n\cup J_1^n) = g_{n-1}^t|(I_1^n\cup J_1^n)$ for some integer $t\geqslant 1$.

A pair of integers (r,t) is called admissible if there exists (r,t)-Fibonacci bimodal maps. Actually, not all (r,t) are admissible; for example, there does not exist $(4,2)$-Fibonacci bimodal maps. It was proven in Ref. [5] that under some conditions, (r,t) is admissible (see Section 2 for detail). The admissible pair (r,t) is just a simplification of Admissibility condition A for stationary combinatorics.

Lemma 1.1. Any pair of integers (r,t) is admissible if either r is even, t is odd with $t<r$, or r is odd, t is odd with $t<r$.

Suppose f is (r,t)-Fibonacci, then we can say f has 'uniformly bounded combinatorics'.

According to Ref. [11], the families of real cubic polynomials P_{ab}^+ and P_{ab}^- are 'full families'. Combined with the rigidity theorem developed in [7], we can obtain the following corollary.

Corollary 1.1. For any admissible pair (r,t), there exists exactly one (r,t)-Fibonacci bimodal map in P_{ab}^+, and one $(r,t)-$ Fibonacci bimodal map in P_{ab}^-.

Let $\mathcal{B}$ denote the class of (r,t)-Fibonacci bimodal maps in $P_{ab}^+\cup P_{ab}^-$.

Theorem 1.1. [5] Suppose $f\in\mathcal{B}$, then there exist constants $C=C(f)>0$ and $0<\lambda=\lambda(f)<1$ such that the scaling factor of f decreases at least exponentially: $\lambda_n(f)\leqslant C\lambda^n$ for all $n\geqslant 1$.

The main result of this paper is the following theorem.

Theorem 1.2. For any $f\in\mathcal{B}$, f admits an acip μ.

This paper is organized as follows. In Section 2, we study the combinatorial properties of (r,t)-Fibonacci bimodal maps. In Section 3, we construct an induced map G and study the metric properties of G. We show that G admits an acip and prove that for any $f\in\mathcal{B}$, f has an acip, which is the main result of this paper. In Section 4, we give a conclusion of this paper.

2 Combinatorics

In this section, we study the combinatorial properties of (r,t)-Fibonacci bimodal maps. For any (r,t)-Fibonacci bimodal map f, we can consider the map $g_1|(I^0\cup J^0)$ and rescale the interval back to $[0,1]$, which is the notion of renormalization. Repeating this procedure is called generalized renormalization. Comparing Ref. [2, Section 4], the analytic extension of a generalized renormalization g_n can be treated as a type Ⅱ special box mapping. Because the maps we consider are combinatorially symmetric, the positions of I_1^n, J_1^n, I^n and J^n have some constraints. Hence, we define 3 types of g_1, the types of g_n are similar.

- Type $\mathcal{A}$: if $g_1(I_1^1)=J^0$, $g_1(I_c^1)\subset J^0$ and $g_1(J_1^1)=I^0$,

DOI: 10.52396/JUSTC-2023-0036
JUSTC, **2024**, 54(8): 0802

$g_1(J_d^1) \subset I^0$;

- Type $\mathcal{B}$: if $g_1(I_1^1) = J^0$, $g_1(I_c^1) \subset I^0$ and $g_1(J_1^1) = I^0$, $g_1(J_d^1) \subset J^0$;
- Type $\mathcal{C}$: if $g_1(I_1^1) = I^0$, $g_1(I_c^1) \subset J^0$ and $g_1(J_1^1) = J^0$, $g_1(J_d^1) \subset I^0$.

Let us subdivide each type $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ in subtypes $\mathcal{A}^{ij}$ $\mathcal{B}^{ij}$ and $\mathcal{C}^{ij}$ with $i, j \in \{+,-\}$, where $i = +$ or $i = -$ if the non-central branches of f are orientation-preserving or orientation-reversing, respectively, and $j = +$ or $j = -$ if f is locally maximal or minimal at c, respectively. Finally, let $\mathcal{A}^+ = \mathcal{A}^{++} \cup \mathcal{A}^{+-}$ and define $\mathcal{A}^-, \mathcal{B}^+, \mathcal{B}^-, \mathcal{C}^+, \mathcal{C}^-, \mathcal{D}^+, \mathcal{D}^-$ analogously.

The proof of the following lemma can be found in Ref. [5].

Lemma 2.1. Suppose (r,t) is admissible, then there exists an (r,t)-Fibonacci bimodal map f, and its first return map sequence $\{g_n\}$ satisfies the following conditions:

(1) If r is even and t is odd, the first return map sequence $\{g_n\}$ exhibits the sequence $\mathcal{A}^+\mathcal{B}^-\mathcal{C}^-\mathcal{A}^-\mathcal{B}^+\mathcal{C}^+\mathcal{A}^+\cdots$ or $\mathcal{C}^-\mathcal{A}^-\mathcal{B}^+\mathcal{C}^+\mathcal{A}^+\mathcal{B}^-\mathcal{C}^-\cdots$.

(2) If r is odd and t is odd, the first return map sequence $\{g_n\}$ exhibits the sequence $\mathcal{A}^+\mathcal{A}^+\mathcal{A}^+\mathcal{A}^+\cdots$ or $\mathcal{C}^-\mathcal{A}^+\mathcal{A}^+\mathcal{A}^+\cdots$.

Let $S_1 = 2, \hat{S}_1 = 1$, for $n \geqslant 1$, define inductively

$$S_{n+1} = S_n + (r-1)\hat{S}_n \text{ and } \hat{S}_{n+1} = S_n + (t-1)\hat{S}_n.$$

Then the return times of critical points c and d to $I^{n-1} \cup J^{n-1}$ are equal to S_n, while the return times of $g_n(c)$ and $g_n(d)$ to $I^{n-1} \cup J^{n-1}$ are equal to $\hat{S}_n$.

Example 2.1. The Fibonacci bimodal maps studied in Ref. [15] are $(2,1)$-Fibonacci, where the first return time of critical points c and d to $I^{n-1} \cup J^{n-1}$ coincides with the Fibonacci sequence. In this case $\hat{S}_n = S_{n-1}$ and hence $S_{n+1} = S_n + S_{n-1}$. In particular, the first return map sequence $\{g_n\}$ exhibits the sequence

$$\mathcal{A}^{++}\mathcal{B}^{-+}\mathcal{C}^{--}\mathcal{A}^{-+}\mathcal{B}^{+-}\mathcal{C}^{+-}\mathcal{A}^{+-}\mathcal{B}^{-}\mathcal{C}^{-+}\mathcal{A}^{--}\mathcal{B}^{++}\mathcal{C}^{++}\mathcal{A}^{++}\cdots$$

or

$$\mathcal{C}^{-+}\mathcal{A}^{--}\mathcal{B}^{++}\mathcal{C}^{++}\mathcal{A}^{++}\mathcal{B}^{-+}\mathcal{C}^{--}\mathcal{A}^{-+}\mathcal{B}^{+-}\mathcal{C}^{+-}\mathcal{A}^{+-}\mathcal{B}^{--}\mathcal{C}^{-+}\cdots$$

depending on $f \in \mathcal{B}^+$ or $f \in \mathcal{B}^-$.

Example 2.2. For $r = 4$ and $t = 3$, we can find the $(4,3)$-Fibonacci bimodal maps, and the first return map sequence $\{g_n\}$ exhibits the sequence $\mathcal{A}^+\mathcal{B}^-\mathcal{C}^-\mathcal{A}^-\mathcal{B}^+\mathcal{C}^+\mathcal{A}^+\cdots$ or $\mathcal{C}^-\mathcal{A}^-\mathcal{B}^+\mathcal{C}^+\mathcal{A}^+\mathcal{B}^-\mathcal{C}^-\cdots$.

For $r = 5$ and $t = 3$, we can find the $(5,3)$-Fibonacci bimodal maps, and the first return map sequence $\{g_n\}$ exhibits the sequence $\mathcal{A}^+\mathcal{A}^+\mathcal{A}^+\mathcal{A}^+\cdots$ or $\mathcal{C}^-\mathcal{A}^+\mathcal{A}^+\mathcal{A}^+\cdots$.

3 Acip for f

Following the strategy in Ref. [1], the idea is to construct a Markov induced map G over f with the intervals I^n and J^n as a countable set of ranges: G is defined on a countable collection of intervals T_i, $G|T_i = f^{s_i}|T_i$ is a diffeomorphism and $G(T_i) = I^n$ or $G(T_i) = J^n$ for some n. We will construct a G-invariant measure $\nu \ll \text{Leb}$, and estimate $\nu(I^n)$ and $\nu(J^n)$, where $\nu \ll \text{Leb}$ means that ν is an absolutely continuous (with respect to the Lebesgue measure) invariant probability measure (acip for short). One result of this section is the following proposition, we will give a proof in Subsection 3.4.

Proposition 3.1. The induced map G admits an acip ν. Moreover, for arbitrarily small $\epsilon > 0$, there exists $C_0 = C_0(f,\epsilon)$ such that $\nu(I^n) + \nu(J^n) \leqslant C_0(\sqrt{\epsilon})^n$.

It is well-known that the nonexistence of a ‘wild attractor’ is based only on the decay of geometry in the unimodal case.

Corollary 3.1. Suppose $f \in \mathcal{B}$, then f has no Cantor attractor.

Proof. This follows from the observation that a Cantor attractor has zero Lebesgue measure, and, disregarding the critical points, is invariant by G. Hence G cannot carry an acip if a Cantor attractor is present.

3.1 Distortion

Given a bounded interval I and a constant $\tau > 0$, let τI denote the open interval that is concentric with I and has length $\tau|I|$. We say a bounded interval J is τ-well inside an interval T if $(1+2\tau)J \subset T$, i.e. both components of $T \setminus J$ have a length of at least $\tau|J|$.

The distortion of a C^1 function $h: J \to h(J)$ is defined as

$$\mathcal{D}(h) := \mathcal{D}(h;J) := \sup_{x,y \in J} \log \frac{|Dh(y)|}{|Dh(x)|}.$$

Let us say a diffeomorphism $h: J \to h(J)$ belongs to the distortion class $\mathcal{F}_p^C$ if it can be written as

$$Q \circ \varphi_q \circ Q \circ \varphi_{q-1} \circ \cdots \circ Q \circ \varphi_1$$

with $q \leqslant p$, where $Q(x) = x^2$ and $\mathcal{D}(\varphi_j) \leqslant C$ for all $1 \leqslant j \leqslant q$.

The following lemma can be found in Ref. [1].

Lemma 3.1. If $h: J \to I$ is a diffeomorphism in $\mathcal{F}_p^1$, and $A \subset J$ is a measurable set, then

$$\frac{1}{(2e)^p} \frac{\text{Leb}(h(A))}{|I|} \leqslant \frac{\text{Leb}(A)}{|J|} \leqslant e^p \left(\frac{\text{Leb}(h(A))}{|I|}\right)^{1/2^p}.$$

The Schwarzian derivative of a C^3 function $\phi: T \to \mathbb{R}$ is defined as

$$S\phi := \frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2.$$

It is well known that if $S\phi \leqslant 0$, then $S\phi^n \leqslant 0$ for all $n \geqslant 1$. Moreover, if f is a real polynomial with only real critical points, then $Sf < 0$; see, for example, Ref. [11, Chapter IV, Exercise 1.7].

We shall use the following version of the Koebe principle, which was proved in Ref. [11].

Proposition 3.2. Assume that $h: T \to h(T)$ is a C^3 diffeomorphism with $Sh < 0$. If J is a subinterval of T such that $h(J)$ is κ-well inside $h(T)$, then,

$$\mathcal{D}(h;J) \leqslant \log K_\kappa, \text{where } K_\kappa = \left(\frac{1+\kappa}{\kappa}\right)^2.$$

The interval J is κ'-well inside T, where $\kappa' = \kappa^2/(1+2\kappa)$.

Suppose $f \in \mathcal{B}$. For any $\epsilon > 0$ small, pick a large positive integer $n_0 = n_0(f)$ such that $\lambda_n \leqslant \epsilon$ for all $n \geqslant n_0$, and such that $f|I^{n_0}$ and $f|J^{n_0}$ can be written as $x \to \varphi \circ x^2$ with $\mathcal{D}(\varphi) \leqslant 1/4$. By Proposition 3.2, it follows that for each $n \geqslant n_0$, if J is a return domain to I^n or J^n, and $f^s|J$ is the return, then $f^s|J$ can be written as $x \to \varphi \circ x^2$ with $\mathcal{D}(\varphi) \leqslant 1/2$ provided ϵ is sufficiently small.

DOI: 10.52396/JUSTC-2023-0036
JUSTC, **2024**, 54(8): 0802

3.2 Construction of induced maps

Let G_0 be the first return map to $I^0 \cup J^0$. Then G_0 has finite number of branches, the two central branches (each contains one critical point) are the branches with return time 2, and each non-central branch maps diffeomorphically onto I^0 or J^0 with return time 1.

We will construct a sequence of maps $G_n : \bigcup_i L_i^{n+1} \cup \bigcup_i R_i^{n+1} \to I^0 \cup J^0$ inductively such that

① $\bigcup_i L_i^{n+1} \subset I^0$ and $\bigcup_i R_i^{n+1} \subset J^0$ are finite unions and for $n \geqslant 1$, $G_n = G_{n-1}$ outside $I^n \cup J^n$;

② the central branches $L_0^{n+1} = I^{n+1}$ and $R_0^{n+1} = J^{n+1}$, $G_n|I^{n+1}$ and $G_n|J^{n+1}$ are the first return maps to I^n or J^n;

③ for each $i \neq 0$, there exists $b_i \leqslant n$ such that $G_n : L_i^{n+1} \to I_{b_i}$ or $G_n : L_i^{n+1} \to J_{b_i}$ is a diffeomorphism; analogously for R_i^{n+1};

④ for each $i \neq 0$, $L_i^{n+1} \subset I^n$ and $\partial L_i^{n+1} \cap \partial I^n \neq 0$ imply $G_n(L_i^{n+1}) = I^0$ or J^0 (and the common boundary of L_i^{n+1} and I^n maps to the fixed point p); analogously for R_i^{n+1};

⑤ $G_n(x) = f^s(x)$ implies that $f(x), \dots, f^{s-1}(x) \notin I^n \cup J^n$.

By definition G_0 satisfies the above statements, so let us assume that by induction G_n exists with the above properties, and construct G_{n+1}. It suffices to construct G_{n+1} inside I^{n+1} since the construction inside J^{n+1} is similar.

Set $G_{n+1}(x) = G_n(x)$ for $x \notin I^{n+1} \cup J^{n+1}$. Let $k_n \in \mathbb{N}$ be minimal so that $G_n^{k_n}(c) \in I^{n+1} \cup J^{n+1}$. Since all the returns are non-central, $k_n \geqslant 2$. Define $K^0 = I^{n+1}$, $K^{k_n} = I^{n+2}$ and, for $0 \leqslant j \leqslant k_n - 1$, let K^j be the component of $\mathrm{dom}(G_n^{j+1})$ which contains c. Next define on $K^j \setminus K^{j+1}$,

$$G_{n+1} = \begin{cases} G_n^{j+1}(x), & \text{if } G_n^{j+1}(x) \in I^{n+1} \cup J^{n+1}; \\ G_n^{j+2}(x), & \text{otherwise.} \end{cases}$$

$G_{n+1}|I^{n+2} = G_n^{k_n}|I^{n+2}$ is the first return map to $I^{n+1} \cup J^{n+1}$.

Properties ① and ② hold by construction for G_{n+1}. Property ③ holds because if $G_n^{j+1}(x) \in I^{n+1}$ (resp. J^{n+1}) for some $x \in I^{n+1} \setminus I^{n+2}$ then $G_{n+1}(L_i^{n+1}) = I^{n+1}$ (resp, J^{n+1}) for the corresponding domain $L_i^{n+1} \ni x$; and if $G_n^{j+1}(x) \notin I^{n+1} \cup J^{n+1}$ then by the induction assumption $G_{n+1}(L_i^{n+1})$ is equal to some I^b or J^b with $b \leqslant n$, because then $G_{n+1}(x) = G_n^{j+2}(x)$. Property ④ holds immediately because ∂I^n and ∂J^n are mapped by G_n into $\partial I^0 \cup \partial J^0$. To show that property ⑤ holds, take $x \in K^j \setminus K^{j+1}$ and let $y = G_n^j(x)$. Note that $G_n^j|K^j$ is inside a component of $\mathrm{dom}(G_n)$ and that all iterates $f(K^j), \dots, G_n^j(K^j) \ni y$ are outside $I^{n+1} \cup J^{n+1}$. Since $G_n^{j+1}(x) = G_n(y)$, we obtain by induction that ⑤ holds for G_{n+1} using that it holds for G_n and y instead of x.

The induced map G is defined as follows: for each $n \geqslant 0$, each component of the domain J of G_n other than the central domains I^{n+1} and J^{n+1} becomes a component of the domain of G, and $G|J = G_n|J$.

Moreover, we compute by induction that if $x \in (I^n \setminus I^{n+1}) \cup (J^n \setminus J^{n+1})$, and $G(x) = f^s(x)$, then

$$s \leqslant t_0 \cdot (k_0 + 1) \dots (k_{n-2} + 1) \cdot (k_{n-1} + 1),$$

where $t_0 = \min\{i > 0, f^i(c), f^i(d) \in I^0 \cup J^0\}$. Note that for $f \in \mathcal{B}$, $t_0 = 2$.

3.3 The measure of the induced map

Note that the assumptions give that there exists a constant B with the following property: if J is any branch of G^k and $G^k(J) = I^n$, then

$$\frac{\mathrm{Leb}(\{x \in J; G^k(x) \in I^{n+m}\})}{|J|} \leqslant B \frac{|I^{n+m}|}{|I^n|}. \tag{1}$$

The same result holds when $G^k(J) = J^n$. This trivially includes the branch of G^0, that is the identity. Note that B is a distortion constant, and $B \leqslant 2$ for ϵ sufficiently small and $n \geqslant n_0$. Therefore, we can assume that $B\sqrt{\epsilon}/(1 - \sqrt{\epsilon}) < 1/3$. Moreover, $|I^n| \leqslant \epsilon^{n-k}|I^k|$ for all $n \geqslant k \geqslant n_0$.

We use the notation $\alpha(y) = n$ if $y \in (I^n \setminus I^{n+1}) \cup (J^n \setminus J^{n+1})$.

Lemma 3.2. If J is a branch of G^{k-1} such that $G^{k-1}(J) = I^{n+1}$, then

$$\mathrm{Leb}(x \in J; \alpha(G^k(x)) \geqslant n + 1) \leqslant \frac{1}{6}|J|, \tag{2}$$

provided $n \geqslant n_0$; analogously for $G^{k-1}(J) = J^{n+1}$.

Proof. Let $I^{n+1} = K^0 \supset K^1 \supset \dots \supset K^{k_n} = I^{n+2}$ be as in Subsec-

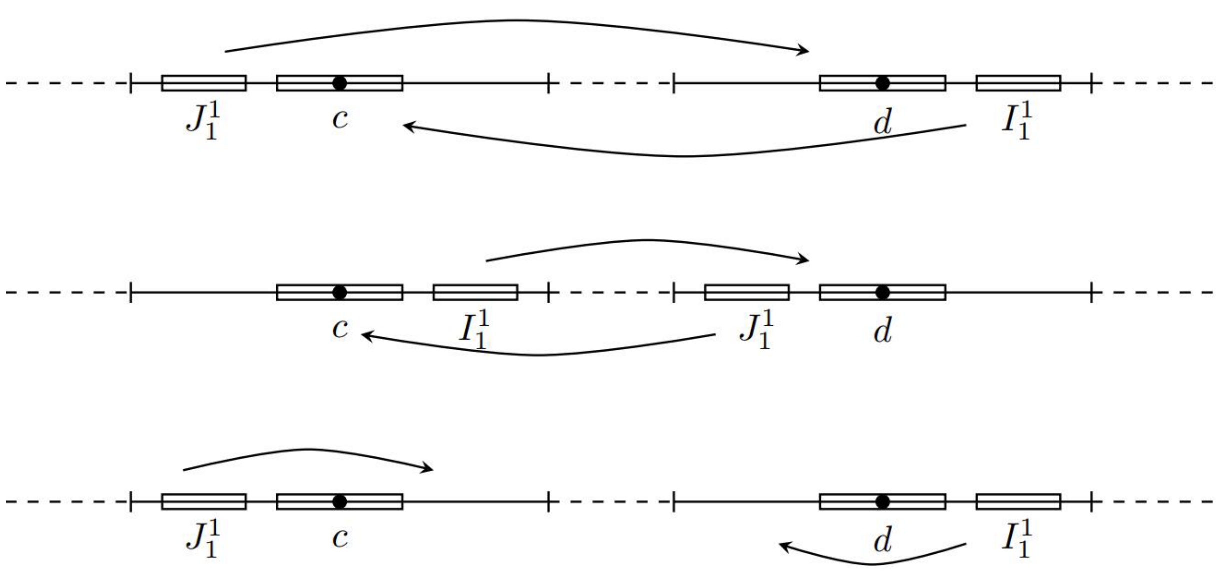


Fig. 1. Examples of types $\mathcal{A}$ $\mathcal{B}$ $\mathcal{C}$.

DOI: 10.52396/JUSTC-2023-0036
JUSTC, **2024**, 54(8): 0802

tion 3.2. For each $0 \leqslant i \leqslant k_n - 1$ with $K^i \neq K^{i+1}$, there can be at most two branches inside K^i, symmetric with respect to the critical point c, which map onto I^{n+1} or J^{n+1}. Let $P \subset K^i \setminus K^{i+1}$ be such a branch (if it exists). We claim that P is $\xi(\epsilon)$-well inside K^i. To see this, let $s \in \mathbb{N}$ be such that $G|P = f^s|P$. We may assume that $G(P) = I^{n+1}$. In particular, $G|P = G_n^{i+1}|K^i$. Then by our construction, f^{s-1} maps an interval $T \ni f(c)$ onto some interval I^j with $j \leqslant n$, and $f^{-1}(T) = K^i$. Since I^{n+1} is $\delta(\epsilon)$-well inside I^j, by Proposition 3.2, $f(P)$ is $\xi'(\epsilon)$-well inside T. Then the claim follows from the non-flatness of the critical point. Moreover, $\xi(\epsilon) \to \infty$ as $\epsilon \to 0$.

Let U_{n+1} be the union of those domains of G inside $I^{n+1} \setminus I^{n+2}$ that are mapped onto I^{n+1} or J^{n+1} by G. Then by the Koebe principle,

$$Leb(\{x \in J; G^{k-1}(x) \in U_{n+1}\}) \leqslant \frac{1}{10}|J|.$$

It remains to consider branches J' of $G^k|J$ for which $G^k(J') = I^{n'}$ or $J^{n'}$ with $n' \leqslant n$. Then, using the remark before this lemma, we have

$$\frac{Leb(\{x \in J'; G^k(x) \in I^{n+1}\})}{|J'|} \leqslant B\frac{|I^{n+1}|}{|I^{n'}|} \leqslant B\epsilon \leqslant \frac{1}{6}.$$

This finishes the proof.

3.4 Acip for G

In this subsection we prove the existence of an acip for the induced map G.

Proof of Proposition 3.1. We will use the result given by Ref. [13] claiming that G has an acip if and only if there exists some $\eta \in (0,1)$ and $\delta > 0$ such that for every measurable set A of measure $\mathrm{Leb}(A) < \delta$ holds $\mathrm{Leb}(G^{-k}(A)) \leqslant \eta$.

Write $y_{n,k} = \mathrm{Leb}(\{x \in I^0 \cup J^0; \alpha(G^k(x)) = n\})$. Take $C_0 > \dfrac{6B}{\min\{|I^{n_0}|, |J^{n_0}|\}} \cdot (\dfrac{1}{\epsilon})^{n_0}$. We prove by induction that $y_{n,k} \leqslant C_0 \cdot (\sqrt{\epsilon})^n$ for all $n,k \geqslant 0$. From the choice of C_0, for all $n \leqslant n_0$ and all k, we have

$$y_{n,k} \leqslant 1 \leqslant C_0 \cdot \epsilon^n \leqslant C_0 \cdot (\sqrt{\epsilon})^n.$$

For $k = 0$, it suffices to prove for $n \geqslant n_0 + 1$. Indeed,

$$y_{n,0} \leqslant (|I^{n_0}| + |J^{n_0}|) \cdot \epsilon^{n-n_0} \leqslant \epsilon^n \cdot \frac{1}{\epsilon^{n_0}} \leqslant C_0 \cdot \epsilon^n \leqslant C_0 \cdot (\sqrt{\epsilon})^n.$$

Now for the inductive step, assume that $y_{n,k-1} \leqslant C_0 \cdot (\sqrt{\epsilon})^n$ for all n. Pick $n \geqslant n_0 + 1$. Write $y_{n,n',k-1} = Leb(\{x \in I^0 \cup J^0; \alpha(G^{k-1}(x)) = n'$ and $\alpha(G^k(x)) = n\})$. Therefore we have

$$y_{n,k} = \sum_{n'<n_0} y_{n,n',k-1} + \sum_{n_0 \leqslant n' < n} y_{n,n',k-1} + y_{n,n,k-1} + \sum_{n'>n} y_{n,n',k-1}.$$

Term 1. Let J be any branch of G^{k-1} such that $G^{k-1}J = I^{n'}$. Then any branch J' of G^k with $J' \subset J$ are mapped onto some I^b with $b \leqslant n'$. By fomula (1),

$$\frac{Leb(\{x \in J'; \alpha(G^k(x)) = n\})}{|J'|} \leqslant B\frac{|I^n|}{|I^{n_0}|}.$$

Summing over all such J', we have

$$\sum_{n'<n_0} y_{n,n',k-1} \leqslant B\frac{|I^n|}{|I^{n_0}|}\sum |J'| \leqslant \frac{C_0}{6} \cdot \epsilon^n \leqslant \frac{C_0}{6} \cdot (\sqrt{\epsilon})^n.$$

Term 2. Since $n_0 \leqslant n' < n$, formula (1) and induction imply

$$y_{n,n'k-1} \leqslant B \cdot \epsilon^{n-n'} \cdot y_{n',k-1} \leqslant B \cdot \epsilon^{n-n'} \cdot C_0(\sqrt{\epsilon})^{n'} \leqslant C_0(\sqrt{\epsilon})^n \cdot B(\sqrt{\epsilon})^{n-n'}.$$

Therefore,

$$\sum_{n_0 \leqslant n' < n} y_{n,n',k-1} \leqslant C_0(\sqrt{\epsilon})^n \sum_{n'<n} B(\sqrt{\epsilon})^{n-n'} \leqslant \frac{C_0}{3}(\sqrt{\epsilon})^n.$$

Term 3. By formula (2) and induction,

$$y_{n,n,k-1} \leqslant \frac{1}{6} y_{n,k-1} \leqslant \frac{C_0}{6}(\sqrt{\epsilon})^n.$$

Term 4. Since $n' > n$,

$$\sum_{n'>n} y_{n,n',k-1} \leqslant \sum_{n'>n} y_{n',k-1} \leqslant \sum_{n'>n} C_0(\sqrt{\epsilon})^{n'} \leqslant C_0(\sqrt{\epsilon})^n \sum_{n'\geqslant n}(\sqrt{\epsilon})^{n'-n} \leqslant \frac{C_0}{3}(\sqrt{\epsilon})^n.$$

Therefore,

$$y_{n,k} \leqslant (\frac{1}{6} + \frac{1}{3} + \frac{1}{6} + \frac{1}{3})C_0 \cdot (\sqrt{\epsilon})^n = C_0 \cdot (\sqrt{\epsilon})^n.$$

If an acip ν exists, then it can be written as $\nu(A) = \lim_{n\to\infty} \dfrac{1}{n}\sum_{i=0}^{n-1} \mathrm{Leb}(G^{-i}A)$. Therefore,

$$\nu(I^n) + \nu(J^n) \leqslant C_0 \cdot (\sqrt{\epsilon})^n.$$

Now take $\eta \in (0,1)$. Fix n_1 such that $\sum_{n \geqslant n_1} y_{n,k} < \eta/2$ for all $k > 0$. We need to show that we can choose $\delta > 0$ so that if $A \subset I^0 \cup J^0$ is a set of measure $\mathrm{Leb}(A) < \delta$, then $\mathrm{Leb}(G^{-k}(A)) < \eta$ for all $k \geqslant 0$. By the choice of n_1, it suffices to show that $\mathrm{Leb}(G^{-k}(A)) < \eta/2$ for any $A \subset (I^0 \setminus I^{n_1}) \cup (J^0 \setminus J^{n_1})$ and all $k \geqslant 0$.

Without loss of generality, assume that $A \subset I^n \setminus I^{n+1}$ for some $n < n_1$. Consider any branch $G^k: J \to I^n$.

Case 1. If $\alpha(G^i(J)) \leqslant n_0$ for all $0 \leqslant i \leqslant k-1$. By Mañé's theorem[10], there exists $C_1 = C_1(f)$ such that $\mathcal{D}(G^k; J) \leqslant C_1$.

Case 2. If $G^k|J$ can be extended to $G^k: \hat{J} \to I^{n-1}$ with $n \geqslant n_0$, then by Proposition 3.2, $\mathcal{D}(G^k; J) \leqslant K_\epsilon$.

In either case we have

$$Leb(G^{-k}A \cap J) \leqslant C'\frac{|A|}{|I^n|}|J| \leqslant C'\frac{|A|}{|I^{n_1}|}|J|.$$

Case 3. There exist $m \geqslant n$ and $i \leqslant k$ that are maximal so that

$$m = \alpha(G^i J) > \alpha(G^{i+1} J) > \ldots > \alpha(G^{k-1} J) > n.$$

By Ref. [1, Proposition 2], any onto branch $G^k: J \to I^n$ can be written as $\psi \circ \varphi$ with

$$\mathcal{D}(\psi) \leqslant \log C_2 \text{ and } \varphi \in \mathcal{F}^1_{2(m-n+1)}.$$

Clearly, $i \geqslant k - m + n - 1$. For such a branch, by Lemma 3.1,

$$\mathrm{Leb}(G^{-k}A \cap J) \leqslant C_2 B(\frac{|A|}{|I^n|})^{1/2^{2(m-n+1)}}|J|.$$

DOI: 10.52396/JUSTC-2023-0036
JUSTC, **2024**, 54(8): 0802

For fixed m, the total measure of the set of points arriving at I^n in this fashion is bounded by $\sum_{i=k-m+n-1}^{k-1} y_{m,i} \leqslant (m-n+1)C_0 \cdot (\sqrt{\epsilon})^m$.

Summing over all branches J and all $m \geqslant n$, we have

$$\mathrm{Leb}(G^{-k}A) \leqslant C' \frac{|A|}{|I^{n_1}|} + \sum_{m \geqslant n} (m-n+1)C_0 \cdot (\sqrt{\epsilon})^m \cdot C_2 B(\frac{|A|}{|I^{n_1}|})^{1/2^{3(m-n+1)}}.$$

Thus $\mathrm{Leb}(G^{-k}A) \leqslant \eta/4n_1$ for any $k \geqslant 0$ and any $A \subset I^n \setminus I^{n+1}$, $n < n_1$, with $|A| < \delta$, provided δ is sufficiently small. It follows that if $A \subset (I^0 \setminus I^{n_1}) \cup (J^0 \setminus J^{n_1})$ with $\mathrm{Leb}(A) < \delta$, then $\mathrm{Leb}(G^{-k}A) < 2n_1\eta/(4n_1) = \eta/2$. This finishes the proof.

3.5 Acip for *f*

In this subsection we prove the main theorem.

Proof of Theorem 1.2. Let $f \in \mathcal{B}$. Fix $r \geqslant 2$. Let G be the Markov induced map of *f*. By Proposition 3.1, G admits an acip ν. Now it sufficed to show that we can pullback ν to obtain an acip for *f*.

Let $I^{n+1} = K^0 \supset K^1 \supset \cdots \supset K^{k_n} = I^{n+2}$ be as in Subsection 3.2. Since $t < r$, we can prove by induction that $I_1^{n+1} \subset K^t \setminus K^{t+1}$ and $G|I_1^{n+1}$ is exactly the first return onto I^n or J^n. The same holds for J_1^{n+1}. Now it is easy to check that $k_n = r$. Therefore, if $x \in (I^n \setminus I^{n+1}) \cup (J^n \setminus J^{n+1})$ and $G(x) = f^s(x)$, then

$$s \leqslant 2(r+1)^n.$$

Summing over all branches $J_j \subset (I^n \setminus I^{n+1}) \cup (J^n \setminus J^{n+1})$, let s_j denote the induced time on J_j. Then we find the partial sum

$$\sum_{J_j} s_j \nu(J_j) \leqslant 2(r+1)^n (\nu(I^n) + \nu(J^n)) \leqslant 2C_0(r+1)^n (\sqrt{\epsilon})^n$$

is exponentially small provided ϵ is sufficiently small.

Now this proposition follows by a standard pullback construction. Define μ by

$$\mu(A) = \sum_i \sum_{j=0}^{s_i-1} \nu(f^{-j}A \cap J_i).$$

As f is non-singular with respect to Lebesgue, μ is absolutely continuous, and the *f*-invariance of μ is a standard result. The finiteness of μ follows directly from the fact that $\sum s_i \nu(J_i) < \infty$.

4 Conclusions

This paper considers the metric properties of a class of non-renormalizable cubic polynomials with generalized Fibonacci combinatorics. The interval maps with Fibonacci combinatorics are the candidate that has wild Cantor attractor ([8]) and have been studied widely for unimodal maps ([4][6]). The situation for multimodal is much more complicated. In this paper, we consider a special class of bimodal maps. The combinatorics of such a class were defined in terms of generalized renormalization based on the twin principal nest. The main result states that maps in this class with bounded combinatorics will have an absolutely continuous invariant measure. Moreover, we prove that any maps from such a class admits no Cantor attractor.

Conflict of interest

The author declares that she has no conflict of interest.

Biography

Wenxiu Ma is currently a graduate student of University of Science and Technology of China. Her research mainly focuses on one dimensional dynamics and complex dynamics.

References

[1] Bruin H, Shen W, van Strien S. Invariant measure exists without a growth condition. *Communications in Mathematical Physics,* **2003**, *241*: 287–306.

[2] Graczyk J, Swiatiek G. The Real Fatou Conjecture. Princeton, USA: Princeton University Press, **1998**.

[3] Jakobson M, Swiatiek G. Metric properties of non-renormalizable *S*-unimodal maps. Part I. Induced expansion and invariant measures. *Ergodic Theory and Dynamical Systems,* **1994**, *14*: 721–755.

[4] Ji H, Li S. On the combinatorics of Fibonacci-like non-renormalizable maps. *Commun. Math. Stat.,* **2020**, *8*: 473–496.

[5] Ji H, Ma W. Decay of geometry for a class of cubic polynomials. arXiv: 2304.10689, **2023**.

[6] Keller G, Nowicki T. Fibonacci maps re(al)-visited. *Ergodic Theory and Dynamical Systems,* **1995**, *15*: 99–120.

[7] Kozlovski O, Shen W, van Strien S. Rigidity for real polynomials. *Annals of Mathematics,* **2007**, *165* (3): 749–841.

[8] Lyubich M, Milnor J. The Fibonacci unimodal map. *J. Amer. Math. Soc.,* **1993**, *6* (2): 425–457.

[9] Lyubich M. Combinatorics, geometry and attractors of quasi-quadratic maps. *Annals of Mathematics,* **1994**, *140*: 345–404.

[10] Mañé R. Hyperbolicity, sinks and measure in one-dimensional dynamics. *Communications in Mathematical Physics,* **1985**, *100*: 495–524.

[11] de Melo W, van Strien S. One-Dimensional Dynamics. Berlin: Springer-Verlag, **1993**.

[12] Shen W. Decay of geometry for unimodal maps: An elementary proof. *Annals of Mathematics,* **2006**, *163*: 383–404.

[13] Straube E. On the existence of invariant absolutely continuous measures. *Communications in Mathematical Physics,* **1981**, *81*: 27–30.

[14] Swiatek G, Vargas E. Decay of geometry in the cubic family. *Ergodic Theory and Dynamical Systems,* **1998**, *18*: 1311–1329.

[15] Vargas E. Fibonacci bimodal maps. *Discrete and Continuous Dynamical Systems,* **2008**, *22* (3): 807–815.

DOI: 10.52396/JUSTC-2023-0036
JUSTC, **2024**, 54(8): 0802