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Curvature estimate of the Yang-Mills-Higgs flow on Kähler manifolds

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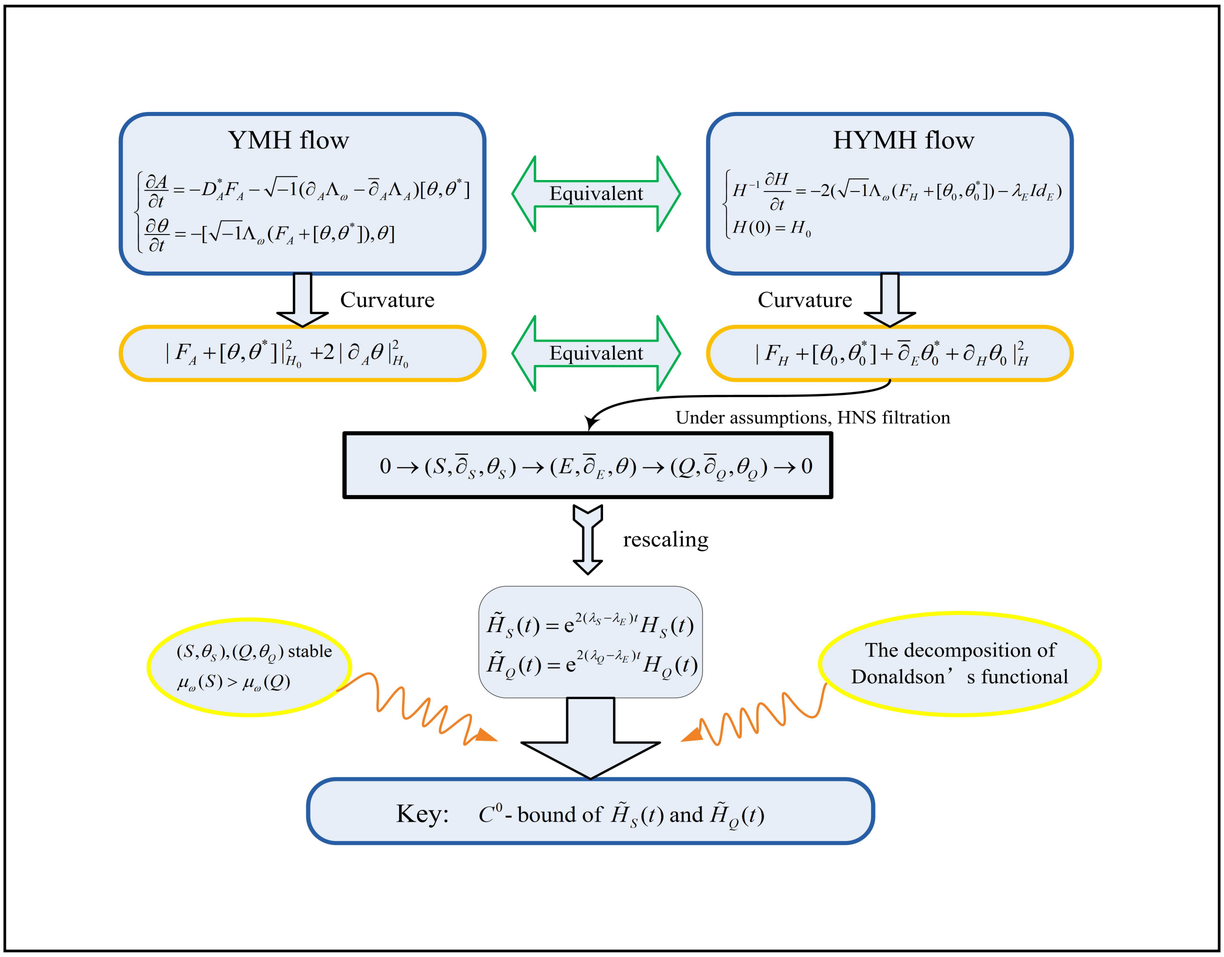
Graphical abstract



Using the decomposition of Donaldson's functional and the properties of Harder-Nasimhan-Seshadri filtration, we get the C^0-bound of the rescaled metrics

Public summary

- By vast calculation and analysis, we drive many evolution equations in Higgs bundles.
- Generalized the decomposition of the Donaldson's functional in Higgs bundle case.
- It provides an analytic method for studying the algebraic singular set and analytic singular set in the future work.

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Abstract: The curvature estimate of the Yang-Mills-Higgs flow on Higgs bundles over compact Kähler manifolds is studied. Under the assumptions that the Higgs bundle is non-semistable and the Harder-Narasimhan-Seshadri filtration has no singularities with length one, it is proved that the curvature of the evolved Hermitian metric is uniformly bounded.

Keywords: Higgs bundle; Harder-Narasimhan-Seshadri filtration; Yang-Mills-Higgs flow; curvature estimate

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1 Introduction

Let $(\mathcal{E},\theta)$ be a Higgs sheaf over a compact Kähler manifold (M,ω). Higgs bundles and Higgs sheaves, which were studied by Hitchin[1] and Simpson[2,3], play an important role in many different areas including gauge theory, Kähler and hyperkähler geometry, group representations, and nonabelian Hodge theory. A Higgs sheaf on (M,ω) is a pair $(\mathcal{E},\theta)$, where $\mathcal{E}$ is a coherent sheaf on M, and the Higgs field $\theta\in\Omega^{1,0}(\mathrm{End}(\mathcal{E}))$ is a holomorphic section such that $\theta\wedge\theta=0$. A torsion-free Higgs sheaf $(\mathcal{E},\theta)$ is ω-stable (resp. ω-semistable) if for every θ-invariant proper coherent sub-sheaf $\mathcal{F}\hookrightarrow\mathcal{E}$ such that

$$\mu_\omega(\mathcal{F})=\frac{\deg_\omega(\mathcal{F})}{\mathrm{rank}(\mathcal{F})}<(\leqslant)\mu_\omega(\mathcal{E})=\frac{\deg_\omega(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}\tag{1}$$

where $\mu_\omega(\mathcal{F})$ is called the ω-slope of $\mathcal{F}$, and ω-degree of $\mathcal{F}$ is defined as follows:

$$\deg_\omega(\mathcal{F})=\int_M c_1(\mathcal{F})\wedge\frac{\omega^{n-1}}{(n-1)!},$$

where $c_1(\mathcal{F})$ is the first Chern class of $\mathcal{F}$.

Let $\Sigma_{\mathcal{E}}$ be a set of singularities, where $(\mathcal{E},\theta)$ is not locally free. If $(\mathcal{E},\theta)$ is locally free on the whole M, i.e., $\Sigma_{\mathcal{E}}=\emptyset$, there is a Higgs bundle $(E,\bar\partial_E,\theta)$ on M such that the Higgs sheaf $(\mathcal{E},\theta)$ is generated by the local holomorphic sections of $(E,\bar\partial_E,\theta)$. A locally free Higgs sheaf $(\mathcal{E},\theta)$ can be considered as a Higgs bundle, i.e., $(\mathcal{E},\theta)=(E,\bar\partial_E,\theta)$.

Given an unstable torsion-free coherent Higgs sheaf $(\mathcal{E},\theta)$, one can associate a filtration by θ-invariant coherent sub-sheaves as follows:

$$0=\mathcal{E}_0\subset\mathcal{E}_1\subset\cdots\subset\mathcal{E}_k=\mathcal{E}\tag{2}$$

such that the quotient $Q_i=\mathcal{E}_i/\mathcal{E}_{i-1}$ is torsion-free, ω-semistable, and $\mu_\omega(Q_i)>\mu_\omega(Q_{i+1})$, which is called the Harder-Narasimhan filtration of $(\mathcal{E},\theta)$. The associated graded object $\mathrm{Gr}^{\mathrm{hn}}(\mathcal{E},\theta)=\oplus_{i=1}^k Q_i$ is uniquely determined by the isomorphism class of $(\mathcal{E},\theta)$ and the Kähler class $[\omega]$. Moreover, for every quotient Q_i, a further filtration by sub-sheaves exists,

$$0=\mathcal{E}_{i,0}\subset\mathcal{E}_{i,1}\subset\cdots\subset\mathcal{E}_{i,k_i}=Q_i\tag{3}$$

such that the quotients $Q_{i,j}=\mathcal{E}_{i,j}/\mathcal{E}_{i,j-1}$ are torsion-free and ω-stable, and $\mu_\omega(Q_{i,j})=\mu_\omega(Q_i)$ for each j. The double filtration $(\mathcal{E}_{i,j})$ is called the Harder-Narasimhan-Seshadri (HNS) filtration of the Higgs sheaf $(\mathcal{E},\theta)$. The associated graded object

$$\mathrm{Gr}^{\mathrm{hns}}(\mathcal{E},\theta)=\bigoplus_{i=1}^k\bigoplus_{j=1}^{k_i}Q_{i,j}\tag{4}$$

is uniquely determined by the isomorphism class of $(\mathcal{E},\theta)$ and Kähler class $[\omega]$. The number $\sum_{i=1}^k k_i-1$ is the length of the HNS filtration.

Given a Hermitian metric H on the Higgs bundle $(E,\bar\partial_E,\theta)$, we consider the Hitchin-Simpson connection:

$$\bar\partial_\theta:=\bar\partial_E+\theta,\ D':=\partial_H+\theta^{*H},\ D_{H,\theta}=\bar\partial+\partial_H\tag{5}$$

where D_H is the Chern connection with respect to the Hermitian metric H and θ^{*H} is the adjoint of θ with respect to H. The curvature of the Hitchin-Simpson connection is expressed as follows:

$$F_{H,\theta}=F_H+[\theta,\theta^{*H}]+\partial_H\theta+\bar\partial_E\theta^{*H}\tag{6}$$

where F_H is the curvature of the Chern connection, denoted by D_H. A Hermitian metric on the Higgs bundle $(E,\bar\partial_E,\theta)$ is said to be ω-Hermitian-Einstein if it satisfies the following Einstein condition on M, i.e.,

$$\sqrt{-1}\Lambda_\omega(F_H+[\theta,\theta^*])=\lambda_{E,\omega}\mathrm{Id}_E\tag{7}$$

where $\lambda_{E,\omega}=\frac{2\pi}{\mathrm{Vol}(M,\omega)}\mu_\omega(E)$ and Λ_ω denotes the contraction with the Kähler metric ω. Hitchin[1] and Simpson[2] proved that a Higgs bundle admits a Hermitian-Einstein metric if and only if it is Higgs poly-stable. Many interesting and important generalizations and extensions can also be found in the literature (see Refs. [1, 4–12]).

Let H_0 be a Hermitian metric on the complex vector bundle E, $\mathcal{A}_{H_0}$ be the space of connections of E compatible with the metric H_0, and $\mathcal{A}^{1,1}_{H_0}$ be the space of the unitary integrable connection of E. A pair $(A,\theta)\in\mathcal{A}^{1,1}_{H_0}\times\Omega^{1,0}(\mathrm{End}(E))$ is a Higgs pair if $\bar\partial_A\theta=0$ and $\theta\wedge\theta=0$. Let $\mathcal{B}_{(E,H_0)}$ denote the space of all Higgs pairs on the Hermitian vector bundle (E,H_0). We consider the following Yang-Mills-Higgs flow on the Hermitian vector bundle (E,H_0) with initial data $(A_0,\theta_0)\in\mathcal{B}_{(E,H_0)}$:

$$\left.\begin{aligned}\frac{\partial A}{\partial t}&=-D_A^*F_A-\sqrt{-1}(\partial_A\Lambda_\omega-\bar\partial_A\Lambda_\omega)[\theta,\theta^*],\\ \frac{\partial\theta}{\partial t}&=-[\sqrt{-1}\Lambda_\omega(F_A+[\theta,\theta^*]),\theta]\end{aligned}\right\}\tag{8}$$

The Yang-Mills flow was first introduced by Atiyah-Bott in Ref. [13]. Simpson[2] induces the following Hermitian-Yang-Mills-Higgs (HYMH) flow for Hermitian metrics $H(t)$ on the Higgs bundle (E,A_0,θ_0) with initial metric H_0:

$$\left.\begin{aligned}&H^{-1}\frac{\partial H}{\partial t}=-2\left(\sqrt{-1}\Lambda_\omega(F_H+[\theta_0,\theta_0^{*H}])-\lambda_E\cdot\mathrm{Id}_E\right),\\ &H(0)=H_0\end{aligned}\right\}\tag{9}$$

Simpson[2] proved the long-time existence of the Hermitian-Yang-Mills-Higgs flow and demonstrated convergence under the condition that the Higgs bundle is stable. In Ref. [14], Li and Zhang showed that by choosing complex gauge transformations $\sigma(t)$ that satisfy $\sigma(t)^*\sigma(t)=H_0^{-1}H(t)$, $(A(t),\theta(t))=(\sigma(t)(A_0),\sigma(t)(\theta_0))$ is the unique long-time solution of the Yang-Mills-Higgs flow Eq. (8).

According to Uhlenbeck's compactness[15,16], for any sequence $A(t_i)$ along the flow, there is a subsequent that weakly converges to a Yang-Mills connection A_∞ in the gauge transformation sense outside a closed subset Σ_{an} of Hausdorff complex co-dimension of at least 2. We call Σ_{an} the bubbling set or the analytic singular set. In contrast, we denote Σ_{alg} as the singular set of the associated graded object $\mathrm{Gr}^{\mathrm{hns}}_\omega(E,\partial_{A_0},\theta_0)$, i.e., $\mathrm{Gr}^{\mathrm{hns}}_\omega(E,\partial_{A_0},\theta_0)$ is locally free away from Σ_{alg}, which is a complex analytic sub-variety of complex co-dimension of at least 2 that we call algebraic singular set. According to the results in Refs. [17, 18], it is not difficult to obtain that $\Sigma_{\mathrm{alg}}\subset\Sigma_{\mathrm{an}}$. It is an interesting problem to demonstrate that $\Sigma_{\mathrm{an}}\subset\Sigma_{\mathrm{alg}}$. This problem was solved by Daskalopoulos and Wentworth in Ref. [19] for Kähler surfaces, and by Sibley and Wentworth in Ref. [20] for Kähler manifolds. In this paper, we derive the curvature estimate in the case where $(E,\bar\partial_{A_0},\theta_0)$ is non-semistable and the Harder-Narasimhan-Seshadri filtration has no singularities with length one, which generalizes Li, Zhang, and Zhang's result[21] in the Higgs bundle case. In fact, we obtain the following theorem:

Theorem 1.1. Let (E,H_0) be a Hermitian vector bundle on a compact Kähler manifold (M,ω), and $(A(t),\theta(t))$ be the solution of the Yang-Mills-Higgs flow (8) with an initial Higgs pair $(A_0,\theta_0)\in\mathcal{A}^{1,1}_{H_0}\times\Omega^{1,0}(\mathrm{End}(E))$. If the Higgs bundle $(\mathcal{E},\theta_0)=(E,A_0,\theta_0)$ is non-semistable and the HNS filtration has no singularities with length one, then there exists a uniform constant C such that

$$\sup_{(x,t)\in M\times[0,+\infty)}\left(|F_A+[\theta,\theta^*]|^2_{H_0}+2|\partial_A\theta|_{H_0}\right)(x,t)\le C\tag{10}$$

If the algebraic singular set $\Sigma_{\mathrm{alg}}\neq\emptyset$, we hypothesize that the theorem holds away from the algebraic singular set. The proof is complicated, and we mainly adapt some Li, Zhang, and Zhang's techniques to our cases of interest. However, the proof is even more complicated in the Higgs bundle case. We next provide an overview of the proposed proof. Let $H(t)$ be the long-time solution of the HYMH flow (9). It is well known that

$$|F_{H(t)}+[\theta_0,\theta_0^{*H}]+\bar\partial_E\theta_0^{*H}+\partial_H\theta_0|^2_H=|F_A+[\theta,\theta^*]|^2_{H_0}+2|\partial_A\theta|^2_{H_0}\tag{11}$$

Therefore, we only need to estimate the curvature tensor $F_{H(t),\theta_0}$ of the Hitchin-Simpson connection $D_{H(t),\theta_0}$ with respect to the evolved Hermitian metric $H(t)$. For simplicity, we denote θ_0 as θ as a fixed Higgs field. According to the assumption of Theorem 1.1, there exists an exact sequence

$$0\to(S,\bar\partial_S,\theta_S)\to(E,\bar\partial_E,\theta)\to(Q,\bar\partial_Q,\theta_Q)\to0\tag{12}$$

such that $(S,\bar\partial_S,\theta_S)$ and $(Q,\bar\partial_Q,\theta_Q)$ become torsion-free, ω-stable Higgs bundles, and

$$\mu_\omega(S)>\mu_\omega(E)>\mu_\omega(Q)\tag{13}$$

Let $H_S(t)$ and $H_Q(t)$ be the Hermitian metrics on the Higgs subbundle (S,θ_S) and quotient bundle (Q,θ_Q), respectively, induced by the evolved metric $H(t)$ on (E,θ), $\gamma(t)$ as the second fundamental form, and let $\beta(t)$ be the $(1,0)$ form induced by the Higgs field θ and $H(t)$.We derive the evolution equations for $H_S(t)$, $H_Q(t)$, $\gamma(t)$, and $\beta(t)$ (see Eqs. (20), (21), (23), and (24)). In Ref. [2], Simpson generalized the Hermitian-Yang-Mills flow to the Higgs bundle case. Under the assumption of stability, using the results of Uhlenbeck and Yau[22], Simpson obtained a uniform C^0-estimate on $H(t)$. This implies uniform higher-order estimates including the uniform curvature estimate. When $(E,\bar\partial_E,\theta)$ is unstable, the C^0-norm of the evolved metric $H(t)$ may be unbounded. In our case, we first obtain a uniform C^0-bound on the rescaled metrics, i.e., $\tilde H_S(t)=\mathrm{e}^{2(\lambda_S-\lambda_E)t}H_S(t)$ and $\tilde H_Q(t)=\mathrm{e}^{2(\lambda_Q-\lambda_E)t}H_Q(t)$. This is crucial in the proof of the proposed theorem, where we use the stabilities of (S,θ_S) and (Q,θ_Q), and the property $\mu_\omega(S)>\mu_\omega(Q)$. Then, we prove that the norms of $|T_S(t)|_{H_S(t)}$, $|T_Q(t)|_{H_Q(t)}$, $|\gamma(t)|_{H(t)}$, and $|\beta(t)|_{H(t)}$ are uniformly bounded using the uniform C^0-estimate. By choosing suitable test functions and using the maximum principle, we obtain a uniform estimate of $|F_{H(t)}+[\theta,\theta^*]+\bar\partial_E\theta^*+\partial_H\theta|_{H(t)}$.

The remainder of this paper is organized as follows. In Section 2, we derive the evolution equations for the induced metrics $H_S(t)$ and $H_Q(t)$, the second fundamental forms $\gamma(t)$, and the induced $(1,0)$ forms $\beta(t)$. We also obtain the decomposition of Donaldson's functional, which is used in Section 3, where we derive a uniform C^0-bound for the rescaled metrics. In Section 4, we obtain the uniform C^1-estimate and complete the proof of Theorem 1.1.

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2 Evolution of the second fundamental form and the Higgs field

Let (M,ω) be a Kähler manifold that may be non-compact, and let $(E,\bar\partial_E,\theta)$ be a Higgs bundle over M. That is, $(E,\bar\partial_E)$ is a holomorphic vector bundle together with a Higgs field $\theta\in\Omega^{1,0}_M(\mathrm{End}(E))$ satisfying

$$\bar\partial_E\theta=0 \quad \text{and} \quad \theta\wedge\theta=0.$$

Let S be a θ-invariant holomorphic sub-bundle of $(E,\bar\partial_E,\theta)$. We assume that an exact sequence of Higgs bundles exists

$$0\to(S,\bar\partial_S,\theta_S)\to(E,\bar\partial_E,\theta)\to(Q,\bar\partial_Q,\theta_Q)\to 0 \tag{14}$$

where θ_S and θ_Q are the Higgs fields on the θ-invariant holomorphic sub-bundle S and the quotient bundle $Q=E/S$ induced by θ, $\bar\partial_S$, respectively, and $\bar\partial_Q$ are holomorphic structures on the holomorphic sub-bundle S and the quotient bundle Q induced by $\bar\partial_E$.

Given a Hermitian metric H on E, we obtain the following bundle isomorphism:

$$f_H:S\oplus Q\to E,\quad (X,[Y])\mapsto i(X)+(\mathrm{Id}_E-\pi_H)(Y) \tag{15}$$

where $X\in S$, $Y\in E$, $i:S\hookrightarrow E$ is the inclusion map and $\pi_H:E\to S$ is the orthogonal projection into S with respect to metric H. Then, the pull-back metric, holomorphic structure, and Higgs field on $S\oplus Q$ respectively are

$$f_H^*(H)=\begin{pmatrix}H_S & 0\\ 0 & H_Q\end{pmatrix},\quad f_H^*(\bar\partial_E)=\begin{pmatrix}\bar\partial_S & \gamma\\ 0 & \bar\partial_Q\end{pmatrix},\quad f_H^*(\theta)=\begin{pmatrix}\theta_S & \beta\\ 0 & \theta_Q\end{pmatrix} \tag{16}$$

where $\gamma(t)\in\Omega^{0,1}_M(S\otimes Q^*)$ and $\beta\in\Omega^{1,0}_M(S\otimes Q^*)$. The corresponding Gauss-Codazzi equation for the Hitchin-Simpson connection is

$$\begin{aligned}
f_H^*(F_{H,\theta})=&f_H^*(F_H+[\theta,\theta^{*H}]+\partial_H\theta+\bar\partial_E\theta^{*H})=\\
&\begin{pmatrix}F_{H_S}-\gamma\wedge\gamma^{*H} & \partial_{S\otimes Q^*}\gamma\\ -\bar\partial_{S^*\otimes Q}\gamma^{*H} & F_{H_Q}-\gamma^{*H}\wedge\gamma\end{pmatrix}+\\
&\begin{pmatrix}\partial_{H_S}\theta_S+\bar\partial_S\theta_S^{*H_S}+\gamma\wedge\beta^{*H}-\beta\wedge\gamma^{*H} & \partial_{S\otimes Q^*}\beta+\theta_S^{*H_S}\wedge\gamma+\gamma\wedge\theta_Q^{*H_Q}\\ \bar\partial_{S^*\otimes Q}\beta^{*H}-\gamma^{*H}\wedge\theta_S-\theta_Q\wedge\gamma^{*H} & \partial_{H_Q}\theta_Q+\bar\partial_Q\theta_Q^{*H}-\gamma^{*H}\wedge\beta+\beta^{*H}\wedge\gamma\end{pmatrix}+\\
&\begin{pmatrix}[\theta_S,\theta_S^{*H_S}]+\beta\wedge\beta^{*H} & \theta_S^{*H_S}\wedge\beta+\beta\wedge\theta_Q^{*H_Q}\\ \beta^{*H}\wedge\theta_S+\theta_Q\wedge\beta^{*H} & [\theta_Q,\theta_Q^{*H_Q}]+\beta^{*H}\wedge\beta\end{pmatrix}
\end{aligned} \tag{17}$$

Let $H(t)$ be the solution of the HYMH flow (9) on the Higgs bundle $(E,\bar\partial_E,\theta)$ with initial metric H_0. Now, we split the HYMH flow into the sub-bundle S and quotient bundle Q. Recall from Ref. [21] that

$$f_H^{-1}\frac{\partial f_{H(t)}}{\partial t}=\begin{pmatrix}0 & \Psi(t)\\ 0 & 0\end{pmatrix} \tag{18}$$

and

$$f_H^*\left(H^{-1}\frac{\partial H}{\partial t}\right)=\begin{pmatrix}H_S^{-1}\dfrac{\partial H_S}{\partial t} & 0\\ 0 & H_Q^{-1}\dfrac{\partial H_Q}{\partial t}\end{pmatrix}-f_H^{-1}\frac{\partial f_H}{\partial t}-\left(f_H^{-1}\frac{\partial f_H}{\partial t}\right)^{*f_H^*(H)} \tag{19}$$

Using the flow (9) and Gauss-Codazzi (17) equations, we have

$$H_S^{-1}\frac{\partial H_S}{\partial t}=-2\left(\sqrt{-1}\Lambda_\omega(F_{H_S}+[\theta_S,\theta_S^{*H_S}]-\gamma\wedge\gamma^{*H}+\beta\wedge\beta^{*H})-\lambda_E\cdot\mathrm{Id}_S\right) \tag{20}$$

$$H_Q^{-1}\frac{\partial H_Q}{\partial t}=-2\left(\sqrt{-1}\Lambda_\omega(F_{H_Q}+[\theta_Q,\theta_Q^{*H_Q}]-\gamma^{*H}\wedge\gamma+\beta^{*H}\wedge\beta)-\lambda_E\cdot\mathrm{Id}_Q\right) \tag{21}$$

$$f_H^{-1}\frac{\partial f_H}{\partial t}=\begin{pmatrix}0 & 2\sqrt{-1}\Lambda_\omega(\partial_{S\otimes Q^*}\gamma+\theta_S^{*H_S}\wedge\beta+\beta\wedge\theta_Q^{*H_Q})\\ 0 & 0\end{pmatrix} \tag{22}$$

Now, we consider the evolution of the second fundamental forms $\gamma(t)$ and $\beta(t)$:

Lemma 2.1. Let $H(t)$ be the solution of the HYMH flow (9) with initial metric H_0 and $\gamma(t)$ be the second fundamental form, and let $\beta(t)$ be induced by θ and $H(t)$. Then we have

$$\frac{\partial\gamma(t)}{\partial t}=2\bar\partial_{S\otimes Q^*}\left(\sqrt{-1}\Lambda_\omega(\partial_{S\otimes Q^*}\gamma+\theta_S^{*H_S}\wedge\beta+\beta\wedge\theta_Q^{*H_Q})\right) \tag{23}$$

and

$$\begin{aligned}\frac{\partial\beta(t)}{\partial t}=&2\theta_S\circ\left(\sqrt{-1}\Lambda_\omega(\partial_{S\otimes Q^*}\gamma+\theta_S^{*H_S}\wedge\beta+\beta\wedge\theta_Q^{*H_Q})\right)-\\ &2\left(\sqrt{-1}\Lambda_\omega(\partial_{S\otimes Q^*}\gamma+\theta_S^{*H_S}\wedge\beta+\beta\wedge\theta_Q^{*H_Q})\right)\circ\theta_Q\end{aligned} \tag{24}$$

Proof. For simplicity, we let $\Psi(t)=2\sqrt{-1}\Lambda_\omega(\partial_{S\otimes Q^*}\gamma+\theta_S^{*H_S}\wedge\beta+\beta\wedge\theta_Q^{*H_Q})$ and denote

$$\bar\partial f_H=\bar\partial_E\circ f_H-f_H\circ\bar\partial_{S\oplus Q} \tag{25}$$

and then

$$f_H^{-1}\bar\partial f_H=f_H^*(\bar\partial_E)-\bar\partial_{S\oplus Q}=\begin{pmatrix}0 & \gamma\\ 0 & 0\end{pmatrix} \tag{26}$$

For the first evolution equation of γ, taking the derivative of the above equation with respect to t, we have that

$$\begin{aligned}\begin{pmatrix}0 & \dfrac{\partial\gamma(t)}{\partial t}\\ 0 & 0\end{pmatrix}=&\frac{\partial}{\partial t}(f_H^{-1}\bar\partial f_H)=\\ &-f_H^{-1}\frac{\partial f_H}{\partial t}f_H^{-1}\bar\partial f_H+f_H^{-1}\circ\bar\partial_E\circ f_H\circ f_H^{-1}\frac{\partial f_H}{\partial t}+\\ &f_H^{-1}\frac{\partial f_H}{\partial t}\circ\bar\partial_{S\oplus Q}=\begin{pmatrix}0 & \bar\partial_{S\otimes Q^*}\Psi(t)\\ 0 & 0\end{pmatrix}.\end{aligned}$$

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Thus, we obtain the first evolution Eq. (23).

Concerning the second evolution equation of β, taking the derivative of $f_H^*(\theta)$ with respect to t, we have that

$$\begin{pmatrix} 0 & \frac{\partial\beta(t)}{\partial t} \\ 0 & 0 \end{pmatrix} = \frac{\partial}{\partial t}(f_H^*(\theta)) = \frac{\partial}{\partial t}(f_H^{-1}\circ\theta\circ f_H) =$$

$$-f_H^{-1}\frac{\partial f_H}{\partial t}f_H^{-1}\circ\theta\circ f_H + f_H^{-1}\circ\theta\circ f_H\circ f_H^{-1}\frac{\partial f_H}{\partial t} =$$

$$\begin{pmatrix} 0 & \theta_S\circ\Psi(t)-\Psi(t)\circ\theta_Q \\ 0 & 0 \end{pmatrix}.$$

As a result, we obtain that $\frac{\partial\beta(t)}{\partial t} = \theta_S\circ\Psi(t)-\Psi(t)\circ\theta_Q$.

Lemma 2.2. Let $f_{H(t)}$ be the bundle isomorphism defined in (15). Then, we have that

$$f_{H_0}^{-1}f_{H(t)} = \begin{pmatrix} \mathrm{Id}_S & G(t) \\ 0 & \mathrm{Id}_Q \end{pmatrix} \tag{27}$$

where $G(t)\in\Gamma(S\otimes Q^*)$ and $G(0)=0$. Furthermore,

$$\frac{\partial G}{\partial t} = 2\sqrt{-1}\Lambda_\omega(\partial_{S\otimes Q^*}\gamma+\theta_S^{*H_S}\wedge\beta+\beta\wedge\theta_Q^{*H_Q}) \tag{28}$$

and

$$\bar\partial_{S\otimes Q^*}G = \gamma-\gamma_0 \tag{29}$$

Proof. Similar to the proof in Ref. [21], we have Eq. (27). For the sake of simplicity, we denote $\Psi(t) = 2\sqrt{-1}\Lambda_\omega(\partial_{S\otimes Q^*}\gamma+\theta_S^{*H_S}\wedge\beta+\beta\wedge\theta_Q^{*H_Q})$.

Taking the derivative of Eq. (27) with respect to t and using Eq. (22), we have that

$$\frac{\partial}{\partial t}(f_{H_0}^{-1}f_H) = f_{H_0}^{-1}f_H\circ f_H^{-1}\frac{\partial f_H}{\partial t} = \begin{pmatrix} 0 & \Psi(t) \\ 0 & 0 \end{pmatrix}.$$

As a result, we obtain Eq. (28).

Finally, taking the derivative of Eq. (28) with respect to $\bar\partial_{S\otimes Q^*}$, we have that

$$\bar\partial_{S\otimes Q^*}\frac{\partial}{\partial t}G = \bar\partial_{S\otimes Q^*}\Psi(t) = \frac{\partial}{\partial t}\gamma \tag{30}$$

Integrating Eq. (30) from 0 to t, we obtain Eq. (29).

In the proof of the C^0-estimate, we need to decompose Donaldson's functional in the Higgs bundle case, which was proved by Donaldson (see Ref. [23]) for the holomorphic vector bundle case.

Let us recall Donaldson's functional in the Higgs bundle case:

$$\mathcal{M}_E^0(H_0,H)=\int_0^1\int_M \mathrm{tr}\left(\sqrt{-1}\Lambda_\omega(F_{H(s)}+[\theta,\theta^*])H^{-1}(s)\frac{\partial H(s)}{\partial s}\right)\frac{\omega^n}{n!}\mathrm{d}s \tag{31}$$

and

$$\mathcal{M}_E(H_0,H) = \mathcal{M}_E^0(H_0,H)-\lambda_E\int_M \log\det(H_0^{-1}H)\frac{\omega^n}{n!} =$$

$$\int_0^1\int_M \mathrm{tr}\left((\sqrt{-1}\Lambda_\omega(F_H+[\theta,\theta^*])-\lambda_E\mathrm{Id_E})H^{-1}\frac{\partial H}{\partial s}\right)\frac{\omega^n}{n!}\mathrm{d}s \tag{32}$$

where $H(s)$ is the path connecting the metrics H_0 and H on E. Donaldson proved that the integral above is independent of the path when the base manifold M is Kähler.

If we have an exact sequence of Higgs bundles, then

$$0\to(S,\theta_S)\to(E,\theta)\to(Q,\theta_Q)\to 0 \tag{33}$$

where S is θ-invariant and θ_S and θ_Q are the Higgs fields on S and Q, respectively, induced by θ. Recall that a Hermitian metric H on E induces Hermitian metrics H_S, H_Q on S, Q, respectively. Note also that $\gamma_H\in\Omega_M^{0,1}(S\otimes Q^*)$ is the second fundamental form, and $\beta_H\in\Omega_M^{1,0}(S\otimes Q^*)$ is determined by the Higgs field θ and Hermitian metric H. Then we have the following lemma:

Lemma 2.3. For any exact sequence of Higgs bundles (33), Donaldson's functional $\mathcal{M}_E(H_0,H(t))$ has the following decomposition:

$$\mathcal{M}_E^0(H_0,H(t)) = \mathcal{M}_S^0(H_{0,S},H_S(t))+\mathcal{M}_Q^0(H_{0,Q},H_Q(t))+$$

$$\|\gamma(t)\|_{L^2}^2-\|\gamma(0)\|_{L^2}^2+\|\beta(t)\|_{L^2}^2-\|\beta(0)\|_{L^2}^2 \tag{34}$$

where $H_{0,S}$ and $H_{0,Q}$ are the Hermitian metrics on S and Q induced by H_0 on E. $H(t)$ is a path-connecting metric between H_0 and $H(t)$.

Proof. Taking derivative of $\mathcal{M}_E^0$ with respect to t, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_E^0(H_0,H(t))=\int_M \mathrm{tr}\left(\sqrt{-1}\Lambda_\omega(F_{H(t)}+[\theta,\theta^{*H(t)}])H^{-1}(t)\frac{\partial H(t)}{\partial t}\right)\frac{\omega^n}{n!} \tag{35}$$

Then, we obtain Donaldson's functional of the pullback metric $f_{H(t)}^*(H(t))$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_E^0(f_H^*(H_0),f_{H(t)}^*(H(t))) =$$

$$\int_M \mathrm{tr}\left(\sqrt{-1}\Lambda_\omega(f_{H(t)}^*(F_{H(t)}+[\theta,\theta^{*H(t)}]))f_{H(t)}^*\left(H^{-1}(t)\frac{\partial H(t)}{\partial t}\right)\right)\frac{\omega^n}{n!} =$$

$$\int_M \mathrm{tr}\left\{\begin{pmatrix} \sqrt{-1}\Lambda_\omega(F_{H_S}+[\theta_S,\theta_S^*]) + \sqrt{-1}\Lambda_\omega(-\gamma\wedge\gamma^*+\beta\wedge\beta^*) & \sqrt{-1}\Lambda_\omega(\partial_{S\otimes Q^*}\gamma) + \sqrt{-1}\Lambda_\omega(\theta_S^*\wedge\beta+\beta\wedge\theta_Q^*) \\ \sqrt{-1}\Lambda_\omega(-\bar\partial_{S^*\otimes Q}\gamma^*) + \sqrt{-1}\Lambda_\omega(\beta^{*H}\wedge\theta_S+\theta_Q\wedge\beta^*) & \sqrt{-1}\Lambda_\omega(F_{H_Q}+[\theta_Q,\theta_Q^*]) + \sqrt{-1}\Lambda_\omega(-\gamma^*\wedge\gamma+\beta^*\wedge\beta) \end{pmatrix}\begin{pmatrix} H_S^{-1}\frac{\partial H_S}{\partial t} & -\Psi(t) \\ -\Psi(t)^* & H_Q^{-1}\frac{\partial H_Q}{\partial t} \end{pmatrix}\right\}\frac{\omega^n}{n!} =$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_S^0(H_{0,S},H_S)+\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_Q^0(H_{0,Q},H_Q)+$$
$$\int_M\sqrt{-1}\Lambda_\omega\mathrm{tr}\left((-\gamma\wedge\gamma^*)H_S^{-1}\frac{\partial H_S}{\partial t}+(-\gamma^*\wedge\gamma)H_Q^{-1}\frac{\partial H_Q}{\partial t}\right)\frac{\omega^n}{n!}+$$
$$\int_M\sqrt{-1}\Lambda_\omega\mathrm{tr}\left((\partial_{S\otimes Q^*}\gamma)(-\Psi(t)^*)+(-\bar\partial_{S^*\otimes Q}\gamma^*)(-\Psi((t))\right)\frac{\omega^n}{n!}+$$
$$\int_M\sqrt{-1}\Lambda_\omega\mathrm{tr}\left((\beta\wedge\beta^*)H_S^{-1}\frac{\partial H_S}{\partial t}+(\beta^*\wedge\beta)H_Q^{-1}\frac{\partial H_Q}{\partial t}\right)\frac{\omega^n}{n!}+$$
$$\int_M\sqrt{-1}\Lambda_\omega\mathrm{tr}\left((\theta_S^*\wedge\beta+\beta\wedge\theta_Q^*)(-\Psi(t)^*)+(\beta^*\wedge\theta_S+\theta_Q\wedge\beta^*)(-\Psi(t))\right)\frac{\omega^n}{n!}=:$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_S^0(H_{0,S},H_S(t))+\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_Q^0(H_{0,Q},H_Q(t))+I_1+I_2+I_3+I_4.$$

Recall that

$$\left(\frac{\partial\gamma}{\partial t}\right)^{*H}=\left(\bar\partial_{S\otimes Q^*}\Psi(t)\right)^{*H}=\partial_{S^*\otimes Q}\Psi(t)^{*H}\tag{36}$$

Using Eq. (36) and Stokes formula, we have

$$I_2=-\int_M\sqrt{-1}\Lambda_\omega\partial\mathrm{tr}(\gamma\Psi(t)^*)\frac{\omega^n}{n!}-\int_M\sqrt{-1}\Lambda_\omega\mathrm{tr}(\gamma\wedge\partial_{S^*\otimes Q}\Psi(t)^*)\frac{\omega^n}{n!}+$$
$$\int_M\sqrt{-1}\Lambda_\omega\bar\partial\mathrm{tr}(\gamma^*\Psi(t))\frac{\omega^n}{n!}+\int_M\sqrt{-1}\Lambda_\omega\mathrm{tr}\left(\gamma^*\wedge\bar\partial_{S\otimes Q^*}\Psi(t)\right)\frac{\omega^n}{n!}=$$
$$-\int_M\sqrt{-1}\Lambda_\omega\mathrm{tr}\left(\gamma\wedge\left(\frac{\partial\gamma}{\partial t}\right)^*\right)\frac{\omega^n}{n!}+\int_M\sqrt{-1}\Lambda_\omega\mathrm{tr}\left(\gamma^*\wedge\left(\frac{\partial\gamma}{\partial t}\right)\right)\frac{\omega^n}{n!}.$$

Given that

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)^{*H(t)}=\frac{\mathrm{d}}{\mathrm{d}t}\left(H_Q^{-1}\bar\gamma^TH_S\right)=$$
$$-H_Q^{-1}\frac{\partial H_Q}{\partial t}H_Q^{-1}\bar\gamma^TH_S+H_Q^{-1}\overline{\left(\frac{\partial\gamma}{\partial t}\right)}^TH_S+H_Q^{-1}\bar\gamma^T\frac{\partial H_S}{\partial t}=$$
$$\left(\frac{\partial\gamma}{\partial t}\right)^*-H_Q^{-1}\frac{\partial H_Q}{\partial t}\gamma^*+\gamma^*H_S^{-1}\frac{\partial H_S}{\partial t}\tag{37}$$

we obtain

$$I_1+I_2=$$
$$\int_M\sqrt{-1}\Lambda_\omega\mathrm{tr}\left(-\gamma\wedge\gamma^*H_S^{-1}\frac{\partial H_S}{\partial t}-\gamma^*\wedge\gamma H_Q^{-1}\frac{\partial H_Q}{\partial t}\right)\frac{\omega^n}{n!}+$$
$$\int_M\sqrt{-1}\Lambda_\omega\mathrm{tr}\left(-\gamma\wedge\left(\frac{\partial\gamma}{\partial t}\right)^*+\gamma^*\wedge\left(\frac{\partial\gamma}{\partial t}\right)\right)\frac{\omega^n}{n!}=$$
$$\int_M\sqrt{-1}\Lambda_\omega\mathrm{tr}\left(\gamma^*H_S^{-1}\frac{\partial H_S}{\partial t}\wedge\gamma-H_Q^{-1}\frac{\partial H_Q}{\partial t}\gamma^*\wedge\gamma+\left(\frac{\partial\gamma}{\partial t}\right)^*\wedge\gamma\right)\frac{\omega^n}{n!}+$$
$$\int_M\sqrt{-1}\Lambda_\omega\mathrm{tr}\left(\gamma^*\wedge\left(\frac{\partial\gamma}{\partial t}\right)\right)\frac{\omega^n}{n!}=$$
$$\int_M\sqrt{-1}\Lambda_\omega\mathrm{tr}\left(\frac{\partial}{\partial t}(\gamma^*)\wedge\gamma+\gamma^*\wedge\left(\frac{\partial\gamma}{\partial t}\right)\right)\frac{\omega^n}{n!}=$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M\sqrt{-1}\Lambda_\omega(\gamma^*(t)\wedge\gamma(t))\frac{\omega^n}{n!}=$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\|\gamma(t)\|_{L^2}^2.\tag{38}$$

Given that

$$\frac{\partial\beta}{\partial t}=\theta_S\circ\Psi(t)-\Psi(t)\circ\theta_Q\tag{39}$$

We can obtain

$$\left(\frac{\partial\beta}{\partial t}\right)^*=\Psi^*(t)\circ\theta_S^*-\theta_Q^*\circ\Psi^*(t)\tag{40}$$

Thus, we have

$$I_4=\int_M\sqrt{-1}\Lambda_\omega\left((\theta_S^*\wedge\beta+\beta\wedge\theta_Q^*)(-\Psi^*(t))+(\beta^*\wedge\theta_S+\theta_Q\wedge\beta^*)(-\Psi(t))\right)\frac{\omega^n}{n!}=$$
$$\int_M\sqrt{-1}\Lambda_\omega\left(\beta\wedge(\theta_S^*\circ\Psi^*(t)-\theta_Q^*\circ\Psi^*(t))+(\theta_S\circ\Psi(t)-\Psi(t)\circ\theta_Q)\wedge\beta^*\right)\frac{\omega^n}{n!}=$$
$$\int_M\sqrt{-1}\Lambda_\omega\mathrm{tr}\left(\beta\wedge\left(\frac{\partial\beta}{\partial t}\right)^*+\frac{\partial\beta}{\partial t}\wedge\beta^*\right)\frac{\omega^n}{n!}.$$

Given that

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta(t)^{*H(t)}=\left(\frac{\partial\beta}{\partial t}\right)^*-H_Q^{-1}\frac{\partial H_Q}{\partial t}\beta^*+\beta^*H_S^{-1}\frac{\partial H_S}{\partial t}\tag{41}$$

we obtain

$$I_3+I_4=\int_M\sqrt{-1}\Lambda_\omega\mathrm{tr}\left((\beta\wedge\beta^*)H_S^{-1}\frac{\partial H_S}{\partial t}+(\beta^*\wedge\beta)H_Q^{-1}\frac{\partial H_Q}{\partial t}\right)\frac{\omega^n}{n!}+$$
$$\int_M\sqrt{-1}\Lambda_\omega\mathrm{tr}\left(\beta\wedge\left(\frac{\partial\beta}{\partial t}\right)^*+\frac{\partial\beta}{\partial t}\wedge\beta^*\right)\frac{\omega^n}{n!}=$$

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$$\int_M \sqrt{-1}\Lambda_\omega \mathrm{tr}\left(\beta\wedge\left(\left(\frac{\partial\beta}{\partial t}\right)^* - H_Q^{-1}\frac{\partial H_Q}{\partial t}\beta^* + \beta^* H_S^{-1}\frac{\partial H_S}{\partial t}\right)\right)\frac{\omega^n}{n!} +$$
$$\int_M \sqrt{-1}\Lambda_\omega \mathrm{tr}\left(\left(\frac{\partial\beta}{\partial t}\right)\wedge\beta^*\right)\frac{\omega^n}{n!} =$$
$$\int_M \sqrt{-1}\Lambda_\omega \mathrm{tr}\left(\beta\wedge\frac{\mathrm{d}}{\mathrm{d}t}(\beta)^* + \left(\frac{\partial\beta}{\partial t}\right)\wedge\beta^*\right)\frac{\omega^n}{n!} =$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M \sqrt{-1}\Lambda_\omega \mathrm{tr}(\beta(t)\wedge\beta^*(t))\frac{\omega^n}{n!} =$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\|\beta(t)\|_{L^2}^2 \tag{42}$$

Combining Eqs. (38) and (42), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_E^0(H_0,H(t)) =$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_S^0(H_{0,S},H_S(t)) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_Q^0(H_{0,Q},H_Q(t)) +$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\|\gamma(t)\|_{L^2}^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|\beta(t)\|_{L^2}^2 \tag{43}$$

Integrating Eq. (43) from 0 to t, we obtain Eq. (34).

At the end of this section, we consider parabolic inequalities for $|\gamma(t)|_{H(t)}^2 = -\sqrt{-1}\Lambda_\omega \mathrm{tr}(\gamma\wedge\gamma^{*H(t)})$ and $|\beta(t)|_{H(t)}^2 = \sqrt{-1}\Lambda_\omega \mathrm{tr}(\beta\wedge\beta^{*H(t)})$, which will be used in the next section. Through direct calculation, we obtain

$$\left(\Delta - \frac{\partial}{\partial t}\right)|\gamma(t)|_{H(t)}^2 =$$
$$2|\nabla^{H(t)}\gamma|_{H(t)}^2 + 2\mathrm{Ric}_\omega(\partial_k,\bar\partial_j)g^{k\bar l}g^{i\bar j}\mathrm{tr}\left(\gamma_{\bar i}H_Q^{-1}\overline{\gamma_{\bar i}}^T H_S\right) +$$
$$4\mathrm{Re}\langle g^{k\bar l}\left(F_{H_S}(\partial_k,\bar\partial_j)\gamma_{\bar i} - \gamma_{\bar i}F_{H_Q}(\partial_k,\bar\partial_j)\right)\mathrm{d}\bar z^j,\gamma\rangle_{H(t)} -$$
$$4\mathrm{Re}\langle\bar\partial_{S\otimes Q^*}(\theta_S^*\wedge\beta + \beta\wedge\theta_Q^*),\gamma\rangle_{H(t)} +$$
$$2\langle\left(-\sqrt{-1}\Lambda_\omega(\gamma\wedge\gamma^* - \beta\wedge\beta^*)\right)\circ\gamma +$$
$$\gamma\circ\left(\sqrt{-1}\Lambda_\omega(\gamma^*\wedge\gamma - \beta^*\wedge\beta)\right)\rangle_{H(t)} +$$
$$2\mathrm{Re}\langle(-\sqrt{-1}\Lambda_\omega[\theta_S,\theta_S^*])\circ\gamma +$$
$$\gamma\circ(\sqrt{-1}\Lambda_\omega[\theta_Q,\theta_Q^*]),\gamma\rangle_{H(t)} \tag{44}$$

and

$$\left(\Delta - \frac{\partial}{\partial t}\right)|\beta(t)|_{H(t)}^2 =$$
$$2|\nabla^{H(t)}\beta(t)|_{H(t)}^2 + 2\mathrm{Ric}_\omega(\partial_i,\bar\partial_l)g^{k\bar l}g^{i\bar j}\mathrm{tr}\left(\beta_k H_Q^{-1}\bar\beta_j^T H_S\right) +$$
$$4\mathrm{Re}\langle g^{k\bar l}(D_{\partial_k}^{H(t)}D_{\bar\partial_l}^{H(t)}\beta_i)\mathrm{d}z^i + (D_{\bar\partial_l}^{H(t)}\beta_i)(\nabla_{\partial_k}\mathrm{d}z^i,\beta)\rangle_{H(t)} -$$
$$4\mathrm{Re}\langle\theta_S\circ\Psi(t) - \Psi(t)\circ\theta_Q,\beta\rangle_{H(t)} +$$
$$2\langle\left(-\sqrt{-1}\Lambda_\omega(\gamma\wedge\gamma - \beta\wedge\beta^*)\right)\circ\beta +$$
$$\beta\circ\left(\sqrt{-1}\Lambda_\omega(\gamma^*\wedge\gamma - \beta^*\wedge\beta)\right),\beta\rangle_{H(t)} +$$
$$2\mathrm{Re}\langle\sqrt{-1}\Lambda_\omega([\theta_S,\theta_S^*])\circ\beta - \beta\circ\sqrt{-1}\Lambda_\omega([\theta_Q,\theta_Q^*]),\beta\rangle_{H(t)} \tag{45}$$

3 C^0-estimate of the rescaled metrics

Note that the Higgs bundles $(S,\bar\partial_S,\theta_S)$ and $(Q,\bar\partial_Q,\theta_Q)$ are ω-stable. According to the Donaldson-Uhlenbeck-Yau theorem, we can suppose that K_S and K_Q are ω-Hermitian-Einstein metrics on $(S,\bar\partial_S,\theta_S)$ and $(Q,\bar\partial_Q,\theta_Q)$, i.e.,

$$\sqrt{-1}\Lambda_\omega(F_{K_S} + [\theta_S,\theta_S^{*K_S}]) = \lambda_S \mathrm{Id}_S \tag{46}$$

and

$$\sqrt{-1}\Lambda_\omega(F_{K_Q} + [\theta_Q,\theta_Q^{*K_Q}]) = \lambda_Q \mathrm{Id}_Q \tag{47}$$

where $\lambda_S = \dfrac{2\pi}{\mathrm{Vol}(M,\omega)}\mu_\omega(S)$ and $\lambda_Q = \dfrac{2\pi}{\mathrm{Vol}(M,\omega)}\mu_\omega(Q)$.

Let us denote $h_S(t) = K_S^{-1}H_S(t)$, $h_Q(t) = K_Q^{-1}H_Q(t)$, and set $\tilde h_S(t) = \mathrm{e}^{2(\lambda_S-\lambda_E)t}h_S(t)$ and $\tilde h_Q(t) = \mathrm{e}^{2(\lambda_Q-\lambda_E)t}h_Q(t)$. Using (20) and (21), we have that

$$(\Delta - \frac{\partial}{\partial t})\mathrm{tr}\tilde h_S = 2\mathrm{tr}\left(-\sqrt{-1}\Lambda_\omega\bar\partial\tilde h_S\tilde h_S^{-1}\partial_{K_S}\tilde h_S\right) - 2(\lambda_S-\lambda_E)\mathrm{tr}(\tilde h_S) -$$
$$2\mathrm{tr}\left(\tilde h_S\sqrt{-1}\Lambda_\omega(F_{H_S} - F_{K_S})\right) - \mathrm{tr}\left(\tilde h_S H_S^{-1}\frac{\partial H_S}{\partial t}\right) =$$
$$2\mathrm{tr}\left(-\sqrt{-1}\Lambda_\omega\bar\partial\tilde h_S\tilde h_S^{-1}\partial_{K_S}\tilde h_S\right) -$$
$$2\mathrm{tr}(\tilde h_S(\sqrt{-1}\Lambda_\omega\gamma\wedge\gamma^*)) + 2\mathrm{tr}(\tilde h_S(\sqrt{-1}\Lambda_\omega\beta\wedge\beta^*)) +$$
$$2\mathrm{tr}\left(\tilde h_S(\sqrt{-1}\Lambda_\omega[\theta_S,\theta_S^{*H_S} - \theta_S^{*K_S}])\right) \geqslant 0 \tag{48}$$

and

$$(\Delta - \frac{\partial}{\partial t})\mathrm{tr}\tilde h_Q^{-1} =$$
$$2\mathrm{tr}\left(-\sqrt{-1}\Lambda_\omega\bar\partial h_Q^{-1}\circ\tilde h_Q\circ\partial_{H_Q}\tilde h_Q^{-1}\right) +$$
$$2\mathrm{tr}\left(\tilde h_Q^{-1}(\sqrt{-1}\Lambda_\omega\gamma^*\wedge\gamma)\right) - 2\mathrm{tr}\left(\tilde h_Q^{-1}(\sqrt{-1}\Lambda_\omega\beta^*\wedge\beta)\right) +$$
$$2\mathrm{tr}\left(\tilde h_Q^{-1}(\sqrt{-1}\Lambda_\omega[\theta_Q,\theta_Q^{*K_Q} - \theta_Q^{*H_Q}])\right) \geqslant 0 \tag{49}$$

where we have applied the non-negativity of $\sqrt{-1}\Lambda_\omega\gamma^*\wedge\gamma$ and $\sqrt{-1}\Lambda_\omega\beta\wedge\beta^*$. We also used the following equalities:

$$\mathrm{tr}\left\{\tilde h_S(\sqrt{-1}\Lambda_\omega[\theta_S,\theta_S^{*H_S} - \theta_S^{*K_S}])\right\} = |\theta_S\tilde h_S^{\frac12} - \tilde h_S\theta_S\tilde h_S^{-\frac12}|_{K_S}^2 \tag{50}$$

and

$$\mathrm{tr}\left\{\tilde h_Q^{-1}(\sqrt{-1}\Lambda_\omega[\theta_Q,\theta_Q^{*K_Q} - \theta_Q^{*H_Q}])\right\} = |\tilde h_Q^{-\frac12}\theta_Q - \tilde h_Q^{\frac12}\theta_Q\tilde h_Q^{-1}|_{K_Q}^2 \tag{51}$$

Using inequalities (48) and (49), and the maximum principle, we can obtain a uniform bound on $\mathrm{tr}\tilde h_S + \mathrm{tr}\tilde h_Q^{-1}$. In fact, we obtain the following lemma:

Lemma 3.1. There exists a uniform constant C_0 such that

$$\sup_{x\in M}\left(\mathrm{tr}\tilde h_S(x,t) + \mathrm{tr}\tilde h_Q^{-1}(x,t)\right) \leqslant C_0 \tag{52}$$

for all $t\in[0,+\infty)$.

In the following, we will derive uniform upper bounds on $\det(\tilde h_S^{-1}(t))$ and $\det(\tilde h_Q(t))$, which imply uniform upper bounds on $\mathrm{tr}\tilde h_S^{-1}(t)$ and $\mathrm{tr}\tilde h_Q(t)$ using (52). Then, we will obtain uniform C^0-bounds on $\tilde h_S(t)$ and $\tilde h_Q(t)$. At the beginning of the proof, the following proposition are required:

Proposition 3.2. Along the heat flow (9), we have

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$$\mathrm{tr}\left(\sqrt{-1}\Lambda_\omega(F_{H_S}+[\theta_S,\theta_S^*]-\gamma\wedge\gamma^*+\beta\wedge\beta^*)-\lambda_S\cdot\mathrm{Id}_S\right)\leqslant v(t) \tag{53}$$

where $v(t)\geqslant 0$ satisfies $\int_1^{+\infty}v(t)\mathrm{d}t\leqslant C_1$, and C_1 is a uniform constant.

Proof. From Lemma 2.3, for any exact sequence of Higgs bundles,

$$0\to(S,\theta_S)\to(E,\theta)\to(Q,\theta_Q)\to 0 \tag{54}$$

we have proved that

$$\mathcal{M}_E^0(H_0,H(t))=\mathcal{M}_S^0(H_{S,0},H_S(t))+\mathcal{M}_Q^0(H_{Q,0},H_Q(t))+ \|\gamma(t)\|_{L^2}^2-\|\gamma(0)\|_{L^2}^2+\|\beta(t)\|_{L^2}^2-\|\beta(0)\|_{L^2}^2 \tag{55}$$

According to the flow equations (20) and (21), we have that

$$\begin{aligned}&\int_M\log\det(H_S^{-1}(0)H_S(t))\frac{\omega^n}{n!}=\\&\int_0^t\frac{\partial}{\partial l}\int_M\log\det(H_S^{-1}(0)H_S(l))\frac{\omega^n}{n!}\mathrm{d}l=\\&\int_0^t\int_M\mathrm{tr}\left(H_S^{-1}(l)\frac{\partial H_S(l)}{\partial t}\right)\frac{\omega^n}{n!}\mathrm{d}l=\\&-2\int_0^t\int_M\mathrm{tr}\left(\sqrt{-1}\Lambda_\omega(F_{H_S}+[\theta_S,\theta_S^*]-\right.\\&\left.\gamma\wedge\gamma^*+\beta\wedge\beta^*)-\lambda_E\mathrm{Id}_S\right)\frac{\omega^n}{n!}\mathrm{d}l=\\&-2\int_0^t\int_M(|\gamma(l)|_{H(l)}^2+|\beta(l)|_{H(l)}^2)\frac{\omega^n}{n!}\mathrm{d}l-\\&2(\lambda_S-\lambda_E)\mathrm{rank}(S)t\mathrm{Vol}(M,\omega)\end{aligned} \tag{56}$$

and

$$\begin{aligned}&\int_M\log\det(H_Q^{-1}(0)H_Q(t))\frac{\omega^n}{n!}=\\&2\int_0^t\int_M\left(|\gamma(l)|_{H(l)}^2+|\beta(l)|_{H(l)}^2\right)\frac{\omega^n}{n!}\mathrm{d}l-\\&2(\lambda_Q-\lambda_E)\mathrm{rank}(Q)t\mathrm{Vol}(M,\omega)\end{aligned} \tag{57}$$

Then,

$$\begin{aligned}\mathcal{M}_E(H_0,H(t))=&\mathcal{M}_S(H_{S,0},H_S(t))+\mathcal{M}_Q(H_{Q,0},H_Q(t))+\\&\|\gamma(t)\|_{L^2}^2-\|\gamma(0)\|_{L^2}^2+\|\beta(t)\|_{L^2}^2-\|\beta(0)\|_{L^2}^2-\\&2(\lambda_S-\lambda_E)^2\mathrm{rank}(S)t\mathrm{Vol}(M,\omega)-\\&2(\lambda_Q-\lambda_E)^2\mathrm{rank}(Q)t\mathrm{Vol}(M,\omega)-\\&2(\lambda_S-\lambda_Q)\int_0^t\int_M\left(|\gamma(l)|_{H(l)}^2+|\beta(l)|_{H(l)}^2\right)\frac{\omega^n}{n!}\mathrm{d}l\end{aligned} \tag{58}$$

Furthermore, from the definition of Donaldson's functional and Gauss-Codazzi Eq. (17), it follows that

$$\begin{aligned}&\mathcal{M}_S(H_{S,0},H_S(t))+\mathcal{M}_Q(H_{Q,0},H_Q(t))=\\&-\|\gamma(t)\|_{L^2}^2+\|\gamma(0)\|_{L^2}^2-\|\beta(t)\|_{L^2}^2+\|\beta(0)\|_{L^2}^2-\\&2(\lambda_S-\lambda_Q)\int_0^t\int_M\left(|\gamma(l)|_{H(l)}^2+|\beta(l)|_{H(l)}^2\right)\frac{\omega^n}{n!}\mathrm{d}l-\\&4\int_0^t\|\partial_{S\otimes Q^*}\Psi(l)\|_{L^2}^2\mathrm{d}l-\\&2\int_0^t\|\sqrt{-1}\Lambda_\omega(F_{H_S}+[\theta_S,\theta_S^*]-\gamma\wedge\gamma^*+\beta\wedge\beta^*)-\\&\lambda_S\cdot\mathrm{Id}_S\|_{L^2}^2(l)\mathrm{d}l-\\&2\int_0^t\|\sqrt{-1}\Lambda_\omega(F_{H_Q}+[\theta_Q,\theta_Q^*]-\gamma^*\wedge\gamma+\beta^*\wedge\beta)-\\&\lambda_Q\cdot\mathrm{Id}_Q\|_{L^2}^2(l)\mathrm{d}l\end{aligned} \tag{59}$$

Now, we set

$$\begin{aligned}\tilde{v}(t)=&2\|\partial_{S\otimes Q^*}\Psi(t)\|_{L^2}^2+(\lambda_S-\lambda_Q)(\|\gamma(t)\|_{L^2}^2+\|\beta(t)\|_{L^2}^2)+\\&\|\sqrt{-1}\Lambda_\omega(F_{H_S}+[\theta_S,\theta_S^*]-\gamma\wedge\gamma^*+\beta\wedge\beta^*)-\lambda_S\cdot\mathrm{Id}_S\|_{L^2}^2(t)+\\&\|\sqrt{-1}\Lambda_\omega(F_{H_Q}+[\theta_Q,\theta_Q^*]-\gamma^*\wedge\gamma+\beta^*\wedge\beta)-\lambda_Q\cdot\mathrm{Id}_Q\|_{L^2}^2(t)\end{aligned} \tag{60}$$

Given that $(S,\bar\partial_S,\theta_S)$ and $(Q,\bar\partial_Q,\theta_Q)$ are ω-stable Higgs bundles over M, Donaldson's functional $\mathcal{M}_S(H_{S,0},\cdot)$ and $\mathcal{M}_Q(H_{Q,0},\cdot)$ are bounded from below. Together with the above Eq. (59), we obtain

$$\int_0^{+\infty}\tilde{v}(t)\mathrm{d}t\leqslant\tilde{C}_1<+\infty \tag{61}$$

where $\tilde{C}_1$ is a uniform positive constant.

By contrast, along with the heat flow Eq. (9), we have

$$\left(\Delta-\frac{\partial}{\partial t}\right)|\sqrt{-1}\Lambda_\omega(F_{H(t)}+[\theta,\theta^*])-\lambda_E\mathrm{Id}_E|_{H(t)}^2\geqslant 0 \tag{62}$$

According to the estimate of the heat kernel $K(x,y,t)$ by Cheng and Li in Ref. [24] (or see Theorem 3.2 in Ref. [25]), a positive constant C_K exists such that

$$0<K(x,y,t)\leqslant C_Kt^{-n} \tag{63}$$

Applying the maximum principle, we have

$$\begin{aligned}&|\sqrt{-1}\Lambda_\omega(F_H(t)+[\theta,\theta^*])-\lambda_E\mathrm{Id}_E|_{H(t)}^2(x,t+s)\leqslant\\&\int_MK(x,y,s)|\sqrt{-1}\Lambda_\omega(F_{H(t)}+[\theta,\theta^*])-\lambda_E\mathrm{Id}_E|_H^2(y,t)\frac{\omega^n}{n!}(y)\leqslant\\&C_Ks^{-n}\int_M|\sqrt{-1}\Lambda_\omega(F_{H(t)}+[\theta,\theta^*])-\lambda_E\mathrm{Id}_E|_{H(t)}^2(y,t)\frac{\omega^n}{n!}(y)\end{aligned} \tag{64}$$

for any $t>0$, $s>0$. Using the Gauss-Codazzi equation (17) and (64), we have

$$\begin{aligned}&2(\lambda_S-\lambda_Q)\cdot\\&\quad\mathrm{tr}\left(\sqrt{-1}\Lambda_\omega(F_{H_S}+[\theta_S,\theta_S^*]-\gamma\wedge\gamma^*+\beta\wedge\beta^*)-\lambda_S\mathrm{Id}_S\right)(t)\leqslant\\&\quad|\sqrt{-1}\Lambda_\omega(F_{H(t)}+[\theta,\theta^*])-\lambda_E\mathrm{Id}_E|_{H(t)}^2-\\&\quad(\lambda_S-\lambda_E)^2\mathrm{rank}(S)-(\lambda_Q-\lambda_E)^2\mathrm{rank}(Q)-\\&\quad2(\lambda_Q-\lambda_E)\mathrm{tr}\left(\sqrt{-1}\Lambda_\omega(F_{H(t)}+[\theta,\theta^*])-\lambda_E\mathrm{Id}_E\right)\leqslant\\&\quad C_K\left(\frac{t}{2}\right)^{-n}\int_M|\sqrt{-1}\Lambda_\omega(F_{H(\frac{t}{2})}+[\theta,\theta^*])-\lambda_E\mathrm{Id}_E|_{H(\frac{t}{2})}^2-\\&\quad(\lambda_S-\lambda_E)\mathrm{rank}(S)-(\lambda_Q-\lambda_E)^2\mathrm{rank}(Q)+\\&\quad(\lambda_E-\lambda_Q)|\mathrm{tr}\left(\sqrt{-1}\Lambda_\omega(F_{H(t)}+[\theta,\theta^*])-\lambda_E\mathrm{Id}_E\right)|\leqslant\\&\quad C_K\left(\frac{t}{2}\right)^{-n}\tilde{v}\left(\frac{t}{2}\right)+\\&\quad2(\lambda_E-\lambda_Q)|\mathrm{tr}\left(\sqrt{-1}\Lambda_\omega(F_{H(t)}+[\theta,\theta^*])-\lambda_E\mathrm{Id}_E\right)|+\\&\quad C_K\left(\frac{t}{2}\right)^{-n}\{(\lambda_S-\lambda_E)^2\mathrm{rank}(S)+(\lambda_Q-\lambda_E)^2\mathrm{rank}(Q)\}\end{aligned} \tag{65}$$

Let us set

$$\begin{aligned}v(t)=&(2(\lambda_S-\lambda_Q))^{-1}\left\{C_K\left(\frac{t}{2}\right)^{-n}\tilde{v}\left(\frac{t}{2}\right)+\right.\\&2(\lambda_E-\lambda_Q)\max_{x\in M}|\mathrm{tr}\left(\sqrt{-1}\Lambda_\omega(F_{H(t)}+[\theta,\theta^*])-\lambda_E\mathrm{Id}_E\right)|+\\&\left.C_K\left(\frac{t}{2}\right)^{-n}\{(\lambda_S-\lambda_E)^2\mathrm{rank}(S)+(\lambda_Q-\lambda_E)^2\mathrm{rank}(Q)\}\right\}\end{aligned} \tag{66}$$

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Combining formulas (61) and (65), and noting that $n \geq 1$, we conclude that $v(t)$ is the targeted function.

Using Proposition 3.2, we obtain a uniform C^0-bound on the rescaled metrics $\tilde{H}_S(t) = \mathrm{e}^{2(\lambda_S-\lambda_E)t}H_S(t)$ and $\tilde{H}_Q(t) = \mathrm{e}^{2(\lambda_Q-\lambda_E)t}H_Q(t)$.

Theorem 3.3. Let $H(t)$ be the solution of the Hermitian-Yang-Mills-Higgs flow (9) on the Higgs bundle $(E,\bar{\partial}_E,\theta)$ with initial metrics H_0, $H_S(t)$, and $H_Q(t)$ be the induced Hermitian metrics on (S,θ_S) and (Q,θ_Q). Set $\hat{h}_S(t) = \mathrm{e}^{2(\lambda_S-\lambda_E)t}H_S^{-1}(0)H_S(t)$ and $\hat{h}_Q(t) = \mathrm{e}^{2(\lambda_Q-\lambda_E)t}H_Q^{-1}(0)H_Q(t)$. Then, there exists a uniform constant $\hat{C}_0$ such that

$$\sup_{x\in M}\{\mathrm{tr}\hat{h}_S(x,t)+\mathrm{tr}\hat{h}_S^{-1}(x,t)+\mathrm{tr}\hat{h}_Q(x,t)+\mathrm{tr}\hat{h}_Q^{-1}(x,t)\} \leqslant \hat{C}_0 \tag{67}$$

for all $t \geqslant 0$.

Proof. Noting that the metrics $H_S(0)$ and $H_Q(0)$ are fixed Hermitian metrics, we can check that

$$-C_1\mathrm{Id}_S \leqslant \sqrt{-1}\Lambda_\omega(F_{H_S(0)}+[\theta_S,\theta_S^*]) \leqslant C_1\mathrm{Id}_S \tag{68}$$

and

$$-C_1\mathrm{Id}_Q \leqslant \sqrt{-1}\Lambda_\omega(F_{H_Q(0)}+[\theta_Q,\theta_Q^*]) \leqslant C_1\mathrm{Id}_Q \tag{69}$$

for the entire M, where C_1 is a uniform constant. Then, we obtain a uniform constant C_2 such that

$$\Delta\log\left(\mathrm{tr}\tilde{h}_S(0)+\mathrm{tr}\tilde{h}_S^{-1}(0)\right) \geqslant -C_2 \tag{70}$$

and

$$\Delta\log\left(\mathrm{tr}\tilde{h}_Q(0)+\mathrm{tr}\tilde{h}_Q^{-1}(0)\right) \geqslant -C_2 \tag{71}$$

According to formulas (70), (71) and Moser's iteration, we have the following mean inequalities, including a uniform constant C_3 such that

$$\sup_M\log\left(\mathrm{tr}\tilde{h}_S(0)+\mathrm{tr}\tilde{h}_S^{-1}(0)\right) \leqslant C_3\int_M\log\left(\mathrm{tr}\tilde{h}_S(0)+\mathrm{tr}\tilde{h}_S^{-1}(0)\right)\frac{\omega^n}{n!} \tag{72}$$

and

$$\sup_M\log\left(\mathrm{tr}\tilde{h}_Q(0)+\mathrm{tr}\tilde{h}_Q^{-1}(0)\right) \leqslant C_3\int_M\log\left(\mathrm{tr}\tilde{h}_Q(0)+\mathrm{tr}\tilde{h}_Q^{-1}(0)\right)\frac{\omega^n}{n!} \tag{73}$$

From the uniform L^1-estimate in Ref. [26, Lemma 5.1], we know that there exists a uniform constant C_4 such that

$$\sup_M\{\log(\mathrm{tr}\tilde{h}_S(0)+\mathrm{tr}\tilde{h}_S^{-1}(0))+\log(\mathrm{tr}\tilde{h}_Q(0)+\mathrm{tr}\tilde{h}_Q^{-1}(0))\} \leqslant C_4 \tag{74}$$

Set

$$\hat{h}_S(t) = \mathrm{e}^{2(\lambda_S-\lambda_E)t}H_S^{-1}(0)H_S(t)$$

and

$$\hat{h}_Q(t) = \mathrm{e}^{2(\lambda_Q-\lambda_E)t}H_Q^{-1}(0)H_Q(t).$$

From (52) and (74), we can obtain that

$$\sup_{(x,t)\in M\times[0,+\infty)}\left(\mathrm{tr}\hat{h}_S(x,t)+\mathrm{tr}\hat{h}_Q(x,t)\right) \leqslant C_5 \tag{75}$$

where C_5 denotes a uniform constant. According to (20), it follws that

$$\frac{\partial}{\partial t}\log\det(\hat{h}_S^{-1})=2\mathrm{tr}\left(\sqrt{-1}\Lambda_\omega(F_{H_S}+[\theta_S,\theta_S^*]-\gamma\wedge\gamma^*+\beta\wedge\beta^*)-\lambda_S\mathrm{Id}_S\right) \tag{76}$$

From (62), the Gauss-Codazzi equation (17), and the maximum principle, we have

$$\sup_{(x,t)\in M\times[0,+\infty)}|\sqrt{-1}\Lambda_\omega(F_{H(t)}+[\theta,\theta^*])-\lambda_E\mathrm{Id}_E|^2_{H(t)} \leqslant C_6 \tag{77}$$

and

$$\sup_{(x,t)\in M\times[0,+\infty)}\left|\mathrm{tr}\left(\sqrt{-1}\Lambda_\omega(F_{H_S}+[\theta_S,\theta_S^*]-\gamma\wedge\gamma^*+\beta\wedge\beta^*)-\lambda_S\mathrm{Id}_S\right)\right| \leqslant C_6 \tag{78}$$

where C_6 denotes a uniform constant. Then, (53) implies that there exists a uniform constant C_7 such that

$$\sup_{(x,t)\in M\times[0,+\infty)}\log\left(\det\hat{h}_S^{-1}(x,t)\right) \leqslant C_7 \tag{79}$$

As a result, we have

$$\det(\hat{h}_S(t))\det(\hat{h}_Q(t)) = \det(H_0^{-1}H(t)) \tag{80}$$

and then

$$\log\det(\hat{h}_Q(x,t)) = \log\det(\hat{h}_S^{-1}(x,t))+\log\det(H_0^{-1}H(x,t)) \leqslant C_8 \tag{81}$$

for all $(x,t)\in M\times[0,+\infty)$, where C_8 denotes a uniform constant. Combining (75), (79) and (81), we can easily conclude that there exists a constant $\hat{C}_0$ such that

$$\sup_{(x,t)\in M\times[0,+\infty)}\left(\mathrm{tr}\hat{h}_S+\mathrm{tr}\hat{h}_S^{-1}+\mathrm{tr}\hat{h}_Q+\mathrm{tr}\hat{h}_Q^{-1}\right)(x,t) \leqslant \hat{C}_0 \tag{82}$$

4 Uniform curvature estimate

Using the above uniform C^0-estimate of $\tilde{H}_S(t)$ and $\tilde{H}_Q(t)$, we first prove that the norms of $|T_S(t)|_{H_S(t)}$, $|T_Q(t)|_{H_Q(t)}$, $|\gamma(t)|_{H(t)}$, and $|\beta(t)|_{H(t)}$ are uniformly bounded.

Let us set

$$T_S(t) = D_{H_S(t),\bar{\partial}_S}-D_{H_0,\bar{\partial}_S} = h_S^{-1}\partial_{H_{0,S}}h_S = (\partial_{H_S(t)}h_S)h_S^{-1} \tag{83}$$

and

$$T_Q(t) = D_{H_Q(t),\bar{\partial}_Q}-D_{H_0,\bar{\partial}_Q} = h_Q^{-1}\partial_{H_{0,Q}}h_Q = (\partial_{H_Q(t)}h_Q)h_Q^{-1} \tag{84}$$

where $h_S(t) = H_{0,S}^{-1}H_S(t)$ and $h_Q(t) = H_{0,Q}^{-1}H_Q(t)$. Direct calculations yield

$$\begin{aligned}&\left(\Delta-\frac{\partial}{\partial t}\right)|T_S|^2_{H_S} = \\ &2|\nabla^{H_S(t)}T_S|^2_{H_S(t)}+2\mathrm{Ric}_\omega(\partial_k,\bar{\partial}_s)g^{k\bar{l}}g^{l\bar{s}}\mathrm{tr}(T_S(\partial_l)H_S^{-1}\overline{T_S(\partial_i)^T}H_S)+\\ &2\mathrm{Re}\langle[\sqrt{-1}\Lambda_\omega([\theta_S,\theta_S^*]-\gamma\wedge\gamma^*+\beta\wedge\beta^*),T_S],T_S\rangle_{H_S(t)}+\\ &4\mathrm{Re}\langle\partial_{H_S}(\sqrt{-1}\Lambda_\omega([\theta_S,\theta_S^*]-\gamma\wedge\gamma^*+\beta\wedge\beta^*)),T_S\rangle_{H_S(t)}+\\ &4\mathrm{Re}\left\{g^{i\bar{j}}g^{k\bar{l}}\langle[T_S(\partial_i),F_{H_{0,S}}(\partial_k,\bar{\partial}_j)],T_S(\partial_l)\rangle_{H_S(t)}\right\}+\\ &4\mathrm{Re}\langle\partial_{H_{0,S}}(\sqrt{-1}\Lambda_\omega F_{H_{0,S}}),T_S\rangle_{H_S(t)}\end{aligned} \tag{85}$$

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and

$$\left(\Delta-\frac{\partial}{\partial t}\right)|T_Q|^2_{H_Q}=$$
$$2|\nabla^{H_{Q(t)}}T_Q|^2_{H_{Q(t)}}+2\mathrm{Ric}_\omega(\partial_k,\bar{\partial}_s)g^{k\bar{l}}g^{l\bar{s}}\mathrm{tr}(T_Q(\partial_l)H_Q^{-1}\overline{T_Q(\partial_i)^T}H_Q)+$$
$$2\mathrm{Re}\langle[\sqrt{-1}\Lambda_\omega([\theta_Q,\theta_Q^*]-\gamma^*\wedge\gamma+\beta^*\wedge\beta),T_Q],T_Q\rangle_{H_{Q(t)}}+$$
$$4\mathrm{Re}\langle\partial_{H_Q}(\sqrt{-1}\Lambda_\omega([\theta_Q,\theta_Q^*]-\gamma^*\wedge\gamma+\beta^*\wedge\beta)),T_Q\rangle_{H_{Q(t)}}+$$
$$4\mathrm{Re}\left\{g^{i\bar{j}}g^{k\bar{l}}\langle[T_Q(\partial_i),F_{H_{0,Q}}(\partial_k,\bar{\partial}_j)],T_Q(\partial_l)\rangle_{H_{Q(t)}}\right\}+$$
$$4\mathrm{Re}\langle\partial_{H_{0,Q}}(\sqrt{-1}\Lambda_\omega F_{H_{0,Q}}),T_Q\rangle_{H_{Q(t)}} \tag{86}$$

However, we can also obtain the following inequality (see (2.5) in Ref. [14] for further details):

$$\left(\Delta-\frac{\partial}{\partial t}\right)|\theta|^2_{H(t)}\geqslant 2|\nabla_{H(t)}\theta|^2_{H(t)}+2|\Lambda_\omega[\theta,\theta^{*H(t)}]|^2_{H(t)}-2|\mathrm{Ric}(\omega)||\theta|^2_{H(t)} \tag{87}$$

From the local C^0-estimate in (67) and the maximum principle, it can be concluded that $|\theta|^2_{H(t)}$ is also uniformly bounded on $M\times[0,+\infty)$, i.e.,

$$\sup_{M\times[0,+\infty)}|\theta|^2_{H(t)}=\sup_{M\times[0,+\infty)}(|\theta_S|^2_{H_S(t)}+|\theta_Q|^2_{H_Q(t)}+|\beta(t)|^2_{H(t)})<\tilde{C}_1 \tag{88}$$

Together with the local uniform C^1-estimate in Ref. [21, Proposition 4.2], we obtain the following proposition:

Proposition 4.1. Let $H(t)$ be the solution of the heat flow (9) with initial metric H_0 on $(E,\bar{\partial}_E,\theta)$, let $T_S(t)$ and $T_Q(t)$ be defined by (83) and (84), and let γ be the second fundamental form. Then, there exists a constant $\tilde{C}_2$ such that

$$\sup_{(x,t)\in M\times[0,+\infty)}(|T_S(t)|^2_{H_S(t)}+|T_Q(t)|^2_{H_Q(t)}+|\gamma(t)|^2_{H(t)}+|\theta|^2_{H(t)})(x,t)<\tilde{C}_2 \tag{89}$$

According to Eqs. (44), (85), and (86), we obtain

$$\left(\Delta-\frac{\partial}{\partial t}\right)(|T_S|^2_{H_S(t)}+|T_Q|^2_{H_Q(t)}+|\gamma(t)|^2_{H(t)})\geqslant$$
$$\frac{1}{4}(|F_{H_S}|^2_{H_S}+|F^2_{H_Q}|+2|\partial_{H(t)}\gamma|^2_{H(t)})-$$
$$\tilde{C}_3(|\gamma|^2_{H(t)}+|\theta|^2_{H(t)})(|T_S|^2_{H_S(t)}+|T_Q|^2_{H_Q(t)}+|\gamma(t)|^2_{H(t)}+|\theta|^2_{H(t)})-$$
$$\tilde{C}_4(|T_S|^2_{H_S(t)}+|T_Q|^2_{H_Q(t)}+|\gamma(t)|^2_{H(t)}+|\theta|^2_{H(t)})-\tilde{C}_5 \tag{90}$$

where constant $\tilde{C}_i$ depends only on the uniform local C^0 bound of $\hat{h}_S$ and $\hat{h}_Q$, $H_{S,0}$, $H_{Q,0}$, and the lower bound of Ricci curvature of (M,ω). Let us consider

$$\Xi(x,t)=|\nabla_{H(t)}(F_{H(t)}+[\theta,\theta^{*H(t)}])|^2_{H(t)}+|\nabla_{H(t)}\theta|^2_{H(t)}(x) \tag{91}$$

Then, it follows that

$$\left(\Delta-\frac{\partial}{\partial t}\right)\Xi\geq$$
$$2(|\nabla^{H(t)}(F_{H(t)}+[\theta,\theta^{*H}])|^2_{H(t)}+|\nabla^2_{H(t)}\theta|^2_{H(t)})-\tilde{C}_6\Xi^{\frac{3}{2}}-$$
$$\tilde{C}_6(|\theta|^2_{H(t)}+|Rm(\omega)|)\Xi(x,t)-\tilde{C}_6|\nabla_\omega\mathrm{Ric}(\omega)|^2 \tag{92}$$

where $\tilde{C}_6$ is a constant, depending only on the complex dimension n and rank r, and Rm is the Riemann curvature of M. The proof of inequality (92) can be found in Ref. [26].

Proof of Theorem 1.1. For the sake of simplicity, we denote

$$\nu=|\theta|^2_{H(t)}+|\gamma|^2_{H(t)}+|T_S|^2_{H_S(t)}+|T_Q|^2_{H_Q(t)} \tag{93}$$

By estimating (89), we can choose a constant $\tilde{C}_7$ such that

$$0\leqslant\frac{1}{2}\tilde{C}_7\leqslant\tilde{C}_7-\nu(x,t)\leqslant\tilde{C}_7 \tag{94}$$

for all $(x,t)\in M\times[0,+\infty)$.

Now we consider the following test function:

$$\zeta_1=\frac{\Xi}{\tilde{C}_7-\nu}+W\nu \tag{95}$$

Let $\zeta_1(p,t_0)=\max_{M\times[0,t_1]}\zeta_1$. Then, at the maximum point (p,t_0), we have

$$\left(\Delta-\frac{\partial}{\partial t}\right)\zeta_1=\frac{1}{\tilde{C}_7-\nu}\left(\Delta-\frac{\partial}{\partial t}\right)\Xi+\frac{\Xi}{(\tilde{C}_7-\nu)^2}\left(\Delta-\frac{\partial}{\partial t}\right)\nu-$$
$$\frac{2}{\tilde{C}_7-\nu}\nabla\left(\frac{\Xi}{\tilde{C}_7-\nu}\right)\cdot\nabla(\tilde{C}_7-\nu)+W\left(\Delta-\frac{\partial}{\partial t}\right)\nu \tag{96}$$

and

$$\nabla\left(\frac{\Xi}{\tilde{C}_7-\nu}\right)+W\nabla\nu=0 \tag{97}$$

Substituting (97) into (96), choosing the constants $\tilde{C}_7$ and W sufficiently large, and using formulas (90) and (92), at the maximum point (p,t_0) we have

$$\left(\Delta-\frac{\partial}{\partial t}\right)\zeta_1=\frac{1}{\tilde{C}_7-\nu}\left(\Delta-\frac{\partial}{\partial t}\right)\Xi+\frac{\Xi}{(\tilde{C}_7-\nu)^2}\left(\Delta-\frac{\partial}{\partial t}\right)\nu-$$
$$\frac{2W}{\tilde{C}_7-\nu}|\nabla\nu|^2+W\left(\Delta-\frac{\partial}{\partial t}\right)\nu\geqslant\Xi-\tilde{C}_8 \tag{98}$$

where $\tilde{C}_8$ is a positive constant that depends only on the local uniform bound of ν and curvature of (M,ω). Thus, we obtain

$$\Xi(p,t_0)\leqslant\tilde{C}_8 \tag{99}$$

Then, there exists a constant $\tilde{C}_9$ such that

$$\sup_{M\times[0,+\infty)}|F_{H(t)}+[\theta,\theta^*]|^2+|\nabla_{H(t)}\theta|^2_{H(t)}\leqslant\tilde{C}_9 \tag{100}$$

This completes the proof of Theorem 1.1.

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Conflict of interest

The author declares that they have no conflict of interest.

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