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Optimal investment in equity and VIX derivatives

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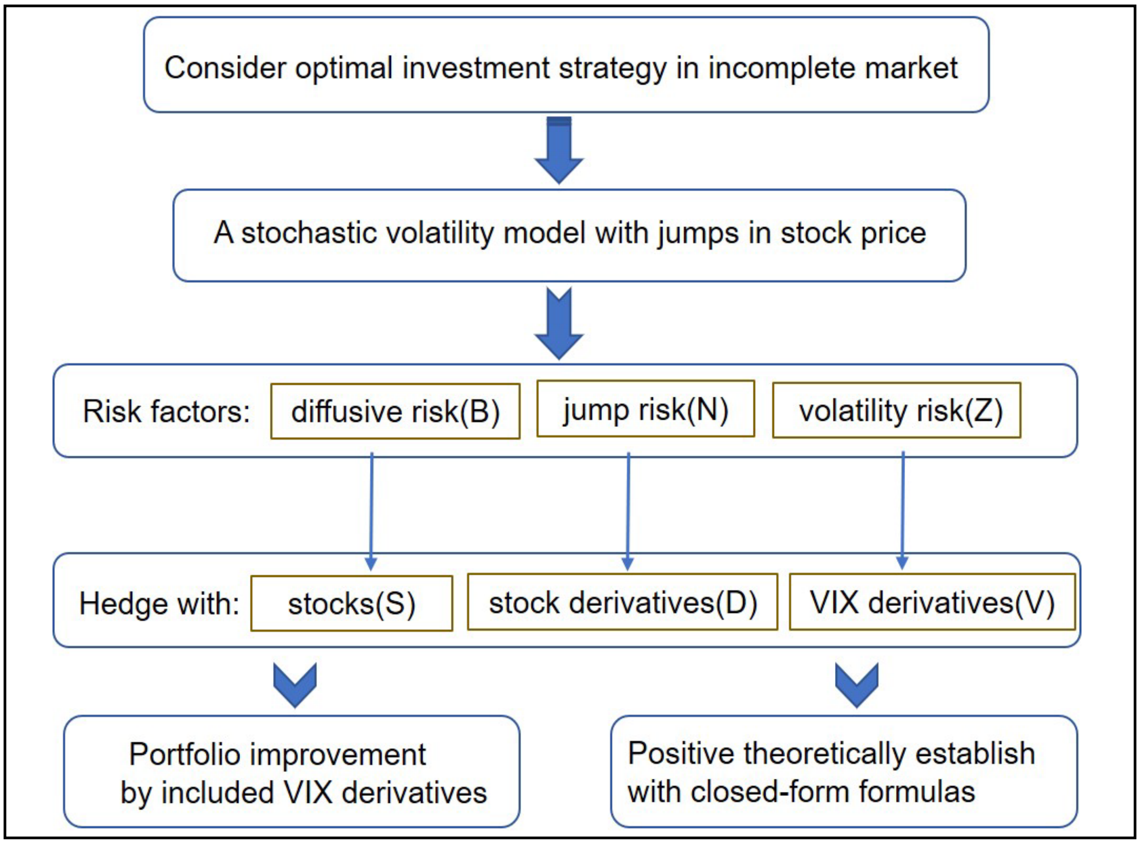
Graphical abstract



Considering the presence of diffusive risk, volatility risk, and jump risk, the inclusion of the VIX derivatives improved the portfolio performance.

Public summary

- Consider the optimal investment problem with equity and VIX derivatives in a stochastic volatility model with jumps.
- Solve in analytical closed-form or semi-closed-form the optimal investment strategies for different combinations of equity and VIX derivatives.
- Establish that there is a strict portfolio improvement when including the VIX derivatives into the investment opportunity set.
- Provide a theoretical justification for the demand of VIX derivatives in portfolio management.

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Abstract: We solve in closed-form the optimal investment strategies in equity and VIX derivatives in a stochastic volatility model with jumps. Our framework includes both complete market and incomplete market cases, when diffusive risk, volatility risk and jump risk are present. VIX derivatives allow for direct exposure to volatility risk compared to equity derivatives. Based on the closed-form formulas, we explicitly determine the portfolio improvements brought by the inclusion of the VIX derivatives and establish that it is theoretically positive. This justifies the economic intuition and observed demand for VIX derivatives in a portfolio management setting. Numerical examples illustrate the results.

Keywords: optimal investment; stochastic control; VIX derivatives; HJB equation; incomplete market

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1 Introduction

Pioneered by Ref. [1]'s seminal work, optimal investment is a classical problem in finance, in which a rational investor wants to maximize her expected utility from terminal wealth by investing in a frictionless market while facing an uncertain evolution of asset prices. The optimal portfolio choice involves stocks and risk-free bonds and does not generally include derivative securities.

However, during a recession, these choices may fail as the markets fell substantially, hence leaving investors exposed to a significant amount of systemic risk. The use of derivatives allows nonlinear exposure to volatility risk and jump risk, and can serve as effective hedging tools during market downturns. In particular, with the introduction of the VIX derivatives, it is possible to have direct access to volatility exposures. This motivates us to consider the inclusion of the VIX derivatives in the optimal investment problem.

The volatility index (VIX) was originally introduced by the Chicago Board Options Exchange (CBOE) in 1993 to measure the market's expectation of 30-day volatility and soon became a leading indicator of the U.S. stock market volatility. Since the VIX is not directly investable, VIX futures and options were launched on the CBOE in 2004 and 2006, respectively. VIX options and futures have become the most actively traded contracts at CBOE and CFE (CBOE Futures Exchange), and the total annual volume was 127 million contracts for VIX options and 62 million for VIX futures in 2019.

There is a fast-growing literature on the valuation of VIX derivatives[2–7]. Ref. [2] valuated VIX futures and VIX options under a general class of jump diffusions. Ref. [3] combined the Ito-Taylor expansion with a Markov chain approximation method to valuate VIX options under a general class of stochastic volatility models. In the realized GARCH model, Ref. [4] first proposed a closed-form valuation formula for VIX options. Ref. [5] proposed a direct approach for the valuation of VIX options in a discrete-time long memory model. Ref. [6] incorporated a stochastic jump intensity factor and time-varying mean of VIX variance in the pricing of VIX derivatives. There are also many literature on how the incorporation of VIX derivatives can improve portfolio performance[7–12]. There is little literature on optimal investment strategies that explicitly incorporate derivatives. Ref. [13] seemed to be the first to consider the problem of optimal investment in derivative securities. Ref. [14] considered a stochastic volatility model with jumps in the asset price, and managed to solve in closed-form the optimal investment strategies in stocks and equity derivatives. To the best of the authors' knowledge, there is no literature that considers the optimal investment problem involving the VIX derivatives, and we aim to fill this gap. More specifically, we consider the optimal investment problem when the investor can choose among the stocks, the equity derivatives, and the VIX derivatives, together with the risk-free assets. We manage to solve the optimal value functions and the optimal investment strategies in closed-form, and provide for the first time a theoretical justification of the advantages of investing in VIX derivatives in an incomplete market setting.

We address the question on the potential benefits and portfolio improvements in incorporating VIX derivatives into the portfolio. For the case of stocks, bonds and equity derivatives, the optimal investment problem has been investigated in Ref. [14]. We further extend their work to incorporate the VIX derivatives. First, this is of practical importance, as VIX derivatives are gradually gaining importance and popularity among investors, and it is natural to include them in the portfolio selection problem. Second, the theoretical problem is challenging as we face the problem cast in an incomplete market, while the framework in Ref. [14] is in a complete market. We manage to solve the corresponding HJB equation in closed-form and analyze in detail the portfolio improvements when we incorporate the volatility risk, jump risk or both.

The contributions of the paper are twofold:

(Ⅰ) We for the first time solve in closed-form the optimal investment problem with equity and VIX derivatives, with different combinations of assets. The framework in Ref. [14] is cast in the complete market framework, and we extend it to incomplete market cases when diffusive risk, volatility risk and jump risk are present, which includes investing solely① in the VIX derivatives. We solve these optimal investment problems in closed-form (see summary in Table 1).

(Ⅱ) We establish that there are strict portfolio improvements when we include the VIX derivatives into the investment opportunity set, which theoretically justifies the demand for VIX derivatives in portfolio management. We also analyze the sensitivity of the optimal investment strategies and portfolio improvements with respect to model parameters.

The remainder of the paper is organized as follows. Section 2 presents the model setting and derives the risk-neutral dynamics. Closed-form formulas for the equity option and the VIX option are derived. Section 3 contains the main results, which are the closed-form solutions for the optimal investment problem with VIX derivatives. Corresponding numerical results are also presented to illustrate the results. Section 4 concludes the paper. The appendix contains a discussion of the two risk factor cases and present additional numerical examples. The supporting information includes the proofs of all the propositions.

2 Model setting and risk-neutral dynamics

2.1 Stock price dynamic model setting

In this section, we present the main model setup, which is cast as a stochastic volatility model with jumps in the stock price, and this is known as Bates' model (see Ref. [15]). This model takes into account the effects of jumps and stochastic volatility, and is consistent with that in Ref. [14]. Let S and V represent the stock price process and the latent stochastic variance process, respectively. More specifically, assume that the stock price dynamic under the physical measure P is given by

$$\begin{aligned}\mathrm{d}S_t &= (r+\eta V_t+\mu(\lambda-\lambda^{\mathrm{Q}})V_t)S_t\mathrm{d}t+\sqrt{V_t}S_t\mathrm{d}B_t+\\ &\quad \mu S_{t-}(\mathrm{d}N_t-\lambda V_t\mathrm{d}t),\\ \mathrm{d}V_t &= \kappa(\bar{v}-V_t)\mathrm{d}t+\sigma\sqrt{V_t}\mathrm{d}B_t^{(2)},\end{aligned} \tag{1}$$

where $E[\mathrm{d}B_t\mathrm{d}B_t^{(2)}]=\rho\mathrm{d}t$, and $\rho\in(-1,1)$ is the correlation between the stock and volatility shocks. Here, r is the risk-free interest rate, and $\kappa>0$ is the mean-reversion rate. N_t is a Poisson process with the intensity given by λV_t. V_t is the variance process with the long-run mean level given by $\bar{v}>0$. Additionally, $\sigma\geq 0$ is the volatility of the volatility coefficient. $\mu>-1$ is a constant multiplier to the jump effect, η is the constant that captures the premium for the Brownian risk factor B, and λ^{Q} is the constant that captures the premium for the jump risk (or Poisson risk) factor N. We assume that Feller's condition $2\kappa\bar{v}>\sigma^2$ holds, so that the volatility stays positive almost surely.

Table 1. Summary of solvable cases.

Portfolio	BZN	BZ	BN
SDV	Complete (Explicit)	–	–
SD	–	Complete (Explicit)	Complete (Explicit)
SV	Incomplete (Implicit)	Complete (Equivalent to BZ-SD)	Degenerate (Equivalent to BN-S)
S	Incomplete (Implicit)	Incomplete (Explicit)	Incomplete (Implicit)
V	Incomplete (Explicit)	Incomplete (Explicit)	Degenerate (Independent of control)

Denote the price of the ith (derivative) security as $O_t^{(i)}=g^{(i)}(t,S_t,V_t), 0\leq t\leq\tau_i$, $i=1,2$, where τ_i is the maturity of the ith security. Then, with the pricing kernel π_t, we have the following pricing relation under the physical measure:

$$O_t^{(i)}=\frac{1}{\pi_t}E_t[\pi_{\tau_i}g^{(i)}(\tau_i,S_{\tau_i},V_{\tau_i})], \tag{2}$$

The stock price dynamic under the risk neutral measure is given by

$$\begin{aligned}\mathrm{d}S_t &= rS_t\mathrm{d}t+\sqrt{V_t}S_t\mathrm{d}B_t^{\mathrm{Q}}+\mu S_{t-}(\mathrm{d}N_t^{\mathrm{Q}}-\lambda^{\mathrm{Q}}V_t\mathrm{d}t),\\ \mathrm{d}V_t &= \kappa^*(\bar{v}^*-V_t)\mathrm{d}t+\sigma\sqrt{V_t}\left(\rho\mathrm{d}B_t^{\mathrm{Q}}+\sqrt{1-\rho^2}\mathrm{d}Z_t^{\mathrm{Q}}\right),\end{aligned} \tag{3}$$

where $\kappa^*:=\kappa+\sigma\left(\rho\eta+\sqrt{1-\rho^2}\xi\right), \bar{v}^*:=\dfrac{\kappa\bar{v}}{\kappa^*}$. The risk-neutral measure is the risk-neutral measure under the Radon-Nikodym and Girsanov Theorem. The B_t^{Q}, Z_t^{Q}, N_t^{Q} is B_t, Z_t, N_t under risk-neutral measure Q, respectively. Under the physical measure, consider that a CRRA (constant relative risk aversion) investor chooses at each time t to invest a fraction ϕ_t of her wealth in stock S_t and fractions $\psi_t^{(i)}, i=1,2$, in the ith derivative. The investor's goal is to maximize the expected utility of her terminal wealth:

$$J(t,W_t,V_t)=\max_{\left\{\phi_t,\psi_t^{(i)},i=1,2,0\leqslant t\leqslant T\right\}}E\left(\frac{W_T^{1-\gamma}}{1-\gamma}\right), \tag{4}$$

where $T\leqslant\tau_1,\tau_2$ and $\gamma>0$ denotes the relative risk-aversion coefficient. $J(\cdot)$ is the value function of the optimization problem, which is a function of the time t, the current wealth W_t and the current level of volatility V_t. Then the wealth process that satisfies the self-financing condition is given by

$$\begin{aligned}\mathrm{d}W_t &= rW_t\mathrm{d}t+\theta_t^{\mathrm{B}}W_t\left(\eta V_t\mathrm{d}t+\sqrt{V_t}\mathrm{d}B_t\right)+\theta_t^{\mathrm{Z}}W_t\left(\xi V_t\mathrm{d}t+\sqrt{V_t}\mathrm{d}Z_t\right)+\\ &\quad \theta_{t-}^{\mathrm{N}}W_{t-}\mu((\lambda-\lambda^{\mathrm{Q}})V_t\mathrm{d}t+\mathrm{d}N_t-\lambda V_t\mathrm{d}t),\end{aligned} \tag{5}$$

where

① This is common practice in volatility trading, where only the VIX futures exchange traded note (ETN) is invested.

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$$
\begin{aligned}
\theta_t^{\mathrm{B}} &= \phi_t + \sum_{i=1}^{2} \psi_t^{(i)} \left(\frac{g_s^{(i)} S_t}{O_t^{(i)}} + \sigma\rho \frac{g_v^{(i)}}{O_t^{(i)}} \right), \\
\theta_t^{\mathrm{Z}} &= \sigma \sqrt{1-\rho^2} \sum_{i=1}^{2} \psi_t^{(i)} \frac{g_v^{(i)}}{O_t^{(i)}}, \\
\theta_t^{\mathrm{N}} &= \phi_t + \sum_{i=1}^{2} \psi_t^{(i)} \frac{\Delta g^{(i)}}{\mu O_t^{(i)}}.
\end{aligned} \tag{6}
$$

Intuitively, the investor invests θ^{B} in diffusive price risk B, θ^{Z} in volatility risk Z, and θ^{N} in jump risk N.

2.2 Prices of equity and VIX options

In this section, we present the valuation formulas for the price of an equity call option① and for the price of a VIX call option under the risk-neutral dynamic (3). We also determine the expressions for the partial derivatives of these formulas to express the optimal portfolio weights using them.

Proposition 2.1. Denote $\tau := T - t$ as the time to maturity, and let $S = S_t$. If the underlying asset satisfies the risk-neutral dynamic as given in Eq. (3), then the price of a call option on the given asset with strike K is given by

$$
c(S, V; K, \tau) = \mathrm{e}^{-r\tau} E_t^{\mathrm{Q}}[(S_T - K)^+] = S P_1 - \mathrm{e}^{-r\tau} K P_2, \tag{7}
$$

where

$$
\begin{aligned}
P_1 &= \frac{1}{2} - \frac{1}{\pi} \int_0^{\infty} \frac{\mathrm{d}u}{u} \mathrm{Im}\left(\mathrm{e}^{A(1-iu)+B(1-iu)V} \mathrm{e}^{iu(\ln K - \ln S - r\tau)}\right), \\
P_2 &= \frac{1}{2} - \frac{1}{\pi} \int_0^{\infty} \frac{\mathrm{d}u}{u} \mathrm{Im}\left(\mathrm{e}^{A(-iu)+B(-iu)V} \mathrm{e}^{iu(\ln K - \ln S - r\tau)}\right), \\
B &= -\frac{a(1-\mathrm{e}^{-q\tau})}{2q - (q+b)(1-\mathrm{e}^{-q\tau})}, \\
A &= -\frac{\kappa^* \bar{v}^*}{\sigma^2} \left((q+b)\tau + 2\ln\left(1 - \frac{q+b}{2q}(1-\mathrm{e}^{-q\tau})\right) \right), \\
b &= \rho\sigma y - \kappa^*, \\
a &= y(1-y) - 2\lambda^{Q}(\mathrm{e}^{y\ln(1+\mu)} - 1 - y\mu), \\
q &= \sqrt{b^2 + a\sigma^2}, \\
\kappa^* &= \kappa + \sigma\left(\rho\eta + \sqrt{1-\rho^2}\xi\right), \quad \bar{v}^* = \frac{\kappa\bar{v}}{\kappa^*}.
\end{aligned}
$$

Next, we introduce the VIX and the VIX derivative. The volatility index (VIX) is an indicator that measures the market's expectation of volatility over the next 30 days. It is defined in the CBOE white paper②, and its square can be determined as follows:

$$
\mathrm{VIX}_t^2 = -\frac{2}{\bar{\tau}} E_t^{\mathrm{Q}}\left[\ln\left(\frac{S_{t+\bar{\tau}}}{F}\right)\right] \times 100^2,
$$

where $F = S_t \mathrm{e}^{r\bar{\tau}}$, and $\bar{\tau} = \frac{1}{12}$.

Proposition 2.2. If S_t follows the risk-neutral dynamic in Eq. (3), then

$$
\mathrm{VIX}_t^2 = (aV_t + b) \times 100^2, \tag{8}
$$

where

$$
\begin{cases}
a = c\dfrac{1-\mathrm{e}^{-\kappa^*\bar{\tau}}}{\kappa^*\bar{\tau}}, \\
b = \bar{v}^*(c-a), \\
c = 2\lambda^{Q}(\mu - \ln(1+\mu)) + 1.
\end{cases}
$$

The VIX options's closed-form valuation formulas can be seen in Ref. [16], and the price for the VIX future is analogously to the VIX option.

3 Main developments

In our model (3), there are three risk factors: diffusive, volatility, and jump factors. We denote them as B, Z, and N. Note that B is the Brownian motion driving the diffusive factor, Z is the Brownian motion driving the (latent) volatility factor, and N denotes the Poisson process driving the jump factor. For our later discussions, we shall denote BZN as the model that contains all three factors B, Z, and N. Similarly, BZ denotes the model that contains the diffusive and volatility factors, and BN denotes the model that contains the diffusive and jump factors.

We denote the three risky assets as the stocks S, the equity derivatives D, and the VIX derivatives V. Then, the notation "SDV" denotes a portfolio containing positions in all three risky assets, and "SD" denotes a portfolio containing stock and equity derivatives (the case in Ref. [14]). Thus, we can see that our consideration contains the result in Ref. [14] as a special case.

We summarize the solutions to different models with different portfolio components and factors in Table 1. In the table, "complete" means that the market is complete, and "incomplete" means that the market is incomplete. "Explicit" means that the final closed-form solution is fully explicit, and "implicit" means that the final formula has some components that have to be determined through implicit relations, e.g., solving the root of an algebraic equation. Take the following entry "BZN-SV" as an example: it means that we have the full stochastic model that has all three factors, but we only choose to invest in the stock and VIX derivatives, while there is no position in equity derivatives. It is clear that the number of factors is larger than that of instruments; hence, the market is incomplete. In this case, we manage to obtain solutions whose intermediate input needs to be determined implicitly.

By explicit closed-form solutions, we mean that we are able to express the unknowns (e.g., optimal control strategy, optimal value function) using basic operations. For implicit solutions, we mean that we have to evaluate some intermediate expressions by numerical methods (e.g., bisection, finite difference method). There are also cases when we cannot determine the explicit or implicit solutions, but we can still establish the existence and uniqueness of the solution. This is because some nonredundancy conditions are needed for explicit solutions. For example, in the case "BZN-SD", we only know whether the optimal control exists or not. We can also show that the second order partial derivative of the value function, i.e., J_{WW}, is strictly less than 0; hence, the optimal solution is unique. However, we fail to express the solution either explicitly or implicitly. These particular cases corres-

① Note that we corrected for some typos in the formula presented in Ref. [14]. In particular, there is a term $\mathrm{e}^{y\ln(1+\mu)}$ in the definition of a, and there is a typo in the original formula in Ref. [14].
② http://www.cboe.com/micro/vix/vixwhite.pdf

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pond to the blank entries in Table 1. There are also "degenerate cases", in which the combination either reduces to some other known combinations, or the optimal value function is independent of the control, i.e., there are an infinite number of optimal solutions.

In this section, we consider the full-scale model BZN as defined in Eq. (3), where all three risk factors are present. In the appendix, we shall consider the case when only two risk factors are present (e.g., BZ and BN), and derive the corresponding results. See the supporting information for the proofs. Many of these derived closed-form expressions are, to the best of the authors' knowledge, new to the existing literature. After specifying a derivative to be the VIX derivatives, we first derive the (explicit or implicit) solution to the corresponding incomplete market case, and then we calculate the portfolio improvements by inclusion of the VIX derivatives in terms of the return in certainty equivalent wealth defined later.

In Ref. [14], a delta-neutral straddle is used to analyze the behavior of investors for the case that the model only contains the diffusive risk B and the volatility risk Z, and not the jump risk N. However, for the model that contains three factors B, Z, and N, the delta-neutral straddle not only contains B and Z but also has contributions to the jump risk N. Since the prices of VIX derivatives do not depend on the initial stock price S_t, they are naturally delta-neutral and have no contributions to the jump risk N. This means that the portfolio with a VIX derivative will have optimal exposure to the volatility risk Z, since we can always adjust the exposure of the volatility risk Z by adjusting the weight for the VIX derivatives.

Similar to Ref. [14], let $\bar{W}$ be the investor's certainty equivalent wealth, which is implicitly defined by

$$\frac{\bar{W}^{1-\gamma}}{1-\gamma}=J(0,W_0,V_0). \tag{9}$$

For a general value function of the exponential affine form

$$J(t,W_t,V_t)=\frac{W_t^{1-\gamma}}{1-\gamma}\exp(\gamma h(T-t)+\gamma H(T-t)V_t),$$

we have

$$\bar{W}=W_0\exp\left(\frac{\gamma}{1-\gamma}[h(T)+H(T)V_0]\right).$$

Denote W^* as the certainty equivalent wealth in the complete market case, and $W^{\rm inc}$ as the certainty equivalent wealth in the incomplete market case, then we define

$$R^{\rm inc}:=\frac{\ln W^*-\ln W^{\rm inc}}{T}=\frac{\gamma}{1-\gamma}\left[\frac{h^*(T)-h^{\rm inc}(T)}{T}+\frac{H^*(T)-H^{\rm inc}(T)}{T}V_0\right] \tag{10}$$

to measure the portfolio improvement of annualized, continuously compounded return in the certainty equivalent wealth for market completion.

Similarly, if we need to compare two incomplete market cases, let $W^{\rm inc_1}$ and $W^{\rm inc_2}$ be the certainty equivalent wealth for the first and second incomplete market cases, respectively; then, we define

$$R^{\rm inc_1-inc_2}:=\frac{\ln W^{\rm inc_1}-\ln W^{\rm inc_2}}{T}=\frac{\gamma}{1-\gamma}\left[\frac{h^{\rm inc_1}-h^{\rm inc_2}}{T}+\frac{H^{\rm inc_1}-H^{\rm inc_2}}{T}V_0\right]. \tag{11}$$

3.1 The complete market case

In this section, we consider the case when the investor invests in the stocks, the equity derivatives and the VIX derivatives in model BZN. Since we have three investment products and three risk factors, the market is inherently complete. This case can be solved by using the results in Ref. [14].

Now, we consider the case when the second derivative security in Ref. [14] is a VIX derivative. Then, the portfolio comprises the stock, the equity derivatives, and the VIX derivative, and we denote it by SDV. The nonredundancy condition is satisfied, as it can be easily calculated by setting $g_s^{(2)}=\Delta g^{(2)}=0$. We can drop the superscript of the portfolio weights and simplify it to the following form:

$$\phi_t=\frac{\eta}{\gamma}+\sigma\rho H(T-t)-\psi_t^{(1)}\left(\frac{g_s^{(1)}S_t}{O_t^{(1)}}+\frac{g_v^{(1)}}{O_t^{(1)}}\sigma\rho\right)-\psi_t^{(2)}\frac{g_v^{(2)}}{O_t^{(2)}}\sigma\rho,$$

$$\psi_t^{(1)}=\left(\frac{\Delta g^{(1)}}{\mu O_t^{(1)}}-\frac{g_s^{(1)}S_t}{O_t^{(1)}}\right)^{-1}\left(\frac{1}{\mu}\left(\left(\frac{\lambda}{\lambda^Q}\right)^{\frac{1}{\gamma}}-1\right)-\frac{\eta}{\gamma}+\frac{\rho}{\sqrt{1-\rho^2}}\frac{\xi}{\gamma}\right),$$

$$\psi_t^{(2)}=\frac{O_t^{(2)}}{g_v^{(2)}}\left(\frac{\xi}{\sigma\gamma\sqrt{1-\rho^2}}+H(T-t)-\frac{g_v^{(1)}}{O_t^{(1)}}\psi_t^{(1)}\right).$$

Similar to Ref. [14], we let the equity derivatives be a put option, which has one month to maturity and $K=0.95S$, as the vehicle to disentangle the jump risk from the diffusive risk. This is because the deep out-of-the money put option is more sensitive to negative jump risk than diffusive risk. We let the VIX derivatives be a one-month VIX future (i.e., VIX call option with $K_{\rm VIX}=0$), which serves as the vehicle for transferring the volatility risk. Then, we perform a sensitivity analysis using the parameters from Ref. [17] as a benchmark.

$$\eta=4,\ \rho=-0.4,\ \bar{v}=0.13^2,\ r=0.05,\ \lambda=0.1;$$
$$\gamma=4,\ \xi=-6,\ \sqrt{V}=0.15,\ \sigma=0.25,\ T=5,\ \kappa=5,$$
$$\mu=-0.15,\ \lambda^Q=0.3.$$

The sensitivities of portfolio weights are plotted in Fig. 1. Some interpretations of the results are in place. In the top right panel of Fig. 1, it can be seen that as the volatility risk premium increases, the holdings in all three risky assets increase, which is consistent with intuition. Note that the demand for the VIX derivatives is higher than that for the equity derivatives since the VIX derivatives provide more direct access to volatility exposure. The top left panel indicates that investors with higher risk-aversion are less likely to invest in derivative securities, which is consistent with the intuition that less risk averse investors are more aggressive in their investment strategies. For the middle panel about the sensitivity to the volatility $\sqrt{V}$, we note that, as the market becomes more volatile, i.e., volatility is high, the cost of purchasing the options becomes high. However, volatility sensitivity decreases, and "volatility exposure per dollar" decreases. Hence, the demand for the VIX derivatives increases, as the absolute value $|\psi_t^{(2)}|$ increases.

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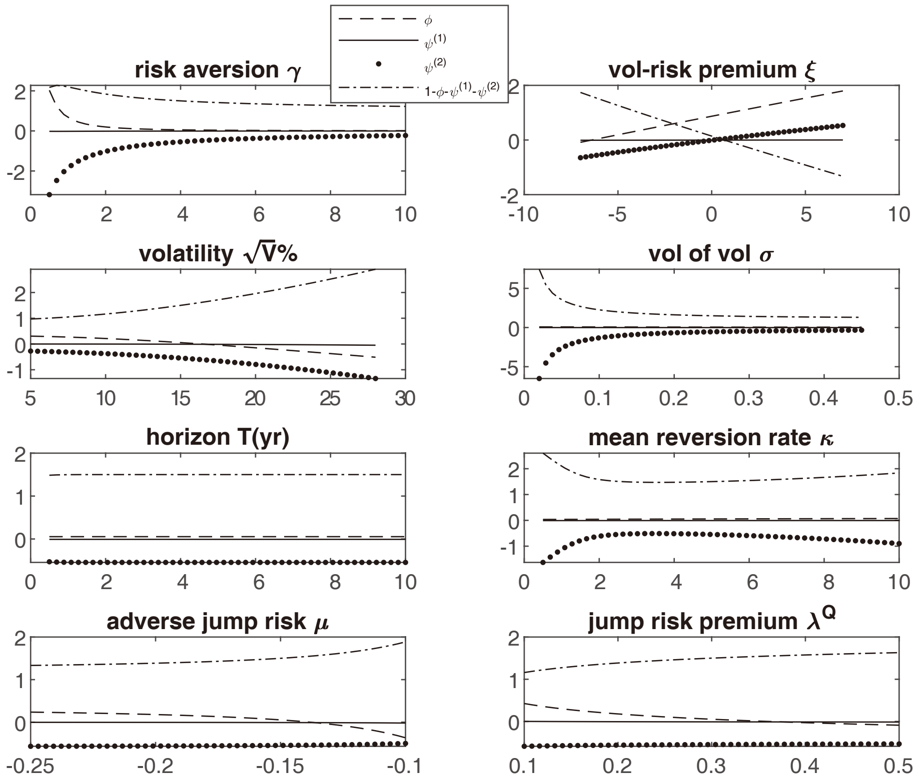

Fig. 1. Sensitivity of portfolio weights for the complete market case. The lines are in the same style of Fig. 1 in Ref. [14] to make comparison, i.e., the y-axes are the optimal portfolio weights ϕ (dash line), $\psi^{(1)}$ (solid line), $\psi^{(2)}$ (dot line), and the remaining portion invested in risk-free asset (dashed-dot line).

3.2 The incomplete market case

Note that Ref. [14, Eq. (12)] guarantees the complete market case by imposing the nonredundancy condition, i.e., it introduces two derivative securities that complete the market with respect to volatility and jump risk. In this section, we consider the case when the investor does not have access to both derivative securities at the same time; thus, the optimal investment problem is considered in an incomplete market. We show that we are still able to obtain closed-form solutions, which comprise the main results of this paper. Based on these results, we later quantify the portfolio improvements and justify the demand for VIX derivatives.

3.2.1 Portfolio with stocks and VIX derivatives

If the investor invests in the stock and only one derivative security (together with trading in the risk-free asset) by the wealth function and solves the HJB equation, the optimal portfolio weights $(\psi_t^{\mathrm{BZN,SV}}, \phi_t^{\mathrm{BZN,SV}})$ are implicitly characterized by the following equations:

$$\sigma\frac{g_v}{O_t}\psi_t^{\mathrm{BZN,SV}} = \frac{1}{\gamma}\left(\rho\eta+\sqrt{1-\rho^2}\xi\right) - \rho\phi_t^{\mathrm{BZN,SV}} + \sigma H^{\mathrm{BZN,SV}}(T-t),$$
$$\eta-\lambda^{\mathrm{Q}}\mu-\gamma(1-\rho^2)\phi_t^{\mathrm{BZN,SV}} - \rho\left(\rho\eta+\sqrt{1-\rho^2}\xi\right)+ \lambda\mu(1+\phi_t^{\mathrm{BZN,SV}}\mu)^{-\gamma} = 0.$$

Note that the solutions of the above equations exist and are unique.

Since the value function has an exponential affine form, we can use Eq. (10) to measure portfolio improvements, as shown in the following result.

Proposition 3.1. In model BZN, for $\gamma \neq 1$, by Eq. (10) the portfolio improvements from including the equity derivatives in the portfolio SV is

$$R^{\mathrm{BZN,SV}} = \frac{\gamma}{1-\gamma}(\Delta r_{SV} + \Delta R_{SV} V_0) \geqslant 0, \tag{12}$$

where

$$\Delta r_{\mathrm{SV}} = \frac{h^{\mathrm{BZN},*}(T)-h^{\mathrm{BZN,SV}}(T)}{T}, \Delta R_{\mathrm{SV}} = \frac{H^{\mathrm{BZN},*}(T)-H^{\mathrm{BZN,SV}}(T)}{T}.$$

The equality is achieved if and only if

$$\frac{1}{\mu}\left(\left(\frac{\lambda}{\lambda^{\mathrm{Q}}}\right)^{\frac{1}{\gamma}}-1\right) = \frac{\eta}{\gamma} - \frac{\rho}{\sqrt{1-\rho^2}}\frac{\xi}{\gamma}.$$

Remark 3.1. Intuitively, $R^{\mathrm{BZN,SV}}$ is calculated based on calculating the difference of the case when the investor has access to all three risky assets (i.e., stocks, VIX derivatives, and equity derivatives) and the case when the investor only has access to two risky assets (i.e., stocks and VIX derivatives). Thus, it is clear that this difference measures the effects of the inclusion of the equity derivatives. Interestingly, we can theoretically establish that $R^{\mathrm{BZN,SV}} \geqslant 0$ always holds, and intuitively, it means that it is beneficial to include the equity derivatives in the opportunity set of the investment problem.

3.2.2 Portfolio with only stocks and risk-free assets

When the investor only trades stocks and the risk-free assets, we have $\psi_t \equiv 0$, and

$$\theta_t^{\mathrm{B}} = \phi_t; \quad \theta_t^{\mathrm{Z}} = 0; \quad \theta_t^{\mathrm{N}} = \phi_t.$$

Then, according to the wealth process and solving the HJB

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equation, we have the following system of three equations:

$$-\gamma\phi_t^{\mathrm{BZN,S}\prime}-\sigma\gamma H^{\mathrm{BZN,S}\prime}(T-t)\rho-\gamma\lambda\mu^2(1+\mu\phi_t^{\mathrm{BZN,S}})^{-\gamma}\phi_t^{\mathrm{BZN,S}\prime}=0,$$
$$\gamma\left(1+\lambda\mu^2(1+\mu\phi_t^{\mathrm{BZN,S}})^{-\gamma}\right)\phi_t^{\mathrm{BZN,S}\prime}=-\sigma\gamma H^{\mathrm{BZN,S}\prime}(\tau)\rho,$$
$$\phi_t^{\mathrm{BZN,S}\prime}=\frac{-\sigma H^{\mathrm{BZN,S}\prime}(\tau)\rho}{1+\lambda\mu^2(1+\mu\phi_t^{BZN,S})^{-\gamma}}.$$

Here,

$$h^{\mathrm{BZN},S}(T)=\frac{1-\gamma}{\gamma}rT+\kappa\bar{v}\int_0^T H^{\mathrm{BZN},S}(s)\,\mathrm{d}s.$$

Remark 3.2. Based on the above solutions of the optimal investment problems, respectively for the case of trading both stocks and VIX derivatives and for the case of trading stocks only, we can determine the improvements to the portfolio when we include the VIX derivatives into our investment choices. The finding is summarized in Proposition 3.2.

Proposition 3.2. In model BZN, for $\gamma\neq 1$, by Eq. (11), the portfolio improvement from including the VIX derivatives in portfolio S is given by

$$R^{\mathrm{BZN,SV-S}}=\frac{\gamma}{1-\gamma}(\Delta r_S+\Delta R_S V_0)>0,$$

where

$$\Delta r_{\mathrm{S}}=\frac{h^{\mathrm{BZN,SV}}(T)-h^{\mathrm{BZN,S}}(T)}{T},\Delta R_{\mathrm{S}}=\frac{H^{\mathrm{BZN,SV}}(T)-H^{\mathrm{BZN,S}}(T)}{T}.$$

Remark 3.3. Intuitively $R^{\mathrm{BZN,SV-S}}$ represents the difference of the optimal portfolio with both stocks and VIX derivatives and the optimal portfolio with stocks only, and it showcases the effect of adding the VIX derivatives into the investment opportunity set. It is clear that the inclusion of the VIX derivatives strictly improves the optimal portfolio. Note that we have strict inequality here, which is different from Eq. (12), in which equality can also hold. The demand for derivatives mainly arises from the need to access volatility risk, and VIX derivatives allow a pure exposure to volatility risk. Here, in a parametric setting, we rigorously demonstrate the fact that it also improves the overall performance of the portfolio.

Note that we have the following identity by definition:

$$R^{\mathrm{BZN,S}}=R^{\mathrm{BZN,SV-S}}+R^{\mathrm{BZN,SV}}>0,\tag{13}$$

where the strict inequality is due to Propositions 3.1 and 3.2.

3.2.3 Portfolio with only VIX derivatives and risk-free assets

If the investor trades only the VIX derivatives[①] (together with risk-free bonds), then $\phi_t\equiv 0$. Let $\theta_t=\sigma\frac{g_v}{O_t}\psi_t$, then we have

$$\theta_t^{\mathrm{B}}=\rho\theta_t;\quad \theta_t^{Z}=\sqrt{1-\rho^2}\theta_t;\quad \theta_t^{\mathrm{N}}\equiv 0.$$

Hence, according to the wealth process and the stochastic variance process, and solving the HJB equation, the optimal portfolio weights are explicitly given by

$$\sigma\frac{g_v}{O_t}\psi_t^{\mathrm{BZN,V}}=\theta_t^{\mathrm{BZN,V}}=\frac{\rho\eta+\sqrt{1-\rho^2}\xi}{\gamma}+\sigma H^{\mathrm{BZN,V}}(T-t).$$

The following two propositions describe how adding the stock and the stock and equity derivatives to a portfolio that only holds the VIX derivatives can lead to portfolio improvements. Intuitively, they shall improve the portfolio, and the results below provide a rigorous theoretical justification of the intuition.

Proposition 3.3. In model BZN, for $\gamma\neq 1$, by Eq. (11), the portfolio improvement from including the stock in portfolio V is

$$R^{\mathrm{BZN,SV-V}}=\frac{\gamma}{1-\gamma}(\Delta r_{\mathrm{VS}}+\Delta R_{\mathrm{VS}}V_0)\geq 0,$$

where

$$\Delta r_{\mathrm{VS}}=\frac{h^{\mathrm{BZN,SV}}(T)-h^{\mathrm{BZN,V}}(T)}{T},\Delta R_{\mathrm{VS}}=\frac{H^{\mathrm{BZN,SV}}(T)-H^{\mathrm{BZN,V}}(T)}{T}.$$

The equality holds if and only if

$$(1-\rho^2)\left(\eta-\frac{\rho}{\sqrt{1-\rho^2}}\xi\right)-\mu(\lambda^{\mathrm{Q}}-\lambda)=0.$$

Proposition 3.4. In model BZN, for $\gamma\neq 1$, by Eq. (10), the portfolio improvements from including the stock and the equity derivatives in portfolio V is

$$R^{\mathrm{BZN,V}}=\frac{\gamma}{1-\gamma}(\Delta r_V+\Delta R_V*V_0)\geq 0,$$

where

$$\Delta r_{\mathrm{V}}=\frac{h^{\mathrm{BZN,*}}(T)-h^{\mathrm{BZN,V}}(T)}{T},\Delta R_{\mathrm{V}}=\frac{H^{\mathrm{BZN,*}}(T)-H^{\mathrm{BZN,V}}(T)}{T}.$$

The equality holds if and only if

$$\frac{\eta}{\rho}=\frac{\xi}{\sqrt{1-\rho^2}},\quad \lambda=\lambda^{\mathrm{Q}}.$$

Remark 3.4. Note the following identity: $R^{\mathrm{BZN,V}}=R^{\mathrm{BZN,SV-V}}+R^{\mathrm{BZN,SV}}$ holds in due to Eqs. (10) and (11). Then we can see that the equality case in Proposition 3.4 is equivalent to the condition that $R^{\mathrm{BZN,SV}}=0$ and $R^{\mathrm{BZN,SV-V}}=0$, since the two inequalities have to be equalities for the final equality to hold, i.e.,

$$\frac{1}{\mu}\left(\left(\frac{\lambda}{\lambda^{\mathrm{Q}}}\right)^{\frac{1}{\gamma}}-1\right)=\frac{\eta}{\gamma}-\frac{\rho}{\sqrt{1-\rho^2}}\frac{\xi}{\gamma},$$
$$(1-\rho^2)\left(\eta-\frac{\rho}{\sqrt{1-\rho^2}}\xi\right)-\mu(\lambda^{\mathrm{Q}}-\lambda)=0,$$

which leads to

$$0\leq\frac{1}{\mu}\left(\left(\frac{\lambda}{\lambda^{\mathrm{Q}}}\right)^{\frac{1}{\gamma}}-1\right)=\frac{\eta}{\gamma}-\frac{\rho}{\sqrt{1-\rho^2}}\frac{\xi}{\gamma}=\frac{\mu(\lambda^{\mathrm{Q}}-\lambda)}{\gamma(1-\rho^2)}\leq 0,$$

and this implies

$$\frac{\eta}{\rho}=\frac{\xi}{\sqrt{1-\rho^2}},\quad \lambda=\lambda^{\mathrm{Q}}.$$

① It is common practice to trade VIX futures, e.g., the VIX short-term futures ETF (ticker VXX), with proper leveraging by borrowing from the bank at a risk-free rate.

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We can define the certainty equivalent wealth for the two portfolios, respectively, the portfolio with only stocks and the portfolio with only VIX derivatives.

$$\frac{(W^{S})^{1-\gamma}}{1-\gamma}=J^{BZN,S}(0,W_0,V_0), \tag{14}$$

and

$$\frac{(W^{V})^{1-\gamma}}{1-\gamma}=J^{BZN,V}(0,W_0,V_0). \tag{15}$$

Since the value function $J^{BZN,V}$ for this case does not depend on the jump parameters μ,λ and λ^{Q}, we can compare the sensitivity of the certainty equivalent wealth W^{S} and W^{V} with respect to the other parameters when we take different values of the jump parameter. Assume $W_0=1$, and we first consider the three cases $\mu=-0.1$, $\mu=-0.15$, and $\mu=-0.25$, which represent three different regimes for the jump magnitude. The larger the absolute value of μ, the more severe the asset price jump is. The sensitivity of W^{S} with respect to other model parameters is shown in Fig. 2. From the figure, it is clear that the certainty equivalent wealth level is the highest when $\mu=-25\%$ and is the lowest when $\mu=-10\%$. They are increasing as we increase the volatility level, the volatility of volatility and the investment horizon. They decrease as the risk-aversion level increases.

Similarly, we can consider the sensitivity of W^{V} with respect to the other model parameters; see Fig. 3.

We can see that under different values of the parameter μ, the certainty equivalent wealth W^{S} changes, but the certainty equivalent wealth W^{V} stays the same under different μ values. This is consistent with our theoretical finding that W^{V} is insensitive to the jump components. Another interesting observation arises when we compare Fig. 2 with Fig. 3 : the certainty equivalent wealth W^{S} does not change as we change the volatility risk premium level ξ, while the certainty equivalent wealth W^{V} exhibits a U-shaped pattern when we vary the volatility risk premium ξ. This indicates that W^{S} does not reflect the volatility risk premium, while W^{V} does and reaches the minimum level when ξ is approximately 4. A similar pattern is also observed in Ref. [14, Fig. 2], but there is no explanation for the shape therein. The reason is that since VIX^2 is linear in V, and V is driven by the Brownian motion $\rho B(t)+\sqrt{1-\rho^2}Z(t)$, which has the premium $\rho\eta+\sqrt{1-\rho^2}\xi$. When ξ is negative with a large absolute value, the premium is negative, and then we can sell VIX futures to obtain the premium. When ξ is positive with a large absolute value, the premium is positive, and we can buy VIX futures to obtain the premium. When $\rho\eta+\sqrt{1-\rho^2}\xi=0$, no premium can be obtained; thus, the demand of VIX is 0, and the certainty equivalent wealth is the same as its degenerated case.

Next, we consider the cases $\frac{\lambda^{Q}}{\lambda}=1$, $\frac{\lambda^{Q}}{\lambda}=3$, and $\frac{\lambda^{Q}}{\lambda}=5$, and we have the sensitivity of W^{S} displayed in Fig. 4. Note that $\frac{\lambda^{Q}}{\lambda}$ is the coefficient for the jump-risk premium.

Similarly, the sensitivity for W^{V} under different $\frac{\lambda^{Q}}{\lambda}$ is given by Fig. 5. We can see that under different values of the parameter $\frac{\lambda^{Q}}{\lambda}$, the certainty equivalent wealth W^{S} changes, but the certainty equivalent wealth W^{V} stays the same. This is consistent with the intuitions.

Remark 3.5. From the above discussions, we can conclude that the certainty equivalent wealth for the investor who only trades VIX derivatives is more robust to the jump parameters compared to the certainty equivalent wealth for the in-

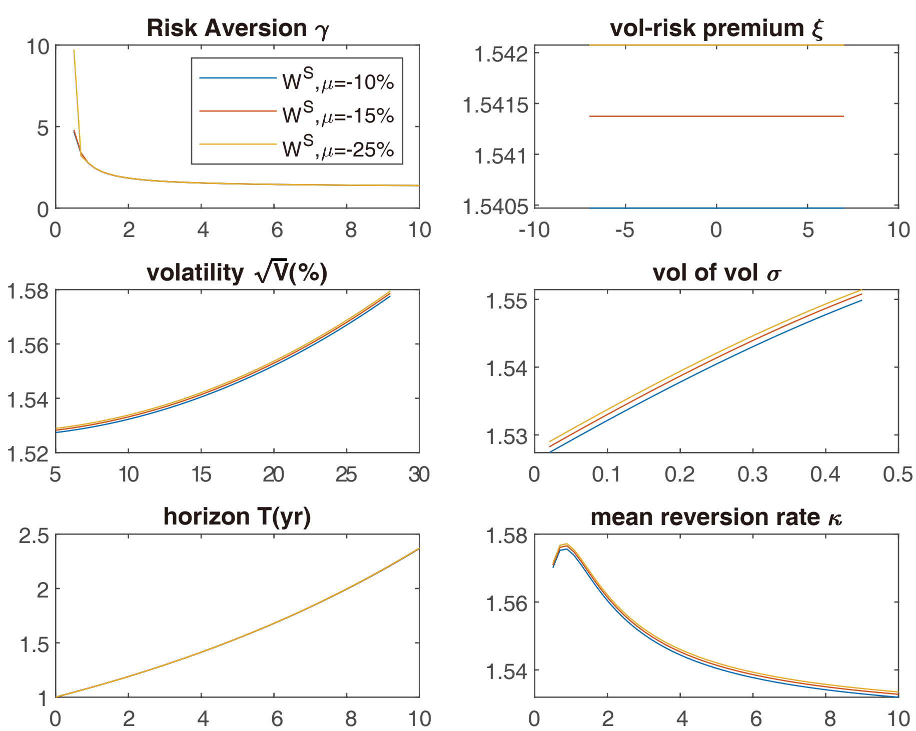

Fig. 2. Sensitivity of certainty equivalent wealth W^{S} for different values of μ. The y-axes represent the certainty equivalent wealth levels for respectively the three cases $\mu=-0.1$, $\mu=-0.15$, and $\mu=-0.25$.

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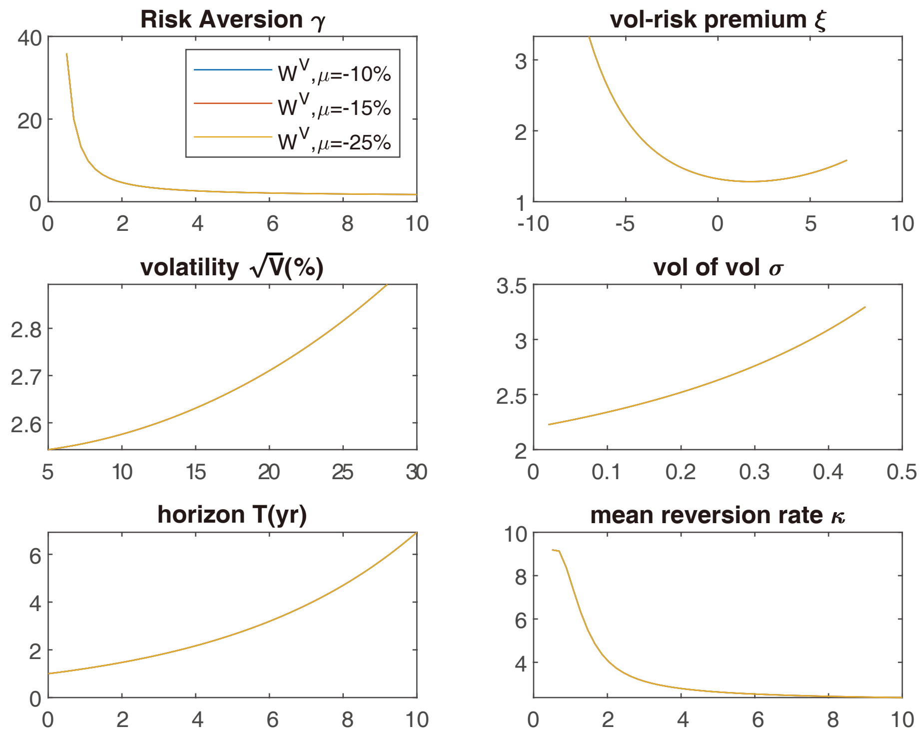

Fig. 3. Sensitivity of certainty equivalent wealth W^V for different values of μ. The y-axes represent the certainty equivalent wealth levels for respectively the three cases $\mu=-0.1$, $\mu=-0.15$, and $\mu=-0.25$.

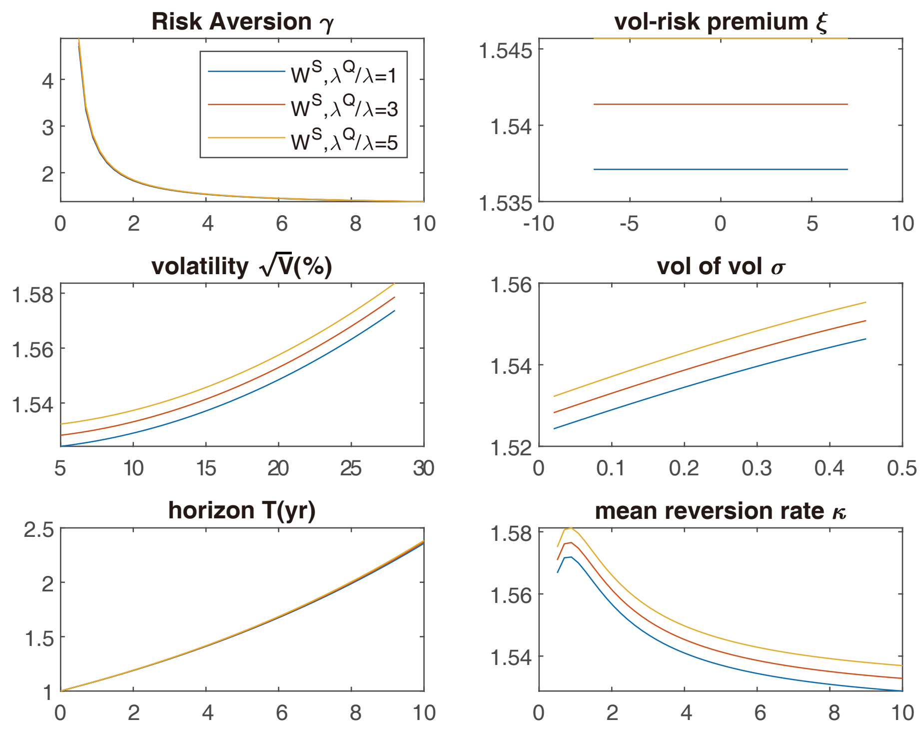

Fig. 4. Sensitivity of certainty equivalent wealth W^S for different values of $\frac{\lambda^Q}{\lambda}$. The y-axes represent the certainty equivalent wealth levels for respectively the three cases $\frac{\lambda^Q}{\lambda}=1$, $\frac{\lambda^Q}{\lambda}=3$ and $\frac{\lambda^Q}{\lambda}=5$.

vestor who only trades stocks. It should also be noted that the optimal strategies (portfolio weights) for the investor who trades only the VIX derivatives are subject to change when the jump parameters change, but the expected terminal utility is not affected by the jump parameters. Put another way, the effects of jumps influence the optimal strategy that an investor takes, but they do not affect the final optimal value of the portfolio.

The reason is that as we can see in Eq. (8), the form of VIX^2 contains jump parameters, which is due to the VIX

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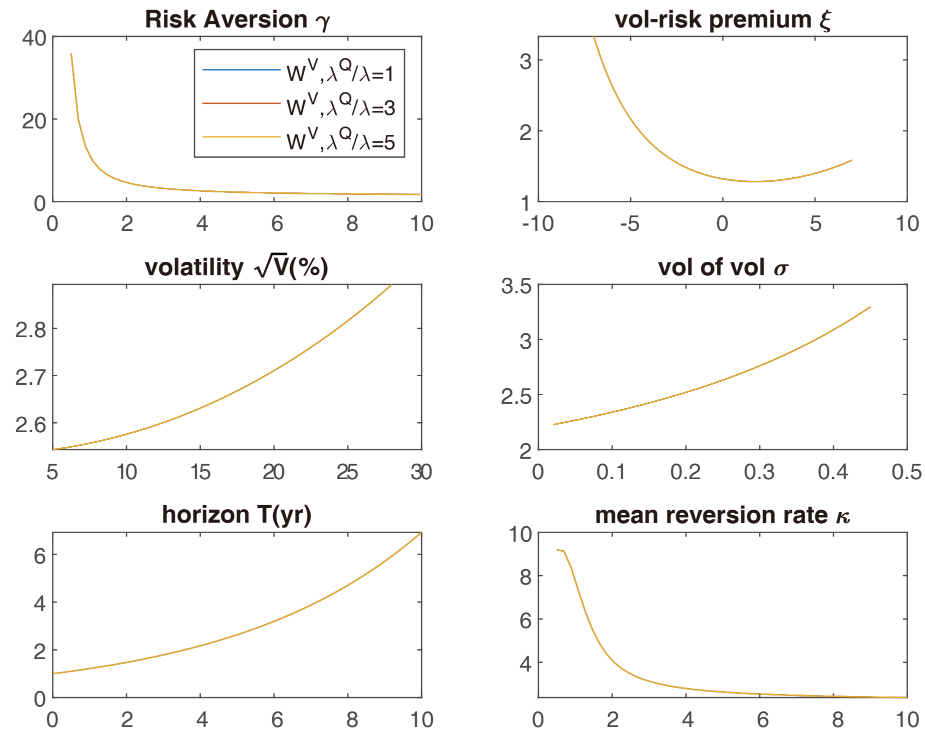

Fig. 5. Sensitivity of certainty equivalent wealth W^V for different values of $\frac{\lambda^Q}{\lambda}$. The y-axes represent the certainty equivalent wealth levels for respectively the three cases $\frac{\lambda^Q}{\lambda}=1$, $\frac{\lambda^Q}{\lambda}=3$ and $\frac{\lambda^Q}{\lambda}=5$.

formulation in Ref. [16]. Hence, the jump parameter will for sure affect the portfolio. However, when we maximize the expected terminal utility, the utility depends on the jump parameters in VIX^2 only through the stochastic variance V, so the optimal expected terminal utility can be determined from the parameters of V (and the premium of the risk factor for the dynamic of V and risk aversion γ without specifying the parameters for VIX).

In practice, an investor who only trades the VIX derivatives will first determine the exposure to the risk factor for V, i.e., $\rho B(t)+\sqrt{1-\rho^2}Z(t)$, then the optimal expected terminal wealth is thus determined from the parameters of V (and the premium of the risk factor for V, i.e., $\rho\eta+\sqrt{1-\rho^2}\xi$ and risk aversion γ). Since the investor cannot directly trade stochastic variance V itself, then the investor needs to determine the portfolio weight for VIX from the exposure to the risk factor for V. Since VIX depends on the jump parameters, then the investor also needs to consider jump parameters to balance the portfolio weight and match the optimal exposure to the risk factor for V.

The implication is that investing in volatility derivatives is not sensitive to sudden jumps in asset prices if we look at the final optimal value achieved. Here, we assume that the investor's optimal strategy takes into account the full information revealed from the asset jumps. In practice, it may be possible that the investor will not be immediately adjusting her strategy when facing an asset price jump, e.g., market crash and asset sell-offs, and in these practical cases, the final optimal wealth will certainly be affected by the jumps.

3.3 Numerical illustrations of portfolio improvement

In this section, we plot the sensitivity of the portfolio improvements measurement R^{inc} (in percentage) for each model and each portfolio in the incomplete market case. We also plot W^* and W^{inc} with the assumption that $W_0=1$. This comparison provides us with intuition about the magnitude of the improvements in portfolio performance when we introduce a particular asset into the portfolio. The parameters for each case (if used) are given below as benchmarks (see Refs. [17]):

$$\eta=4,\rho=-0.4,\bar{v}=0.13^2,r=0.05,\lambda=0.1;$$
$$\gamma=4,\xi=-6,\sqrt{V}=0.15,\sigma=0.25,T=5,\kappa=5,$$
$$\mu=-0.15,\lambda^Q=0.3.$$

For model BZN, the three different portfolio improvements in the return of certainty equivalent wealth R are given in Fig. 6. Here, we use the shorthand notation R^{SV} to denote $R^{\mathrm{BZN,SV}}$, as it is clear that we are considering model BZN throughout. This applies to the other notations R^{S} and R^{V}. Similarly, the three improvements in the certainty equivalent wealth W are given in Fig. 7. Recall that the certainty equivalent represents the amount of guaranteed money an investor would accept now instead of taking a risk of getting more money at a future date. It can be seen that in most cases $W^{\mathrm{V}}>W^{\mathrm{S}}$ holds, and intuitively this means that investors would prefer to invest purely in VIX derivatives as compared with investing in the stock, given that we use the certainty equivalent wealth as the comparison criteria. It can also be seen that R^{SV} is always close to 0 (approximately 0 to 0.15 and reaches 0.15 as λ^Q reaches 0.5), which is also reflected in Fig. 7 that W^{SV} is very close to W^* as they overlap. This intuitively implies that the certainty equivalent wealth for the investors who only trade stocks S and VIX derivatives V almost reaches that for the complete market cases under the

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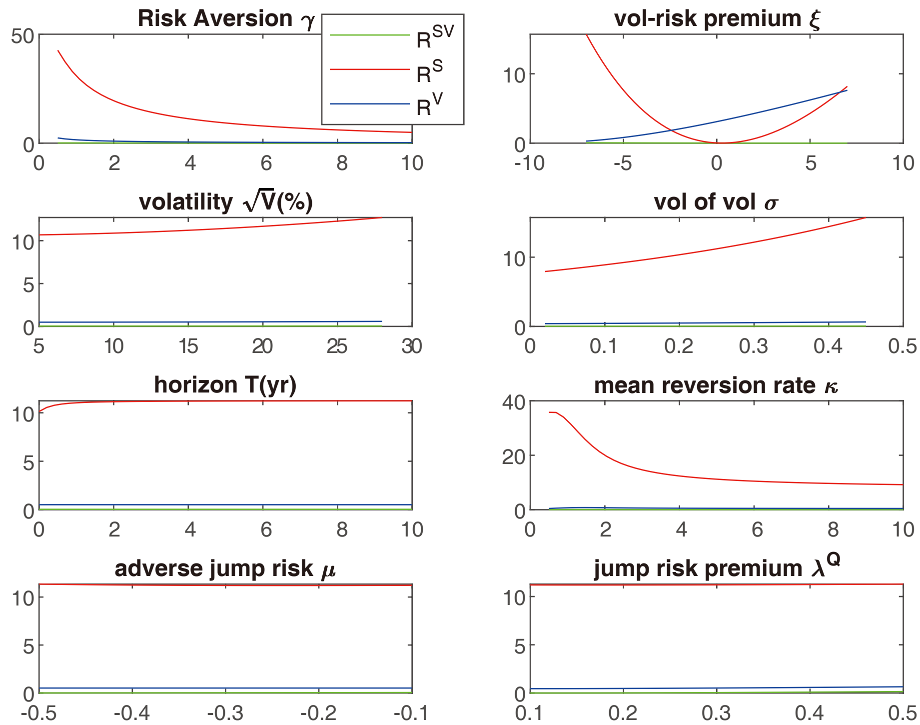

Fig. 6. Portfolio improvement in R in model BZN. The y-axes represent the return of certainty equivalent wealth levels.

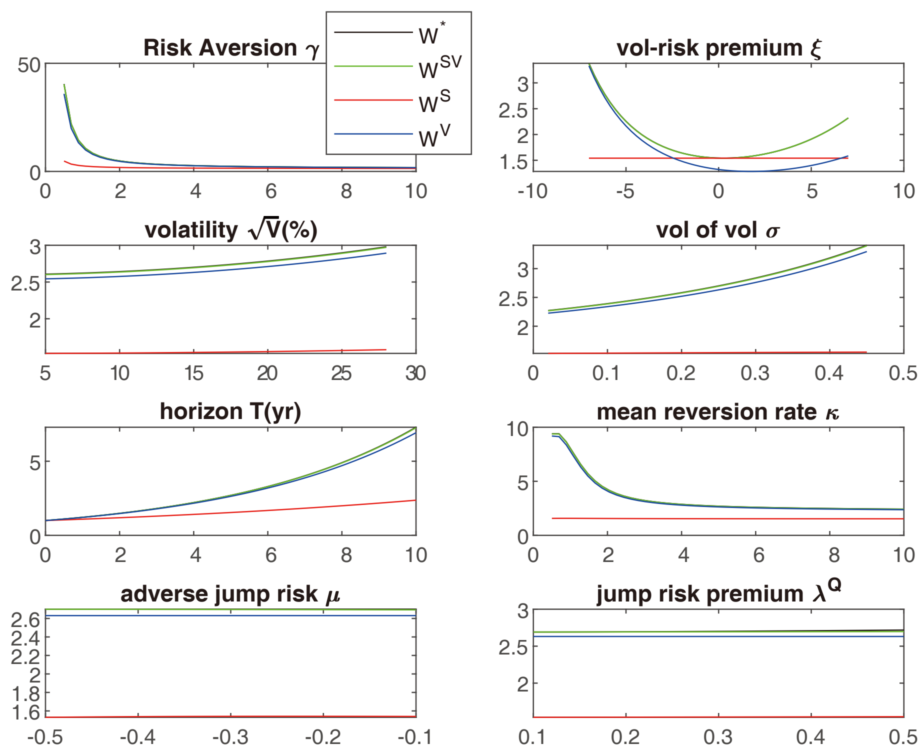

Fig. 7. Portfolio improvement in W in model BZN. The y-axes represent the certainty equivalent wealth levels.

current parameter setting. In other words, there is not much demand for including equity derivatives to complete the market, since the improvements is insignificant.

4 Conclusions

In this paper, we consider the optimal investment problem when the investor has access not only to the equity derivatives, but also to the VIX derivatives. We solve the optimal investment in closed-form or semi-closed-form for different combinations of investment products and risk factors in both complete and incomplete markets. We demonstrate through theoretical results that the VIX derivatives, when they are included in the opportunity set of the portfolio, strictly improve

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the portfolio when measured using certainty equivalent wealth. We believe that this theoretical study on the inclusion of VIX derivatives into the optimal investment problem will shed light on how to maintain a volatility-managed portfolio[18], which are portfolios that actively manage exposure to volatility risk and take less risk during recessions. Future research should include indifference pricing when both equity derivatives and VIX derivatives are present. Another possible extension is to consider a recently introduced volatility model named the rough volatility model[19], and solve the corresponding optimal investment problem.

Supporting information

The supporting information for this article can be found online at http://doi.org/10.52396/JUSTC-2022-0095. It includes the proofs of all the propositions.

Conflict of interest

The authors declare that they have no conflict of interest.

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References

[1] Merton R C. Optimal consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory,* **1971**, *3*: 373–413.

[2] Ye W, Wu B, Chen P. Pricing VIX derivatives using a stochastic volatility model with a flexible jump structure. *Probability in the Engineering and Informational Sciences,* **2023**, *37* (1): 245–274.

[3] Cui Z, Lee C, Liu M. Valuation of VIX derivatives through combined Ito-Taylor expansion and Markov chain approximation. Hoboken, NJ: Stevens Institute of Technology, **2021**.

[4] Chen T, Zhuo H. Pricing VIX options with realized volatility. *Journal of Futures Markets,* **2021**, *41* (8): 1180–1200.

[5] Yin F, Bian Y, Wang T. A short cut: Directly pricing VIX futures with discrete-time long memory model and asymmetric jumps. *Journal of Futures Markets,* **2021**, *41* (4): 458–477.

[6] Yuan P. Time-varying skew in VIX derivatives pricing. *Management Science,* **2021**, *68* (10): 7065–7791.

[7] Mencía J, Sentana E. Valuation of VIX derivatives. *Journal of Financial Economics,* **2013**, *108* (2): 367–391.

[8] Simon D P. Trading the VIX futures roll and volatility premiums with VIX options. *Journal of Futures Markets,* **2017**, *37* (2): 184–208.

[9] Simon D P, Campasano J. The VIX futures basis: Evidence and trading strategies. *The Journal of Derivatives,* **2014**, *21* (3): 54–69.

[10] Chen H C, Chung S L, Ho K Y. The diversification effects of volatility-related assets. *Journal of Banking & Finance,* **2011**, *35* (5): 1179–1189.

[11] Black K H. Improving hedge fund risk exposures by hedging equity market volatility, or how the VIX ate my kurtosis. *The Journal of Trading,* **2006**, *1* (2): 6–15.

[12] Daigler R T, Rossi L. A portfolio of stocks and volatility. *The Journal of Investing,* **2006**, *15* (2): 99–106.

[13] Carr P, Jin X, Madan D P. Optimal investment in derivative securities. *Finance and Stochastics,* **2001**, *5* (1): 33–59.

[14] Liu J, Pan J. Dynamic derivative strategies. *Journal of Financial Economics,* **2003**, *69* (3): 401–430.

[15] Bates D S. Jumps and stochastic volatility: Exchange rate processes implicit in deutsche mark options. *The Review of Financial Studies,* **1996**, *9* (1): 69–107.

[16] Lian G H, Zhu S P. Pricing VIX options with stochastic volatility and random jumps. *Decisions in Economics and Finance,* **2013**, *36*: 71–88.

[17] Escobar M, Ferrando S, Rubtsov A. Robust portfolio choice with derivative trading under stochastic volatility. *Journal of Banking & Finance,* **2015**, *61*: 142–157.

[18] Moreira A, Muir T. Volatility-managed portfolios. *The Journal of Finance,* **2017**, *72* (4): 1611–1644.

[19] Gatheral J, Jaisson T, Rosenbaum M. Volatility is rough. *Quantitative Finance,* **2018**, *18* (6): 933–949.

Appendix

A.1 Model with two risk factors B and Z

In this section, we consider an economy with the diffusive risk B and the volatility risk Z, but no jump risk N. Similar to Ref. [14, Example 1], we can disable the jump component by setting $\mu = 0, \lambda = \lambda^{Q} = 0$ in the cases of model with three risk factors, then we end up with a model with only two risk factor B and Z.

A.1.1 Complete market case

After substituting $\mu = 0, \lambda = \lambda^{Q} = 0$ into Eq. (5), we can obtain the wealth process, and by solving the HJB equation, the optimal portfolio weights on the risk factors B, Z, and N are given by

$$\theta_t^{B^{BZ,*}} = \frac{\eta}{\gamma} + \sigma\rho H^{BZ,*}(T-t), \quad \theta_t^{Z^{BZ,*}} = \frac{\xi}{\gamma} + \sigma\sqrt{1-\rho^2} H^{BZ,*}(T-t).$$

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Transforming to the optimal portfolio weights, we have

$$\psi_t^{\mathrm{BZ},*} = \frac{O_t}{g_v}\left(\frac{\xi}{\gamma\sigma\sqrt{1-\rho^2}} + H^{\mathrm{BZ},*}(T-t)\right), \quad \phi_t^{\mathrm{BZ},*} = \frac{\eta}{\gamma} - \frac{\rho}{\sqrt{1-\rho^2}}\frac{\xi}{\gamma} - \frac{g_s S_t}{O_t}\psi_t^{\mathrm{BZ},*}.$$

A.1.2 Portfolio with only stocks and risk-free assets

For portfolio with only stocks (and risk-free assets), we have $\psi_t \equiv 0, \theta_t^B = \theta_t^N = \phi_t$. Then according the wealth process and stochastic variance process, and by solving the HJB equation, the optimal portfolio weight is given by $\phi_t^{\mathrm{BZ,S}} = \frac{\eta}{\gamma} + \sigma\rho H^{\mathrm{BZ,S}}(T-t)$.

Note that this result has also been derived in Ref. [14], as it considered the case of equity derivatives. Next, we consider the portfolio improvement.

Proposition A.1. In model BZ, for $\gamma \neq 1$, the portfolio improvement from including VIX derivatives in portfolio S is

$$R^{\mathrm{BZ,S}} = \frac{\gamma}{1-\gamma}\left(\frac{h^{\mathrm{BZ},*}(T) - h^{\mathrm{BZ,S}}(T)}{T} + \frac{H^{\mathrm{BZ},*}(T) - H^{\mathrm{BZ,S}}(T)}{T}V_0\right) > 0.$$

Remark A.1. Alternatively, we can argue that $R^{\mathrm{BZ,S}} \geq 0$ since

$$J^{\mathrm{BZ,S}}(t,w,v) = \max_{\phi_t,\psi_t} E\left[\frac{W^{1-\gamma}}{1-\gamma}|W_t = w, V_t = v\right]$$
$$\text{s.t. } \psi_t \equiv 0.$$

Recall that $J^{\mathrm{BZ},*}(t,w,v)$ is the solution of the corresponding non-constraint optimization problem. And the equality $R^{\mathrm{BZ,S}} = 0$ holds if and only if $\psi_t^{\mathrm{BZ},*} \equiv 0$, since

$$\psi_t^{\mathrm{BZ},*} = \frac{\xi}{\gamma} + \sigma\sqrt{1-\rho^2}H^{\mathrm{BZ},*}(T-t).$$

Here, $H^{\mathrm{BZ},*}(\tau)$ is a constant only if $k_2^{\mathrm{BZ},*} = 0$ or $\delta^{\mathrm{BZ},*} = 0$, which results in $H^{\mathrm{BZ},*}(\tau) \equiv 0$, and $H^{\mathrm{BZ},*\prime}(\tau) \equiv 0$, then $\delta^{\mathrm{BZ},*} = 0$. But we also have $\delta^{\mathrm{BZ},*} = \frac{1-\gamma}{\gamma^2}(\eta^2+\xi^2)$, so we get a contradiction, thus the equality $R^{\mathrm{BZ,S}} = 0$ cannot be achieved.

A.1.3 Portfolio with only VIX derivatives and risk-free assets

For portfolio with only VIX derivatives, let $\phi_t = 0$, $\theta_t = \sigma\frac{g_v}{O_t}\psi_t$, then $\theta_t^{\mathrm{B}} = \rho\theta_t$; $\theta_t^{\mathrm{Z}} = \sqrt{1-\rho^2}\theta_t$. According the wealth process and stochastic variance process, and by solving the HJB equation, the optimal portfolio weight is given by

$$\sigma\frac{g_v}{O_t}\psi_t^{\mathrm{BZ,V}} = \theta_t^{\mathrm{BZ,V}} = \frac{\rho\eta + \sqrt{1-\rho^2}\xi}{\gamma} + \sigma H^{\mathrm{BZ,V}}(T-t).$$

Proposition A.2. In model BZ, for $\gamma \neq 1$, the portfolio improvement from including the stock in the portfolio V is

$$R^{\mathrm{BZ,V}} = \frac{\gamma}{1-\gamma}\left(\frac{h^{\mathrm{BZ},*}(T) - h^{\mathrm{BZ,V}}(T)}{T} + \frac{H^{\mathrm{BZ},*}(T) - H^{\mathrm{BZ,V}}(T)}{T}V_0\right) \geqslant 0.$$

The equality holds if and only if $\frac{\eta}{\rho} = \frac{\xi}{\sqrt{1-\rho^2}}$.

Remark A.2. Similarly we can argue that $R^{\mathrm{BZ,V}} \geqslant 0$, since portfolio V has more constraints and equality holds if and only if $\phi_t^{\mathrm{BZ},*} \equiv 0$, i.e., $\phi_t^{\mathrm{BZ},*} = \frac{\eta}{\gamma} - \frac{\rho}{\sqrt{1-\rho^2}}\frac{\xi}{\gamma} = 0$, or $\frac{\eta}{\rho} = \frac{\xi}{\sqrt{1-\rho^2}}$.

A.2 Model with two risk factors B and N for complete market cases

In this section, we consider the model with diffusive risk B and jump risk N, but no volatility risk Z, i.e., the volatility is constant. Similar to Ref. [14, Example 2], we can make the stochastic variance V_t constant at $\bar{v}$ by setting $\sigma = 0, V_0 = \bar{v}$ in the previous case of three factors model. Then the factor Z disappeared and the model only has two factors B and N. Since VIX derivatives do not have exposure to the jump risk , the portfolio containing VIX derivatives would degenerate to the portfolio with remaining assets.

Since $\theta_t^{\mathrm{Z}} = \sigma\sqrt{1-\rho^2}\frac{g_v}{O_t}\psi_t \equiv 0$, according to the wealth process and the stochastic variance process, and solving the HJB equation, then the optimal portfolio weights are given by

$$\phi_t^{\mathrm{BN},*} = \frac{\eta}{\gamma} - \frac{g_s S_t}{O_t}\psi_t^{\mathrm{BN},*}, \quad \psi_t^{\mathrm{BN},*} = \left(\frac{g_s S_t}{O_t} - \frac{\Delta g}{\mu O_t}\right)^{-1}\left(\frac{\eta}{\gamma} - \frac{1}{\mu}\left(\left(\frac{\lambda}{\lambda^{\mathrm{Q}}}\right)^{\frac{1}{\gamma}} - 1\right)\right).$$

It should be noted that, if we trade stock and VIX derivatives, the non-redundant condition is not satisfied, thus the market is

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incomplete. Because $g_s \equiv 0$ and $\Delta g \equiv 0$, ψ has no contribution to the risk factors θ_t^{B} and θ_t^{N}, thus it reduces to the case that the investor only trade the stock.

A.3 Portfolio with only stocks and risk-free assets

For a portfolio with one stock, we have $\theta_t^{\mathrm{B}} = \theta_t^{\mathrm{N}} = \phi_t$. Then according to the wealth process and stochastic variance process, and solving the HJB equation, and the optimal weight $\phi_t^{\mathrm{BN,S}}$ are determined by

$$\eta - \lambda^{\mathrm{Q}}\mu - \gamma\phi_t^{\mathrm{BN,S}} + \lambda\mu(1+\mu\phi_t^{\mathrm{BN,S}})^{-\gamma} = 0$$

and the solution of above equation exists and unique. Note that the above result has also been derived in Ref. [14], and we demonstrate that it can be derived based on our general framework.

Proposition A.3. In model BN, for $\gamma \neq 1$, the portfolio improvement from including equity derivatives is

$$R^{\mathrm{BN,S}} = \frac{\gamma}{1-\gamma}\left(\frac{1}{2}\gamma\bar{v}(\delta^{\mathrm{BN,*}} - \delta^{\mathrm{BN,S}})\right) \geqslant 0.$$

The equality holds if and only if $\dfrac{\lambda^{\mathrm{Q}}}{\lambda} = \left(1+\mu\dfrac{\eta}{\gamma}\right)^{-\gamma}$.

A.4 Portfolio with only VIX derivatives and risk-free assets

For a portfolio with VIX derivative, denote $\theta_t = \sigma\dfrac{g_v}{O_t}\psi_t = 0$, so that

$$\theta_t^{\mathrm{B}} = \rho = 0; \quad \theta_t^{\mathrm{Z}} = \sqrt{1-\rho^2}\theta_t = 0; \quad \theta_t^{\mathrm{N}} = 0.$$

Then according to the wealth process and stochastic variance process, and solving the HJB equation, the optimal value does not depend on the optimal weight ψ_t, i.e., $\psi_t^{\mathrm{BN,S}}$ can be any real number.

Proposition A.4. In model BN, for $\gamma \neq 1$, the portfolio improvement from including the stock into portfolio V is given by

$$R^{\mathrm{BN,S-V}} = \frac{\gamma}{1-\gamma}\left(\frac{1}{2}\gamma\bar{v}\delta^{\mathrm{BN,S}}\right) > 0.$$

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