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Intersection complex via residues

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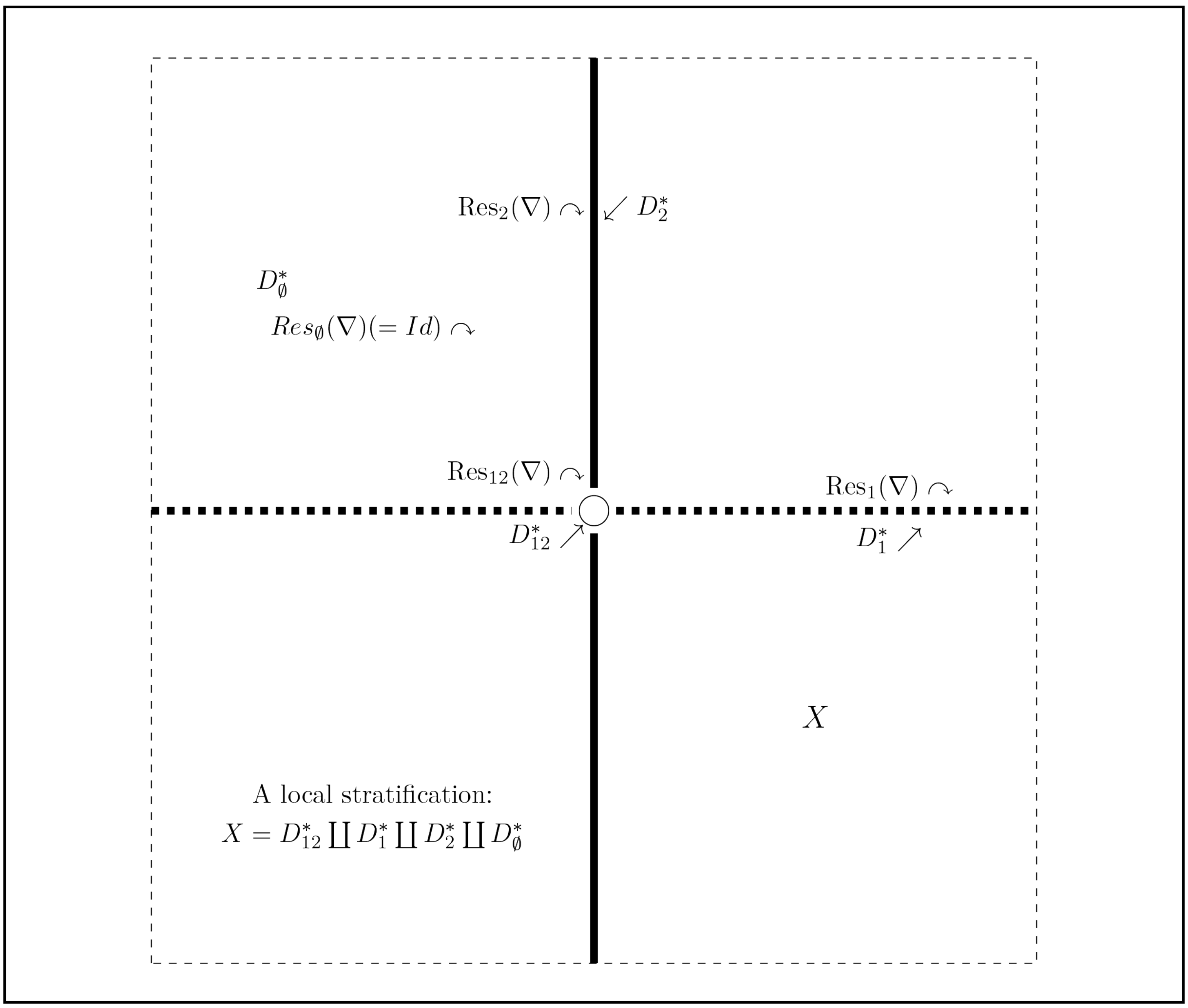
Graphical abstract



A normal crossing divisor gives rise to a stratification of a smooth scheme, and a logarithmic connection of a vector bundle along the divisor induces residue maps along each stratums.

Public summary

■ We provide an intrinsic definition of intersection subcomplex via these residues.

■ We present its explicit geometric description.

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Abstract: We provide an intrinsic algebraic definition of the intersection complex for a variety.
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1 Introduction

Intersection homology theory is a generalization of singular homology for singular algebraic varieties. In Ref. [1], Sheng and Zhang established a positive characteristic analog of an intersection cohomology theory for polarised variations of Hodge structures and proposed an algebraic definition of the intersection complex, but with the help of coordinate systems. Here, we provide an intrinsic definition of the intersection complex via residues and provide a geometric description of it.

The remainder of this paper is organized as follows. Section 2 establishes notations and presents key definitions. Section 3 provides the main theorem and its proof. Finally, in Section 4, an explicit computation following the spirit of proof in surface case is made, and a counterexample is discussed.

2 Intersection complex

Let (X,D) be a smooth scheme over a regular locally Noetherian scheme S with a reduced smooth normal crossing divisor $D=\sum_{i\in\mathcal{I}} D_i$, where $\mathcal{I}$ is a finite index set, and ε be a locally free coherent sheaf with an integrable logarithmic λ-connection ∇ along D.

We introduce some natural morphisms of log-differential sheaves before providing our definitions.

Suppose X is of relative dimension n over S. Owing to smoothness of X and the definition of simple normal crossing divisors, for any $x\in X$, there exists a neighborhood U of x such that we can find a coordinate system $(t_1,\cdots,t_r;\ t_{r+1},\cdots,t_n)$ such that $D\cap U$ is defined by the equation $t_1\cdot t_2\cdots t_r=0$. As an immediate result, $\Omega^1_{U/S}(\log D)$ admits an O_U basis

$$\{\omega_1=\mathrm{dlog}\,t_1,\cdots,\omega_r=\mathrm{dlog}\,t_r;\ \omega_{r+1}=\mathrm{d}\,t_{r+1},\cdots,\omega_n=\mathrm{d}\,t_n\}.$$

Moreover, it induces a free system of generators for $\Omega^a_{U/S}(\log D)$.

$$\{\omega_I=\omega_{i_1}\wedge\cdots\wedge\omega_{i_a}|I=\{i_1,i_2\cdots,i_a\},\ \text{with}\ 1\leqslant i_1<\cdots<i_a\leqslant n\}.$$

For $1\leqslant i,j\leqslant r$ and $a\geqslant 1$, we define

$$\beta_i^a:\Omega^a_X(\log D)\longrightarrow\Omega^{a-1}_{D_i}(\log(D-D_i)|_{D_i}),$$
$$\phi'+\phi_i\wedge\mathrm{dlog}\,t_i\longrightarrow\phi_i|_{D_i},$$

where ϕ' lies in span ω_I with $i\notin I$.

$$\gamma_j^a:\Omega^a_{X/S}(\log(D-D_j))\longrightarrow\Omega^a_{D_j/S}\left(\log(D-D_j)|_{D_j}\right),$$
$$\sum_{j\in I}a_I\omega_I+\sum_{j\notin I}a_I\omega_I\longrightarrow\sum_{j\notin I}a_I\omega_I|_{D_j}.$$

One can consider β_i^a as taking the residual part of a log differential form along D_i, and γ_j^a is the restriction of the D_i regular log differential forms to D_i. Obviously, β_i^a and γ_j^a are surjective and independent of the coordinate system, respectively. For simplicity, we omit the upper symbol a.

Clearly, for any log connection ∇, the composite map $(\beta_i\otimes Id)\circ\nabla$ factors through γ_i

$$\varepsilon\xrightarrow{\gamma_i}O_{D_i}\otimes\varepsilon\longrightarrow O_{D_i}\otimes\varepsilon.$$

We call the second map the residue map of ∇ along D_i, and denote it as $\mathrm{Res}_i(\nabla)$.

We can generalize morphisms above to the multi-indices case as follows. For a subset $I=\{j_1,\cdots,j_a\}\subseteq\{1,2,\cdots,r\}$ with $j_1<j_2<\cdots<j_a$, set $D_I=\cap_{i\in I}D_i$, and define the residue Res_I of the connection ∇ along D_I as follows:

$$\mathrm{Res}_{j_1}(\nabla)\circ\mathrm{Res}_{j_2}(\nabla)\circ\cdots\circ\mathrm{Res}_{j_a}(\nabla).$$

We define β_I and γ_I in a similar manner.

The following diagram naturally commutes.

$$\begin{array}{ccc}\varepsilon\otimes\Omega^a_{X/S}(\log(D-D_j)) & \xrightarrow{\nabla\circ(\text{inclusion})} & \varepsilon\otimes\Omega^{a+1}_{X/S}(\log(D)),\\ \downarrow l_i\otimes\gamma_j & & \downarrow l_I\otimes\beta_j\\ \varepsilon\otimes\Omega^a_{D_j/S}(\log(D-D_j))|_{D_j} & \xrightarrow{(-id)^a\otimes\mathrm{Res}_{D_j}} & \varepsilon\otimes\Omega^a_{D_j/S}(\log(D-D_j))|_{D_j},\end{array}\tag{1}$$

where $l_i:\varepsilon\to\varepsilon|_{D_i}$ is the canonical restriction map.

Now we can define the intersection complex. Set

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$$X_{n-s}=\cup_{J\subset I,|J|=k}\cap_{j\in J}D_j,\ 1\leqslant s\leqslant|\mathcal{I}|,$$

then the following descending chain gives rise to a stratification of X:

$$X_n:=X\supset X_{n-1}\supset\cdots\supset X_{n-|\mathcal{I}|}\supset X_{n-|\mathcal{I}|-1}=\emptyset.$$

And let $j_s:U_s:=X-X_s\to U_{s-1}=X-X_{s-1}$ be the natural inclusion for $n-|\mathcal{I}|\leqslant s\leqslant n-1$.

Definition 2.1. Notations as above. We inductively define res-intersection complex IC_r as follows:

• $IC_r^*(\varepsilon,\nabla)|_{U_{n-1}}=IC_r^*(\varepsilon,\nabla)|_{U_{n-1}}$;

• Assume $IC_r^*(H,\nabla)(U_s)$ is defined. A section $\beta\in j_{s*}IC_r^*(H,\nabla)(U_s)$ belongs to $IC_r^*(\varepsilon,\nabla)(U_{s-1})$ when the following two conditions are satisfied:

① β has log pole along $D|_{U_{s-1}}$;

② $\mathrm{Res}_{D_J}\beta\in\mathrm{Im}(\mathrm{Res}_{D_J}\nabla:\varepsilon|_{D_J}\to\varepsilon|_{D_J})\otimes\Omega^{l-n+s}_{D_J/S}$, $\forall J\subset I$ with $|J|=n-s$.

Then we provide a geometric description of res-intersection complex in the sequel of this section. For any subset I of $\mathcal{I}$, let $D_I=\cap_{i\in I}D_i$, $D_I^*=D_I-\cup_{j\notin I}(D_i\cap D_I)$ and let $D_\emptyset^*=X-D$. Set theoretically, we have $X=\coprod_{I\subset\mathcal{I}}D_I^*$. Each D_I^* is a locally closed subspace of X, and thus we can endow D_I^* with reduced subscheme structure.

Proposition 2.1. If $\mathrm{Res}_i(\nabla)$ are bundle morphisms for all $i\in\{1,2,\cdots,r\}$, then the res-intersection complex is a complex of locally free sheaves if it is restricted to each stratum D_I^*, where I is an index subset of $\{1,2,\cdots,r\}$ and D_I^* is endowed with a reduced subscheme structure.

We employ the following lemma to prove Proposition 2.1, see Ref. [2] for details.

Lemma 2.1. Let X be a reduced Noetherian scheme, and let $\mathcal{F}$ be a coherent sheaf on X. Consider the function

$$\phi_{\mathcal{F}}(x)=\dim_{k(x)}\mathcal{F}_x\otimes_{O_x}k(x),$$

where $k(x)=O_x/m_x$ is the residue field at point x. If ϕ is constant, then $\mathcal{F}$ is locally free.

Proof of Proposition 2.1. Consider the reduced scheme D_I^* and its associated coherent sheaf $IC_I^i=IC_r^i(X,\varepsilon)|_{D_I^*}$. Because of the assumption the divisors are reduced, Lemma 2.1, the proposition is proven if we can show that the dimension of the fibre of sheaf, which is $\phi_{IC_I^i}(x)$, is constant over D_I^*.

For each $x\in D_I^*$, $IC_r^*(X,\varepsilon)(x)$ is an $\Omega^*_{X,x}$ module spanned by basis

$$\left\{\overline{\mathrm{Res}}_J(\nabla)(e)\otimes\mathrm{dlog}\,t_J|e \text{ is the fibre of } \varepsilon \text{ and } J \text{ is a subset of } I\right\},$$

where $\overline{\mathrm{Res}}_J(\nabla)$ represents the restriction of $\mathrm{Res}_J(\nabla)$ on fibre. Due to that $\mathrm{Res}_i(\nabla)$ are bundle morphisms, $\phi_{IC_I^i}$ is constant if we restrict it to each degree i and stratum D_I^*. Therefore, $IC_r^*(X,\varepsilon)|_{D_I^*}$ is a complex of locally free sheaves.

Proof of Lemma 2.1. It is a local problem, we may assume $X=\mathrm{Spec}\,A$ and $\mathcal{F}=\tilde{M}$, where A is a reduced commutative local ring with maximal ideal $\mathfrak{m}$ and M is a finite A-module.

We only have to show M is a free A-module. Assume that $k(\mathfrak{m})$ vector space $M/\mathfrak{m}M$ has dimension n. We use Nakayama's lemma to lift the basis for $M/\mathfrak{m}M$ into a set of generators $m_1,m_2,\cdots,m_n$. It is sufficient to demonstrate that m_i is linearly independent. Suppose that $\sum_i a_im_i=0$, where $a_i\in A$. In addition, a_i must lie in $\mathfrak{m}$ for all i, because the generators m_i form the basis of the fibre $M/\mathfrak{m}M$. Choose $\mathfrak{q}\in\mathrm{Spec}\,A$ arbitrarily; then, the images of m_i in $M_{\mathfrak{q}}/\mathfrak{q}M_{\mathfrak{q}}$ generate vector space. In addition, ϕ is constant, implying that they are, in fact, a basis, similarly to $a_i\in\mathfrak{q}$ for all i.

Therefore, a_i lies in the intersection of the prime ideals of A, which is the nilradical of A, and thus $a_i=0$ because A is assumed to be reduced. This completes this proof.

It is interesting to investigate the case where (ε,∇) comes from the polarized variation of Hodge structures. Let us consider a quick recall of this (cf. Ref. [3]). If X is a complex variety, and E is a local system over $X-D$ underlies a polarized variation of Hodge structures, then we obtain a vector bundle (ε,∇) equipped with a flat connection via a Riemann-Hilbert correspondence over $X-D$. There is a canonical extension of ε to a vector bundle with a logarithmic flat connection over X, with the residue of the connection along divisor D_i being the log of the monodromy of the divisor (up to a scalar), which we denote as N_i. It can be observed that N_i is topologically defined.

In Refs. [4, 5], the intermediate extension complex can be fibre-wisely expressed as follows: for $x\in D_I^*$ and a set of coordinates z_i, the fibre of intermediate extension complex at x is an $\Omega^*_{X,x}$ sub-module generated by the sections $\tilde{v}\wedge_{j\in J}\dfrac{\mathrm{d}z_j}{z_j}$ for $v\in N_J\varepsilon$ and $J\subset I$. The differential map of the complex at fibre is defined as

$$\mathrm{d}(\tilde{v}\otimes\mathrm{dlog}\,t_I)=\sum(N_i(v)\otimes\mathrm{dlog}\,t_i)\wedge\mathrm{dlog}\,t_I.$$

Note that the residue of the connection ∇_I is exactly (up to a scalar) the endomorphism N_I if it is restricted to the stratum D_I. It can be easily seen that the res-intersection complex coincides with the intermediate extension complex. From this perspective, we provide an algebraic definition of the intermediate extension complex.

3 Main theorem

In the following, we show that the res-intersection subcomplex above coincides with the intersection subcomplex defined in Ref. [1].

Let X,D,ε be as in the previous section. Given a coordinate system

$$\{t_1,t_2,\cdots,t_r;\ t_{r+1},\cdots,t_n\}$$

of U, locally we can write

$$\nabla=\sum_{i\leqslant r}\nabla_i\mathrm{dlog}\,t_i+\sum_{k>r}\nabla_k\mathrm{d}t_k,$$

due to that the set $\{\mathrm{dlog}\,t_i,\mathrm{d}t_k|i\leqslant r,r+1\leqslant k\leqslant n\}$ forms a basis of the log sheaf. For subset $I=\{j_1,\cdots,j_a\}\subseteq\mathcal{I}$ with $j_1<j_2<\cdots<j_a$, let $\nabla_I=\nabla_{j_1}\circ\nabla_{j_2}\circ\cdots\circ\nabla_{j_a}$. We can generalize diagram (1) as follows:

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$$\begin{array}{ccc} \varepsilon\otimes\Omega_X^{m-|I|}(\log(D-D_I)) & \xrightarrow{\nabla_I\omega_I} & \varepsilon\otimes\Omega_X^m(\log(D)), \\ \gamma_{m-|I|}\downarrow & & \beta_I\downarrow \\ \varepsilon\otimes\Omega_{\cap D_I}^{m-|I|}(\log(D-D_I))|_{\cap D_I} & \xrightarrow{(-id)^{m-|I|}\otimes\mathrm{Res}_{D_I}} & \varepsilon\otimes\Omega_X^{m-|I|}(\log(D-D_I))|_{D_I}. \end{array} \quad (2)$$

In Ref. [1], the intersection complex is defined as follows:

Definition 3.1. $IC^*(X,\varepsilon)$ is an Ω_U^* graded submodule of $\Omega_X^*(\log D)\otimes\varepsilon$ generated by the abelian subsheaf

$$\sum_{I\subseteq M}\nabla_I\varepsilon_U\otimes\omega_I,$$

where U is an open subset of X, and $M=\{1,2,\cdots,r\}$.

Our main theorem is as as follows:

Theorem 3.1. If $\mathrm{Res}_i(\nabla):\varepsilon|_{D_i}\to\varepsilon|_{D_i}$ are bundle morphisms for all $i\in\mathcal{I}$, then $IC_r^*(X,\varepsilon)=IC^*(X,\varepsilon)$.

This proof makes essential use of the weight filtration of the log complex.

Definition 3.2. Weight filtration $W_.$ of the logarithmic complex is defined as follows:

$$W_m\Omega_X^p(\log D)=\begin{cases}0, & \text{for } m<0;\\ \Omega_X^p(\log D), & \text{for } m\geqslant p;\\ \Omega_X^{p-m}\otimes\Omega_X^m(\log D), & \text{for } 0\leqslant m\leqslant p.\end{cases} \quad (3)$$

We establish the following lemma for weight filtration to prove the theorem.

Lemma 3.1. ① One has exact sequence:

$$0\to W_{m-1}(\Omega_X^a(\log D))\to W_m(\Omega_X^a(\log D))\xrightarrow{\oplus_{|I|=m}\beta_I}\alpha_{m*}\Omega_{D_{(m)}}^{a-m}\to 0, \quad (4)$$

where $D_{(m)}$ is the disjoint union over subset $K\subset I$ of cardinal m of $D_K:=\cap_{i\in K}D_i$, and α_m is the natural immersion $D_{(m)}\to X$.

② The following diagram is commutative.

$$\begin{array}{ccc} \varepsilon\otimes\Omega_X^{a-|I|} & \xrightarrow{\nabla_I\omega_I} & \varepsilon\otimes W_{|I|}\Omega_X^a(\log D), \\ l_I\otimes\gamma_I\downarrow & & \downarrow l_I\otimes\beta_I \\ \varepsilon|_{D_I}\otimes\Omega_{D_I}^{a-|I|} & \xrightarrow{\mathrm{Res}_I\otimes(-Id)^{a-|I|}} & \varepsilon|_{D_I}\otimes\Omega_{D_I}^{a-I}, \end{array} \quad (5)$$

where $l_i:\varepsilon\to\varepsilon|_{D_i}$ is the canonical restriction map. And if Res_I is a bundle morphism, then

$$(l_I\otimes\beta_I)\circ(\nabla_I\omega_I)(\varepsilon\otimes\Omega_{X/S}^{a-|I|})=(\mathrm{Res}_I(\nabla))(\varepsilon|_{D_1})\otimes\Omega_{D_I/S}. \quad (6)$$

Proof. It is a basic fact of weight filtration, for a rigorous proof of this lemma the reader is referred to Refs. [6, 7] .

Firstly, one have to verify that the upper arrow is well defined. That is, one have to show $\nabla_I\omega_I(\varepsilon\otimes\Omega_X^{a-|I|})$ is contained in $W^{|I|}\Omega_X^a(\log D)\otimes\varepsilon$. It is straightforward because the source the map $\varepsilon\otimes\Omega_X^{a-|I|}$ is weight zero and the map $\nabla_I\omega_I$ is of weight $|I|$.

Note that we have $\beta_I(W_{|I|}\Omega_X^a(\log D))=\Omega_{D_I}^{a-|I|}$, hence the vertical arrow on the right is well-defined. The commutativity of the diagram follows from restricting diagram (1) on subbundle $\varepsilon\otimes\Omega_X^{a-|I|}\subset\varepsilon\otimes\Omega_{D_I/S}(\log((D-\sum_{i\in I}D_i)|_{D_1})$. It remains to show Eq. (6). It is easy to see the sheaf on right side is contained in left side. By the commutativity of diagram (1) again, one has the left side of Eq. (6) is contained in

$$(\mathrm{Res}_I(\nabla))(\varepsilon|_{D_I})\otimes\Omega_{D_I/S}(\log((D-\sum_{i\in I}D_i)|_{D_1}).$$

Therefore, Eq. (6) follows from the following claim.

Claim. If Res_I is a bundle morphism, then we have the following equation:

$$(\mathrm{Res}_I(\nabla))(\varepsilon|_{D_I})\otimes\Omega_{D_I/S}^{a-|I|}(\log((D-\sum_{i\in I}D_i)|_{D_1})\cap(\varepsilon|_{D_1})\otimes\Omega_{D_I/S}^{a-|I|}=$$

$$(\mathrm{Res}_I(\nabla))(\varepsilon|_{D_I})\otimes\Omega_{D_I/S}^{a-|I|}.$$

For the "$\supseteq$" direction, it is obvious. For the other direction, the sheaf $\mathrm{Res}_I(\nabla)(\varepsilon|_{D_I})$ is locally free due to the assumption that Res_I is a bundle morphism, thus it has no torsion along $D_j\cap D_I$, where $j\in\mathcal{I}-I$. This completes the proof of the claim.

Remark 3.1. The first sublemma illustrates that weight equals to the number of times of the irreducible components of the divisor intersect (in other words, the depth of the stratification). The second provides a local description of the weight filtration along divisor in terms of the coordinates.

We now return to the proof of the main theorem.

Proof of Theorem 3.1. Without loss of generality, we assume that $r=n$.

Clearly, we have $IC^*(X,\varepsilon)\subset IC_r^*(X,\varepsilon)$, because the restriction of ∇_i on D_i is exactly the residue map Res_i.

Conversely, we consider any $s\in IC_r^*(X,\varepsilon)(U)\bigcap\Omega_X^a(\log D)\otimes\varepsilon(U)$. Taking $a=|I|$ in the diagram (5), we will obtain a section $e\in\varepsilon(U)$ such that $\mathrm{Res}_I\circ\gamma_I(e)=\beta_I(s)$, by the comutitivity of the diagram we have $s-\nabla_I(e)\otimes\omega_I\in\varepsilon\otimes\ker\beta_I$, and we denote it as s_1. Taking $m=a$ in the exact sequence (4), we know that $\ker\beta_I$ is equal to $W^{a-1}(\Omega_X^a(\log D))$; hence, $s_1\in\varepsilon\otimes W^{a-1}(\Omega_X^a(\log D))$.

Replacing s with s_1, by the definition of res-intersection complex, one has $l_I\otimes\beta_I(s_1)\in\mathrm{Im}(\mathrm{Res}_{D_I}\nabla:\varepsilon|_{D_I}\to\varepsilon|_{D_I})\otimes\Omega_{D_I/S}^a(\log(D-\sum_{i\in I}D_i)|_{D_I})$ and one has $s_1\in\varepsilon\otimes W^{a-1}(\Omega_X^a(\log D))$ by the construction. Then we can chase in the diagram (5) for index set I with $|I|=a-1$, by Eq. (6), we obtain a section $e_I\in\varepsilon\otimes\Omega_{X/S}^{a-1}$ such that $s_1-\nabla_I(e_I)\omega_I\in\varepsilon\otimes\ker\beta_I$, therefore, we have

$$s_2:=s_1-\sum_I\nabla_I(e_I)\mathrm{d}\log t_I\in\varepsilon\otimes\ker(\oplus\beta_I),$$

where I is a of cardinal $(a-1)$. Therefore, we have $s_2\in W^{a-2}(\Omega_X^a(\log D))\otimes\varepsilon$ by (4).

Repeat the processes above a times, we obtain $s_a\in\varepsilon\otimes W^0=\Omega_X^a\otimes E$ and $e_I\in\varepsilon\otimes\Omega_{X/S}^{a-|I|}$ with $I\subseteq\mathcal{I}$, satisfying

$$s=\sum_{I\subseteq\mathcal{I}}\nabla_I(e_I)\omega_I+s_a,$$

which implies $s\in IC(X,\varepsilon)\bigcap\Omega_X^a(\log D)\otimes\varepsilon$. This completes our proof.

4 Surface case

In this section, we provide an explicit calculation for the surface case and present an example to the main theorem without bundle morphism conditions.

Let $X=\mathrm{Spec}(k[t_1,t_2])$ be a surface and let $D=D_1+D_2$ defined by $t_1t_2=0$. The divisor gives rise to a stratification the

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surface as $X = D^*_\emptyset \coprod D^*_1 \coprod D^*_2 \coprod D^*_{12}$. With the help of the coordinates t_i, we can write $\nabla = \nabla_1 \mathrm{dlog}\, t_1 + \nabla_2 \mathrm{dlog}\, t_2$.

① $IC^0_r(X,\varepsilon) = IC^0(X,\varepsilon)$, because both are equal to ε.

② In the one-degree term, note that the sections of the sheaf $IC^1(X,\varepsilon)$ are of the form:

$$\tilde{s} = \nabla_1(e_1)\mathrm{dlog}\, t_1 + \nabla_2(e_2)\mathrm{dlog}\, t_2 + f_1 \mathrm{d}t_1 + f_2 \mathrm{d}t_2.$$

Let $s \in IC(X,\varepsilon)^1(X)$. To verify $s \in IC^1_c(X,\varepsilon)$, we aim to find e_1 and e_2. By definition, s satisfies $\beta_i(s) \in \mathrm{Im}(\mathrm{Res}_{D_i})$: Consider a commutative diagram

$$\begin{array}{ccc} \varepsilon & \xrightarrow{\nabla} & \varepsilon \otimes \Omega^1_X(\log(D)), \\ l_1 \downarrow & & \downarrow l_1 \otimes \beta_1 \\ \varepsilon|_{D_1} & \xrightarrow{\mathrm{Res}_{D_1}(\nabla)} & \varepsilon|_{D_1}. \end{array}$$

The first vertical arrow is surjective, so we can find a section $e_1 \in \varepsilon$ such that $s - \nabla(e_1)\mathrm{dlog}\, t_1$ in the kernel of the second vertical, which is $\varepsilon \otimes \Omega^1_X(\log(D_2))$, replacing 1 with 2, and we obtain a section e_2 of ε, by the exact sequence (4), $s_1 = s - (\nabla_1(e_1)\mathrm{dlog}\, t_1 + \nabla_2(e_2)\mathrm{dlog}\, t_2)$ is of weight zero. In other words, it is regular, which allows us to write:

$$s = \nabla_1(e_1)\mathrm{dlog}\, t_1 + \nabla_2(e_2)\mathrm{dlog}\, t_2 + f_1 \mathrm{d}t_1 + f_2 \mathrm{d}t_2,$$

where $e_1, e_2, f_1, f_2 \in \varepsilon(U)$.

③ Using the same pattern, a section ω in $IC^2(X,\varepsilon)(X)$ is of the form

$$\begin{aligned} t = &(\nabla_{12}(e_{12}))\,\mathrm{dlog}\, t_1 \wedge \mathrm{dlog}\, t_2 + \nabla_1(e_1)\mathrm{dlog}\, t_1 \wedge \mathrm{d}t_2 + \\ &\nabla_2(e_2)\mathrm{dlog}\, t_2 \wedge \mathrm{d}t_1 + f \mathrm{d}t_1 \wedge \mathrm{d}t_2 \end{aligned} \tag{7}$$

by definition. For any section $s \in IC^2_r(X,\varepsilon)(X)$, we aim to get section e_{12}, e_1, e_2, f. By (5), we have the following commutative diagram:

$$\begin{array}{ccc} \varepsilon & \xrightarrow{\nabla_{12}\omega_{12}} & \varepsilon \otimes \Omega^2_X(\log(D)), \\ l_{12} \downarrow & & \downarrow l \otimes \beta_{12} \\ \varepsilon|_{D_1 \cap D_2} & \xrightarrow{\mathrm{Res}_{12}} & \varepsilon|_{D_1 \cap D_2}, \end{array}$$

where both vertical arrows are surjective, we obtain $e_{12} \in \varepsilon$ such that $s_1 := s - (\nabla(e_{12}))\omega_{12} \in \ker \varepsilon \otimes \beta_{12}$, which is of weight one by short exact sequence (4).

Then consider diagram (5) and set $a = 2$, $I = \{1,2\}$. Using the same argument as above, we obtain $\tilde{e}_1$ and $\tilde{e}_1 \in \varepsilon \otimes \Omega^1_X$, such that $s_1 - \nabla_1(e_1)\mathrm{dlog}\, t_1 \wedge \mathrm{d}t_2 - \nabla_2(e_2)\mathrm{dlog}\, t_2 \wedge \mathrm{d}t_1$ in $\ker(\beta_2 \oplus \beta_2)$, which is exactly $\Omega^2_X \otimes \varepsilon$ by (4). Therefore, e_{12}, e_1, e_2, f are the section we want. This completes our proof.

In the sequel of the section we will present an example, which is provided by Zebao Zhang, to show that the main theorem will be wrong if the residue morphisms are not assumed to be bundle morphisms. Let k be a perfect field of character p, (X,D) is as above. Define logarithmic connection over

$$\begin{aligned} \nabla : O_X &\to \Omega^1_{X/k}(\log D), \\ f &\to \mathrm{d}f + (f t_2^p) \cdot \frac{\mathrm{d}t_1}{t_1}. \end{aligned}$$

One can verify that ∇ is integral, and the residue morphisms are as follows:

$$\mathrm{Res}_1(\nabla) = t_2^p, \quad \mathrm{Res}_2(\nabla) = 0. \tag{8}$$

One can see that $\mathrm{Res}_1(\nabla)$ has torsion at $t_2 = 0$, hence it is not a bundle morphism. By definition, we have $IC^2 = (t_2^p \cdot \mathrm{dlog}\, t_1 \wedge \mathrm{d}t_2)O_X + (\mathrm{d}t_1 \wedge \mathrm{d}t_2)O_X$. Consider the section $t_2^p \mathrm{dlog}\, t_1 \wedge \mathrm{dlog}\, t_2$, it is a section in IC^2_r but not in IC^2.

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Conflict of interest

The author declares that he has no conflict of interest.

Biographies

Xiaojin Lin is currently a graduate student under the tutelage of Prof. Mao Sheng at the University of Science and Technology of China. His research interests focus on Hodge theory and vector bundle.

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