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Yang–Mills bar connection and holomorphic structure

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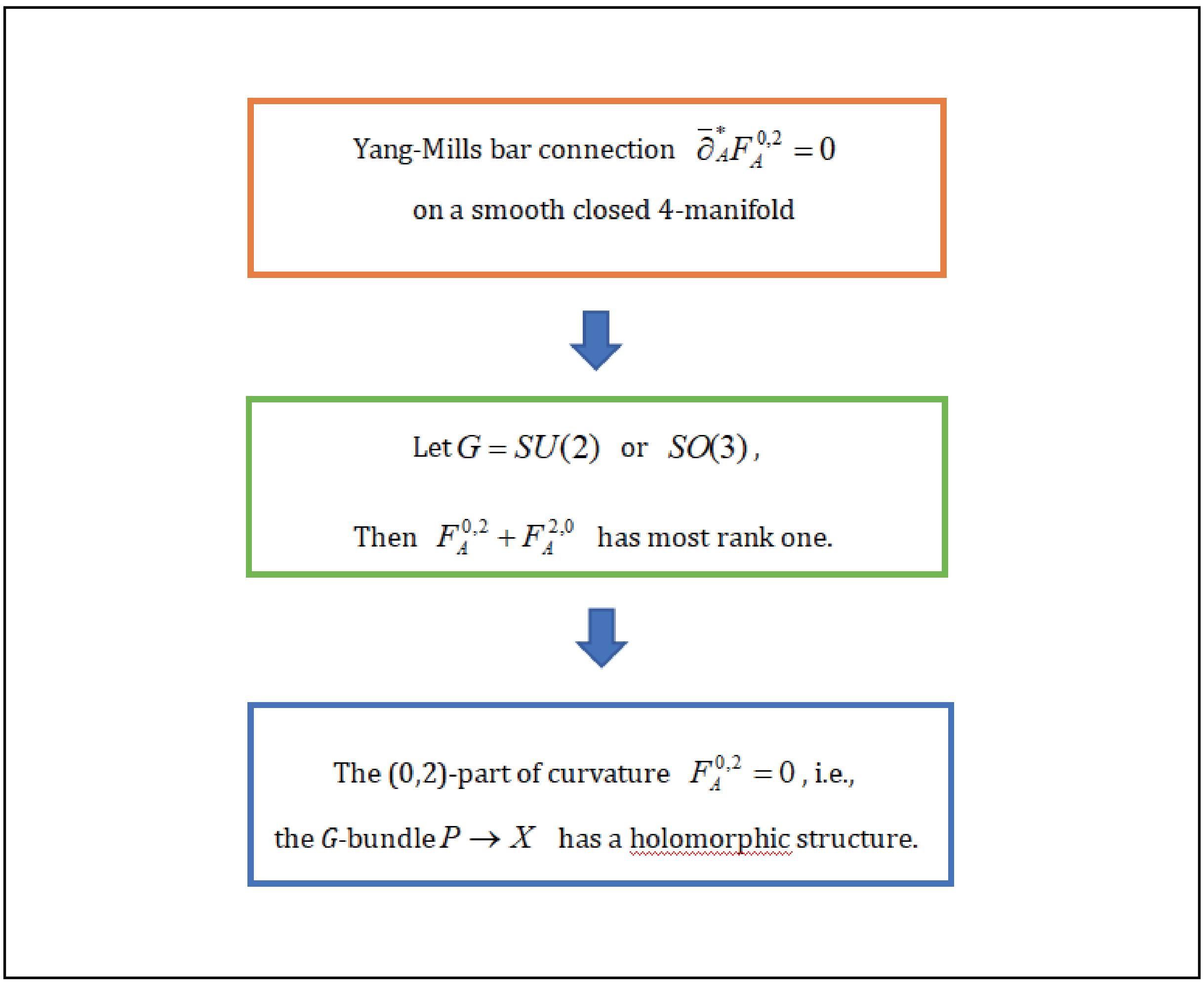
Graphical abstract



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Public summary

- A connection A is called Yang–Mills bar connection if the curvature of the connection A satisfies $\bar{\partial}_A^* F_A^{0,2} = 0$.
- When the structure group $G = SU(2)$ or $SO(3)$, we show that $\operatorname{rank}(F_A^{0,2} + F_A^{2,0}) \le 1$.
- Suppose that $H^1(X, \mathbb{Z}_2) = 0$, following an idea from Donaldson, we prove that $F_A^{0,2} = 0$.

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Abstract: In this note, we study the Yang–Mills bar connection A, i.e., the curvature of A obeys $\bar{\partial}^*_A F^{0,2}_A = 0$, on a principal G-bundle P over a compact complex manifold X. According to the Koszul–Malgrange criterion, any holomorphic structure on P can be seen as a solution to this equation. Suppose that $G = SU(2)$ or $SO(3)$ and X is a complex surface with $H^1(X,\mathbb{Z}_2) = 0$. We then prove that the $(0,2)$-part curvature of an irreducible Yang–Mills bar connection vanishes, i.e., $(P,\bar{\partial}_A)$ is holomorphic.

Keywords: Yang–Mills bar connection; holomorphic structure; Kähler surface

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2020 Mathematics Subject Classification: 53C05; 53C55

1 Introduction

Let E be a C^∞ complex vector bundle of rank r over a compact complex manifold X and H be some reference Hermitian inner product in the fibres of E, i.e., (E,H) defines an Hermitian vector bundle. We shall sometimes consider E issued from its associated $GL_n(\mathbb{C})$ principal bundle or from its associated unitary principal bundle. The classical Newlander–Nirenberg theorem[9] states that given an almost complex structure J over an even dimensional smooth manifold X then the torsion of J (also called the Nijenhuis tensor) vanishes if and only if J defines a complex structure. We denote by F_A the curvature 2-form of a smooth connection A of (E,H) over a complex manifold X. We will be interested in the bundle version of the Newlander-Nirenberg theorem as first proven in Ref. [8] (also shown in Ref. [3, Theorem 2.1.53]). It states that unitary connections satisfying $F^{0,2}_A = 0$ are in one to one correspondence with holomorphic structures:

Koszul–Malgrange criterion. Let A be a smooth unitary connection of a C^∞ Hermitian bundle (E,H) over a complex manifold X. Then E has a holomorphic structure if and only if $F^{0,2}_A = 0$.

The calculus of variations of Yang–Mills in four-dimensions has naturally led to the definition of Sobolev connections. One of the goals of Ref. [10] is to extend this identification to Sobolev connections. More precisely, the authors analyzed the weak holomorphic structures, that is Sobolev connections (see Ref. [10, Definition 1.1]) satisfying the integrability condition $F^{0,2}_A = 0$.

We note that in the decomposition for the curvature of unitary connection A:

$$F_A = F^{0,2}_A + F^{1,1}_A + F^{2,0}_A.$$

The Kozsul–Malgrange criterion suggests that we consider the Yang–Mills bar functional

$$E'(A) = \|F^{0,2}_A\|^2$$

which is the square of the L^2-norm of the $(0,2)$-component $F^{0,2}_A$ of the curvature on (E,H). Ref. [6] introduced the Yang–Mills bar equation as the Euler-Lagrange equation for the Yang–Mills bar functional. The solutions of the Yang–Mills bar equation are called Yang–Mills bar connections.

Definition 1.1.[5, 6] A connection A on a compact complex manifold is said to be a Yang–Mills bar connection if the $(0,2)$-part of its curvature is harmonic, i.e.,

$$\bar{\partial}^*_A F^{0,2}_A = 0.$$

Since a holomorphic connection on a complex bundle of rank $r \geq 2$ over a compact complex manifold X, $\dim_{\mathbb{C}} X \geq 2$, is overdetermined, and the Yang–Mills bar connection is moduli invariant under the complex gauge group of the complex vector bundle E. The Yang–Mills bar connection has an advantage over the holomorphic connection. Thus, Ref. [6] suggested that we can use the Yang–Mills bar equation to find useful sufficient conditions under which a complex vector bundle carries a holomorphic structure. The existence of a holomorphic structure on complex vector bundles over projective algebraic manifolds could be a key step in solving the Hodge conjecture. A particular result (see Ref. [6, Theorem 4.25]) which states that any Yang–Mills bar connection on a compact Kähler surface with positive Ricci curvature is holomorphic. In higher-dimensional cases, Stern proved that on a compact Calabi–Yau 3-fold X with $\mathrm{Hol}(X) = SO(3)$, if the connection A on a principal G-bundle P over X is a stable critical point of $E'(A)$, then $F^{0,2}_A = 0$, i.e., $(P,\bar{\partial}_A)$ is holomorphic (see Ref. [11, Theorem 6.21]).

In this note we consider the Yang–Mills bar connection A on a $SU(2)$ or $SO(3)$-bundle P over a compact complex surface. A self-dual two-form $B\in\Omega^+(X,\mathfrak{g}_P)$ which takes value in $\mathfrak{g}_P$ is said to be of rank r if, when considered as a section of $\mathrm{Hom}(\Lambda^{+,*})$, $B(x)$ has rank less than or equal to r at every point $x\in X$ (see Ref. [12, Defintion 1.5]). If the structure group of the principal bundle P is either $SU(2)$ or $SO(3)$, then the rank of any B must be less than or equal to 3. The key point in the proof of the following result is that $F_A^{0,2}+F_A^{2,0}$ has at most rank one (see Proposition 3.1).

Theorem 1.1. Let (X,g) be a compact complex surface with $H^1(X,\mathbb{Z}_2)=0$, P be a $SU(2)$ or $SO(3)$-bundle over X and A be a connection on P. Suppose that A is an irreducible Yang–Mills bar connection. Then $F_A^{0,2}=0$, i.e., $(P,\bar\partial_A)$ is holomorphic

Remark 1.1. The following example shows that the condition for irreducible connection in Theorem 1.1 is necessary. Let T^4 be a 2-dimensional complex torus with coordinates $z^1=x^1+\sqrt{-1}y^1$, $z^2=x^2+\sqrt{-1}y^2$. It is easy to see that $\pi_1(T^4)\cong\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$. Therefore, $H^1(T^4,\mathbb{Z}_2)=0$. Let $L\to T^4$ be a complex line bundle whose Chern class is represented by the cohomology class $c_1(L)$ of $dz^1\wedge dz^2+d\bar z^1\wedge d\bar z^2$. Let A be a unitary connection of L. Then the curvature

$$F_A=\sqrt{-1}(dz^1\wedge dz^2+d\bar z^1\wedge d\bar z^2)+\sqrt{-1}da,$$

where $a\in\Omega^1(T^4)$. The new connection $A'=A-a$ has the curvature $\sqrt{-1}(dz^1\wedge dz^2+d\bar z^1\wedge d\bar z^2)$. However, according to the Hodge theorem, we observe that L has no holomorphic structure (see Ref. [6, 3.21]). One can see that the bundle $L\oplus L^{-1}$ also carries no holomorphic structure. In general, the Hodge theory implies that on any Hermitian complex line bundle over a compact complex manifold there is a Yang–Mills bar connection which realizes the infimum of the energy $E'(A)$.

2 Preliminaries

2.1 Yang–Mills bar connection

Let (X,g) be a smooth complex surface with a $(1,1)$-form ω and P be a principal G-bundle over X with G being a compact Lie group. We denote by $\mathcal{A}_P$ the set of all connections. For any connection A on P. We have the covariant exterior derivatives $d_A:\Omega^k(X,\mathfrak{g}_P)\to\Omega^{k+1}(X,\mathfrak{g}_P)$. Like the canonical splitting the exterior derivatives $d=\partial+\bar\partial$, decomposes over X into $d_A=\partial_A+\bar\partial_A$. We also denote by $\Omega^{p,q}(X,\mathfrak{g}_P^{\mathbb{C}})$ the space of C^∞-(p,q) forms on $\mathfrak{g}_P^{\mathbb{C}}:=\mathfrak{g}_P\otimes\mathbb{C}$.

We define a Hermitian inner product $\langle\cdot,\cdot\rangle$ on $\Omega^{p,q}(X,\mathfrak{g}_P^{\mathbb{C}})$ by

$$\langle\alpha,\beta\rangle_{L^2(X)}=\int_X\langle\alpha,\beta\rangle(x)dvol_g,$$

$$\langle\alpha,\beta\rangle(x)dvol_g=\langle\alpha\wedge*\bar\beta\rangle,$$

where $*$ is the $\mathbb{C}$-linear extension of the Hodge operator over complex forms and $\bar\bullet$ is the conjugation on the bundle $\mathfrak{g}_P\otimes\mathbb{C}$-forms which is defined naturally. One can also see Ref. [5, Page 99] or Ref. [4]. Denote by L_ω the operator of exterior multiplication by the Kähler form ω:

$$L_\omega=\omega\wedge\alpha,\quad \alpha\in\Omega^{p,q}(X,\mathfrak{g}_P^{\mathbb{C}}),$$

and, as usual, let Λ_ω denote its pointwise adjoint, i.e.,

$$\langle\Lambda_\omega\alpha,\beta\rangle=\langle\alpha,L_\omega\beta\rangle.$$

It is well known that $\Lambda_\omega=*^{-1}\circ L_\omega\circ *$. We can decompose the curvature, F_A, as

$$F_A=F_A^{0,2}+F_{A0}^{1,1}+\frac{1}{2}\Lambda_\omega F_A\otimes\omega+F_A^{2,0}.$$

where $F_{A0}^{1,1}=F_A^{1,1}-\frac{1}{2}\Lambda_\omega F_A\otimes\omega$. Ref. [11] defined two new energies:

$$E'(A):=\|F_A^{0,2}\|^2,\quad E''(A):=\|\Lambda_\omega F_A\|^2.$$

We can write the Yang–Mills functional as

$$YM(A)=4\|F_A^{0,2}\|^2+\|\Lambda_\omega F_A\|^2+\int_X\mathrm{tr}(F_A\wedge F_A)=$$
$$4E'(A)+E''(A)+\text{topological constant}.$$

The energy functional $\|\Lambda_\omega F_A\|^2$ plays an important role in the study of Hermitian-Einstein connections, see Refs. [2, 3, 13]. If the connection A is a Hermitian–Yang–Mills connection, i.e.,

$$F_A^{0,2}=0,\quad \Lambda_\omega F_A=\Lambda\mathrm{Id},$$

where Λ is a constant, then the Yang–Mills functional is minimum. Suppose that an integrable connection $A\in\mathcal{A}_P^{1,1}$ on a holomorphic bundle over a Kähler surface is Yang–Mills, then $\Lambda_\omega F_A$ is parallel, i.e., $\nabla_A\Lambda_\omega F_A=0$.

Using the formula

$$F_{A+ta}^{0,2}=F_A^{0,2}+t\bar\partial_A a^{0,1}+t^2a^{0,1}\wedge a^{0,1},$$

we get the first variation of energy $E'(A)$ is given by

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E'(A+ta)|_{t=0}=\int_X\langle\bar\partial_A^*F_A^{0,2},a\rangle.$$

One can see that the Yang–Mills bar connection

$$\bar\partial_A^*F_A^{0,2}=0$$

is a critical point of $E'(A)$. Using the Bianchi identity $\bar\partial_A F_A^{0,2}=0$, a Yang–Mills bar connection A is equivalent to the $(0,2)$-part, $F_A^{0,2}$, of the curvature of the connection A is harmonic with respect to the Laplacian operator $\Delta_{\bar\partial_A}$, i.e.,

$$\Delta_{\bar\partial_A}F_A^{0,2}=0.$$

2.2 Irreducible connection

In this section, we first recall a definition of irreducible connection on a principal G-bundle P, where G being a compact, semisimple Lie group. Given a connection A on a principal G-bundle P over X. We can define the stabilizer Γ_A of A in the gauge group $\mathcal{G}_P$ by

$$\Gamma_A:=\{g\in\mathcal{G}_P|g(A)=A\}.$$

One can also see Ref. [3, Section 4.2.2]. A connection A is called reducible if the connection A whose stabilizer Γ_A is larger than the center $C(G)$ of G. Otherwise, the connections

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are irreducible, they satisfy $\Gamma_A \cong C(G)$. It is easy to see that a connection A is irreducible when it admits no nontrivial covariantly constant Lie algebra-value 0-form, i.e.,

$$\ker d_A \cap \Omega^0(X, \mathfrak{g}_P) = 0.$$

The most useful definition of reducibility in our note is the following.

Definition 2.1.[12, Definition B.1] A connection A on a principal $SU(2)$ or $SO(3)$ bundle $P \to X$ is reducible if one of the following equivalent conditions is satisfied:

① The stabilizer of A under the group of gauge transformations has a positive dimension.

② There exists a nonzero $\Gamma \in \Omega^0(X, \mathfrak{g}_P)$ such that $d_A \Gamma = 0$.

③ The holonomy of A is contained in some $SO(2)$ subgroup.

We recall the definition of locally reducible connection on a principal $SU(2)$ or $SO(3)$ bundle P.

Definition 2.2.[12, Definition 2.1] A connection A on a principal $SU(2)$ or $SO(3)$ bundle P over a smooth closed Riemanian manifold X is locally reducible if there is an open cover of X such that on each of the open subsets, there is a nonzero, covariantly constant section of $\mathfrak{g}_P$.

With regard to the local reducibility as in Definition 2.2, Tanaka observed that

Proposition 2.1.[12, Proposition B.3] A connection A on a principal $SU(2)$ or $SO(3)$-bundle $P \to X$ is locally reducible if and only if the holonomy of A is contained in some $O(2)$ subgroup.

Remark 2.1. Let A be a connection on a principal $SU(2)$ or $SO(3)$ bundle P. If $\pi_1(X)$ has no subgroup of index two, it follows that $H^1(X, \mathbb{Z}_2) = 0$, then every locally reducible connection A is reducible (see Ref. [12, Remark B.5]). In particular, a locally reducible connection on a closed simply connected manifold is reducible.

3 Proof of main theorem

Let (X, ω) be a compact Kähler surface with a smooth Kähler $(1,1)$-form ω. Given an orthonormal coframe $\{e_0, e_1, e_2, e_3\}$ on X for which $\omega = e^{01} + e^{23}$, where $e^{ij} = e^i \wedge e^j$. We define

$$dz^1 = e^0 + \sqrt{-1}e^1, \quad dz^2 = e^2 + \sqrt{-1}e^3,$$

$$d\bar{z}^1 = e^0 - \sqrt{-1}e^1, \quad d\bar{z}^2 = e^2 - \sqrt{-1}e^3,$$

so that

$$\omega = \frac{\sqrt{-1}}{2}(dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2).$$

For any $B \in \Omega^+(X, \mathfrak{g}_P)$, we can write

$$B = B_1(e^{01} + e^{23}) + B_2(e^{02} + e^{31}) + B_3(e^{03} + e^{12}),$$

where B_i, $i = 1,2,3$, takes value in $\mathfrak{g}_P$. Then we can define β as follows:

$$\beta := \frac{1}{2}(B_2 - \sqrt{-1}B_3)dz^1 \wedge dz^2.$$

Hence

$$\beta^* := -\frac{1}{2}(B_2 + \sqrt{-1}B_3)d\bar{z}^1 \wedge d\bar{z}^2.$$

We can rewrite B as

$$B := B_1\omega + \beta - \beta^*.$$

We define a bilinear map

$$[\bullet.\bullet] : \Omega^{2,+}(X, \mathfrak{g}_P) \otimes \Omega^{2,+}(X, \mathfrak{g}_P) \to \Omega^{2,+}(X, \mathfrak{g}_P)$$

by $\frac{1}{2}[\cdot,\cdot]_{\Omega^{2,+}} \otimes [\cdot,\cdot]_{\mathfrak{g}_P}$ (see Ref. [7, Appendxi A]). In a direct calculation (see Ref. [7, Section 7.1]),

$$-\frac{1}{4}[B.B] = [B_2, B_3](e^{01} + e^{23}) + [B_3, B_1](e^{02} + e^{31}) + [B_1, B_2](e^{03} + e^{12}).$$

Let P be a principal G-bundle over a closed, smooth Riemannian four-dimensional manifold (X, g) with Riemannian metric g. We recall a notion of rank of a section $B \in \Omega^+(X, \mathfrak{g}_P)$ (see Refs. [1] or [12, Definiton 1.5]. We denote $d = \dim G$. Choose local frames for $\mathfrak{g}_P$ and $\Lambda^+(T^*X)$, $(\dim \Lambda^+(T^*X) = 3)$, then the section B is represented by a $d \times 3$ matrix-valued function with respect to the local frames. The rank of B at a point $x \in X$ is the rank of the matrix at x. We denote by $\mathrm{rank}(B)$ the maximum of the pointwise rank over X. The pointwise rank of B also provides a stratification of the manifold X, namely,

$$X^i(B) = \{x \in X : \mathrm{rank}(B(x)) = i\}, \quad 0 \leqslant i \leqslant \mathrm{rank}(B). \qquad (1)$$

The top rank stratum is a nonempty open subset of X. If the structure group of the principal bundle P is either $SU(2)$ or $SO(3)$, then the possibilities for the rank of B are less than or equal to 3. We next recall the following from Ref. [7, Section 4.11]

Lemma 3.1.[12, Lemma 1.6] Let $P \to X$ be a principal $SU(2)$ or $SO(3)$ bundle over a closed four-dimensional Riemannian manifold X. If $B \in \Omega^+(X, \mathfrak{g}_P)$ satisfies $[B.B] = 0$, then the rank of B is at most one. Furthermore,

$$X^1(B) = \{x \in X : B(x) \neq 0\}.$$

Proof. Since the rank of B is at most one, it is easy to see $X = X_1(B) \cup X_0(B)$. Noting that $X_0(B)$ is the zero set of B. Therefore, $X^1(B) = \{x \in X : B(x) \neq 0\}$.

We then obtain that

Proposition 3.1. Let A be a connection on a principal $SU(2)$ or $SO(3)$-bundle over a compact Kähler surface. If the connection A is a Yang–Mills bar connection, then $F_A^{0,2} + F_A^{2,0}$ has at most rank one.

Proof. Since $\bar{\partial}_A^* F_A^{0,2} = 0$, we have

$$0 = \bar{\partial}_A \bar{\partial}_A^* F_A^{0,2} = -*[F_A^{0,2} \wedge *F_A^{0,2}]. \qquad (2)$$

In an orthonormal coframe, we can write $F_A^{0,2}$ as

$$F_A^{0,2} = (B_1 + \sqrt{-1}B_2)d\bar{z}^1 \wedge d\bar{z}^2,$$

where B_1 and B_2 take value in Lie algebra $\mathfrak{su}(2)$ or $\mathfrak{so}(3)$. Thus

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$$*F_A^{0,2} = (-B_1 + \sqrt{-1}B_2)dz^1 \wedge dz^2.$$

Following Eq. (2), we obtain that

$$0 = [B_1 + \sqrt{-1}B_2, -B_1 + \sqrt{-1}B_2] = -\sqrt{-1}[B_2, B_1] + \sqrt{-1}[B_1, B_2] = 2\sqrt{-1}[B_1, B_2].$$

Hence

$$0 = [B_1, B_2]. \tag{3}$$

Note that $F_A^{0,2} + F_A^{2,0}$ is a self-dual two form that takes value in $\mathfrak{g}_P$. Following Eq. (3), we obtain

$$[F_A^{0,2} + F_A^{2,0}.F_A^{0,2} + F_A^{2,0}] = 0.$$

Therefore, following Lemma 3.1, $[F_A^{0,2} + F_A^{2,0}]$ has at most rank one.

We now prove a useful lemma that will be crucial in the proof of Theorem 1.1. The idea follows from Ref. [3, Lemma 4.3.25].

Lemma 3.2. Let X be a smooth closed Riemannian four-manifold with $H^1(X,\mathbb{Z}_2) = 0$, and let $P \to X$ be a principal G-bundle with structure group G being either $SU(2)$ or $SO(3)$. Let $B \in \ker d_A^{+,*} \cap \Omega^+(X, \mathfrak{g}_P)$ be a nonzero self-dual 2-form that takes value in $\mathfrak{g}_P$. If B has at most rank 1, the connection A is reducible.

Proof. Let Z^c denote the complement of the zero set of B. By unique continuation of the elliptic equation $d_A^{+,*}B = 0$, Z^c is either empty or dense. If Z^c is not empty, then ϕ has rank one and is nowhere vanishing on Z^c (see Lemma 3.1). We denote by $\{U_\alpha\}$ a finite open cover of X. Locally, in each open set U_α, we can write

$$B \upharpoonright_{U_\alpha} = s_\alpha \otimes \omega_\alpha.$$

where s_α is a section $\Gamma(\mathfrak{g}_P, U_\alpha)$ with $|s_\alpha| = 1$ and $\omega_\alpha \in \Omega^+(U_\alpha)$. It is easy to see $\omega_\alpha(x) \neq 0$, $x \in Z^c \cap U_\alpha$. Now the condition $|s_\alpha| = 1$ implies that $\langle d_A s_\alpha, s_\alpha \rangle = 0$ on U_α. The equation $d_A^{+,*}B = 0$ implies that

$$d_A s_\alpha \wedge \omega_\alpha + s_\alpha \otimes d\omega_\alpha = 0.$$

Therefore, we obtain

$$d_A s_\alpha \wedge \omega_\alpha = 0, \quad d\omega_\alpha = 0.$$

Since a nonvanishing, pure self-dual 2-form ω_α gives an isomorphism from $\Omega^1(Z^c \cap U_\alpha)$ to $\Omega^3(Z^c \cap U_\alpha)$, s_α is covariant, i.e., $\nabla_A s_\alpha = 0$ on $Z^c \cap U_\alpha$. Since Z^c is dense on X, $\nabla_A s_\alpha = 0$ all over U_α. Hence, the connection A is locally reducible. Since $H^1(X,\mathbb{Z}_2) = 0$, A is reducible (see Remark 2.1).

Following Lemma 3.2, we then have

Corollary 3.1. Let X be a smooth closed Riemannian four-manifold with $H^1(X,\mathbb{Z}_2) = 0$, and let $P \to X$ be a principal G-bundle with structure group G being either $SU(2)$ or $SO(3)$. Let $B \in \ker d_A^{+,*} \cap \Omega^+(X, \mathfrak{g}_P)$. If $d_A^{+,*}B = 0$ and $[B.B] = 0$, the either B vanishes or the connection A is reducible.

Proof of Theorem 1.1. We now begin to prove Theorem 1.1. Following Proposition 3.1, $(F_A^{0,2} + F_A^{2,0}) \in \Omega^+(X, \mathfrak{g}_P)$ has at most rank one. Noting that

$$d_A^{+,*}(F_A^{0,2} + F_A^{2,0}) = \bar{\partial}_A^* F_A^{0,2} + \partial_A^* F_A^{2,0} = 0.$$

According to Corollary 3.1, $F_A^{0,2}$ vanishes over all of X since connection A is irreducible.

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Conflict of interest

The author declares that he has no conflict of interest.

Biography

Teng Huang is an Associate Professor at the School of Mathematical Sciences, University of Science and Technology of China (USTC). He received his Ph.D. degree in Mathematics from USTC in 2016. His research mainly focuses on mathematical physics and differential geometry.

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