JUSTC
Article
http://justc.ustc.edu.cn
Received: October 09, 2021; Accepted: January 21, 2022

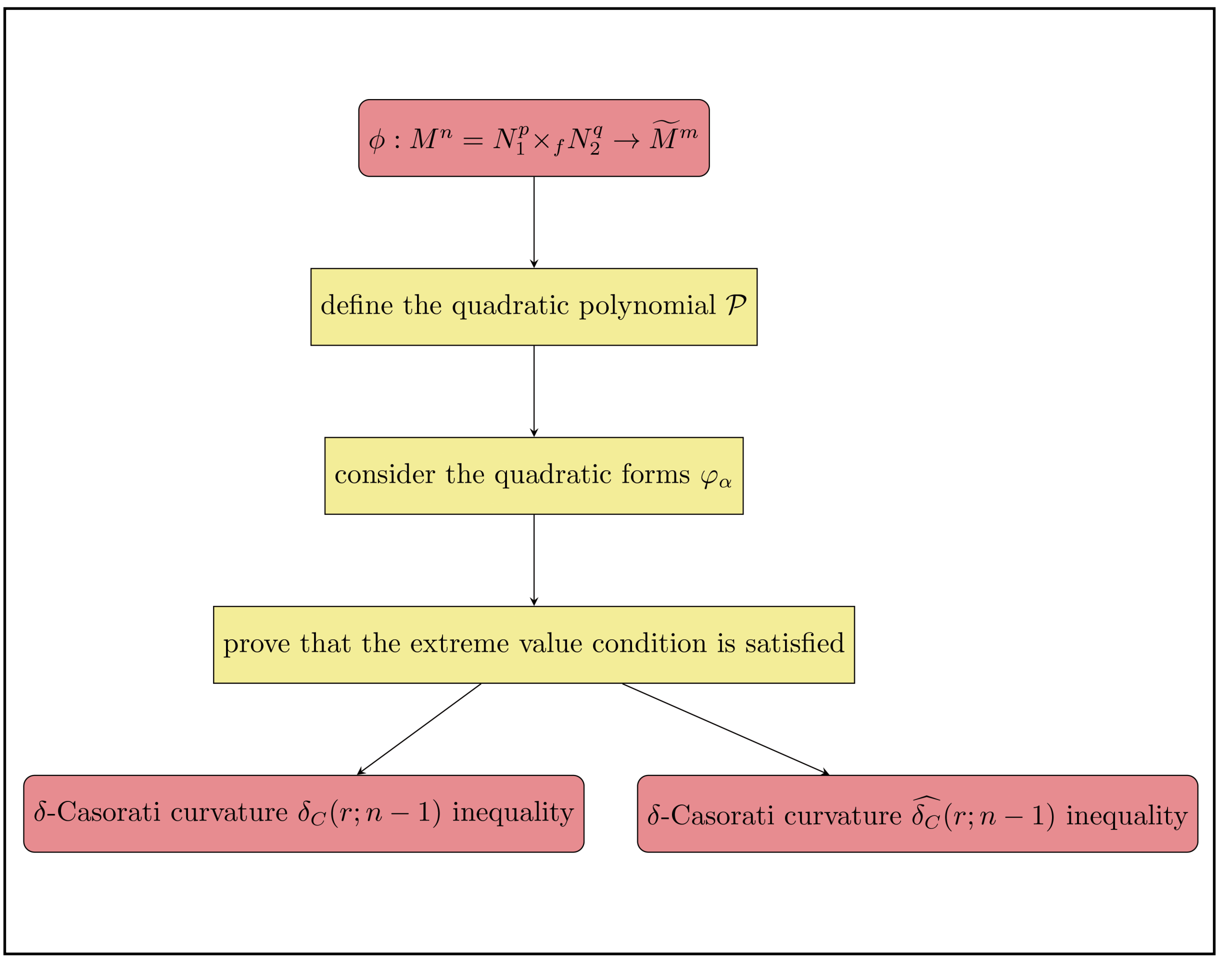
Inequalities of warped product submanifolds in a Riemannian manifold of quasi-constant curvature

Jiahui Wang, Lijuan Cheng, and Yecheng Zhu ✉

School of Mathematics and Physics, Anhui University of Technology, Maanshan 243002, China

✉Correspondence: Yecheng Zhu, E-mail: zhuyc929@mail.ustc.edu.cn

Graphical abstract



The process of establishing the generalized normalized δ-Casorati curvatures inequality.

Public summary

■ We establish Chen-like inequalities for generalized normalized δ-Casorati curvatures of warped product submanifolds in a Riemannian manifold of quasi-constant curvature.

■ Our inequalities extend the optimal inequalities involving the scalar curvature and the Casorati curvature of a Riemannian submanifold in a real space form.

Citation: Wang J H, Cheng L J, Zhu Y C. Inequalities of warped product submanifolds in a Riemannian manifold of quasi-constant curvature. *JUSTC*, **2022**, 52(3): 6. DOI: 10.52396/JUSTC-2021-0217

JUSTC

Article

http://justc.ustc.edu.cn

Received: October 09, 2021; Accepted: January 21, 2022

Inequalities of warped product submanifolds in a Riemannian manifold of quasi-constant curvature

Jiahui Wang, Lijuan Cheng, and Yecheng Zhu ✉

School of Mathematics and Physics, Anhui University of Technology, Maanshan 243002, China

✉Correspondence: Yecheng Zhu, E-mail: zhuyc929@mail.ustc.edu.cn

Cite This: *JUSTC*, **2022**, 52(3): 6 (7pp)   Read Online

Abstract: By optimization methods on Riemannian submanifolds, we establish two inequalities between the intrinsic and extrinsic invariants, for generalized normalized δ-Casorati curvatures of warped product submanifolds in a Riemannian manifold of quasi-constant curvature. We generalize the conclusions of the optimal inequalities of submanifolds in real space forms.

Keywords: Casorati curvature; optimization methods; scalar curvature

CLC number: O186.1   **Document code:** A

2020 Mathematics Subject Classification: 53C42

1 Introduction

In 1993, Chen[1] introduced δ-invariants, and established relationships between intrinsic invariants and extrinsic invariants for minimal submanifolds. In 1995, Chen [2] found Chen-like inequalities for Riemannian submanifolds and gave some applications of δ-invariants. Submanifolds are ideal submanifolds when Chen-like inequalities are equal and they receive the least possible tension at each point from ambient spaces.

The Casorati curvature was originally introduced in 1980 for surfaces in 3-dimensional Euclidean space and is defined as the normalized square of the length of the second fundamental form (see Ref. [3]). In 2007, Decu et al. [4] introduced the normalized δ-Casorati curvatures $\widehat{\delta_C}(n-1)$ and $\delta_C(n-1)$ and established two optimal inequalities involving the scalar curvature and the normalized δ-Casorati curvature. In 2008, Decu et al.[5] introduced the generalized normalized δ-Casorati curvatures $\widehat{\delta_C}(r;n-1)$ and $\delta_C(r;n-1)$ and proved two sharp inequalities. In 2017, Park[6] obtained two types of optimal inequalities for the real hypersurfaces of complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians. In 2020, Choudhary and Blaga[7] established some sharp inequalities involving generalized normalized δ-Casorati curvatures for invariant, anti-invariant and slant submanifolds in metallic Riemannian space forms and characterized the submanifolds for which the equality holds.

In this study, we establish Chen-like inequalities for generalized normalized δ-Casorati curvatures of warped product submanifolds in a Riemannian manifold of quasi-constant curvature.

Let N_1^p and N_2^q be two Riemannian manifolds with positive dimensions equipped with Riemannian metrics $g_{N_1^p}$ and $g_{N_2^q}$, respectively. Let f be a positive function on N_1^p. Consider the product manifold $N_1^p\times N_2^q$, with its projections $\pi: N_1^p\times N_2^q\to N_1^p$ and $\eta: N_1^p\times N_2^q\to N_2^q$. The warped product manifold $M^n=N_1^p\times_f N_2^q$ is the product manifold $N_1^p\times N_2^q$ equipped with a Riemannian structure such that

$$\|X\|^2=\|\pi_*(X)\|^2+f^2(\pi(x))\|\eta_*(X)\|^2 \tag{1}$$

for any tangent vector $X\in T_xM^n$. Thus, we have $g=g_{N_1^p}+f^2g_{N_2^q}$. The function f is called the warping function of the warped product manifold.

A Riemannian manifold $(\widetilde{M}^m,\tilde{g})$ is called a Riemannian manifold of quasi-constant curvature if the curvature tensor satisfies the following condition (see Ref. [8]):

$$\begin{aligned}\widetilde{R}(X,Y,Z,W)=&a[\tilde{g}(X,Z)\tilde{g}(Y,W)-\tilde{g}(Y,Z)\tilde{g}(X,W)]+\\&b[\tilde{g}(X,Z)T(Y)T(W)-\tilde{g}(X,W)T(Y)T(Z)+\\&\tilde{g}(Y,W)T(X)T(Z)-\tilde{g}(Y,Z)T(X)T(W)]\end{aligned} \tag{2}$$

where a and b are scalar functions, T is a 1-form defined by

$$T(X)=\tilde{g}(X,P) \tag{3}$$

where P denotes the unit vector field. We uniquely decompose the vector field P on M^n into its tangent component P^T and normal component $P^\perp$, that is,

$$P=P^T+P^\perp \tag{4}$$

Theorem 1.1. Let $\phi: M^n=N_1^p\times_f N_2^q\to\widetilde{M}^m$ be an isometric immersion of an n-dimensional warped product submanifold M^n into an m-dimensional Riemannian manifold of a quasi-constant curvature $\widetilde{M}^m$. Then

(i) the generalized normalized δ-Casorati curvature $\delta_C(r;n-1)$ satisfies

$$\begin{aligned}&\frac{2}{n(n-1)}\times\left\{\frac{q\Delta f}{f}+\frac{p(p-1)a}{2}+b(p-1)\|P^T\|^2_{N_1^p}+\frac{q(q-1)a}{2}+\right.\\&\left.b(q-1)\|P^T\|^2_{N_2^q}\right\}+\frac{\delta_C(r;n-1)}{n(n-1)}-\frac{npr(n^2-n+qr-r)\|H\|^2}{(n-1)\{(n^2-n-r+qr)^2+p^2r^2\}}\geqslant\rho\end{aligned} \tag{5}$$

for any real number r such that $0<r<n(n-1)$, where $\|P^T\|^2_{N_1^p}=\sum_{i=1}^p g(P^T,e_i)^2$, $\|P^T\|^2_{N_2^q}=\sum_{s=p+1}^n g(P^T,e_s)^2$, ρ is the normalized scalar curvature, $\|H\|^2$ is the squared mean curvature, a and b are scalar functions;

(ii) the generalized normalized δ-Casorati curvature $\widehat{\delta_C}(r;n-1)$ satisfies

$$\frac{2}{n(n-1)}\times\left\{\frac{q\Delta f}{f}+\frac{p(p-1)a}{2}+b(p-1)\|P^T\|^2_{N_1^p}+\frac{q(q-1)a}{2}+b(q-1)\|P^T\|^2_{N_2^q}\right\}+\frac{\widehat{\delta_C}(r;n-1)}{n(n-1)}-\frac{npr(n^2-n+qr-r)\|H\|^2}{(n-1)\{(n^2-n-r+qr)^2+p^2r^2\}}\geqslant\rho \tag{6}$$

for any real number $r>n(n-1)$.

Equalities hold in (5) and (6) if and only if the shape operators for the suitable tangent and normal orthonormal frames are given by

$$\left.\begin{array}{l}
A_{e_{n+1}}=\\
\begin{pmatrix}
h_{11}^{n+1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & h_{22}^{n+1} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & h_{pp}^{n+1} & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & \cdots & 0 & h_{p+1p+1}^{n+1} & 0 & \cdots & 0 & 0\\
0 & 0 & \cdots & 0 & 0 & h_{p+2p+2}^{n+1} & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & h_{n-1n-1}^{n+1} & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & h_{nn}^{n+1}
\end{pmatrix},\\
h_{11}^{n+1}=\cdots=h_{pp}^{n+1}=pr^2f_1e_{n+1},\\
h_{p+1p+1}^{n+1}=\cdots=h_{n-1n-1}^{n+1}=(n^2-n+qr-r)rf_1e_{n+1},h_{nn}^{n+1}=\\
n(n-1)(n^2-n+qr-r)f_1e_{n+1},\\
h_{ij}^{n+1}=0,i\neq j,\\
A_{e_{n+2}}=\cdots=A_{e_{n+m}}=0
\end{array}\right\} \tag{7}$$

where f_1 is a function on M^n.

Let $b=0$ and $a=\text{const}$. Then we have

Corollary 1.1. Let $\phi: M^n=N_1^p\times_f N_2^q\to\widetilde{M}^m(a)$ be an isometric immersion of an n-dimensional warped product submanifold M^n into an m-dimensional Riemannian manifold of a constant sectional curvature a. Then

(i) the generalized normalized δ-Casorati curvature $\delta_C(r;n-1)$ satisfies

$$\frac{2}{n(n-1)}\times\left\{\frac{q\Delta f}{f}+\frac{p(p-1)a}{2}+\frac{q(q-1)a}{2}\right\}+\frac{\delta_C(r;n-1)}{n(n-1)}-\frac{npr(n^2-n+qr-r)\|H\|^2}{(n-1)\{(n^2-n-r+qr)^2+p^2r^2\}}\geqslant\rho \tag{8}$$

for any real number r such that $0<r<n(n-1)$;

(ii) the generalized normalized δ-Casorati curvature $\widehat{\delta_C}(r;n-1)$ satisfies

$$\frac{2}{n(n-1)}\times\left\{\frac{q\Delta f}{f}+\frac{p(p-1)a}{2}+\frac{q(q-1)a}{2}\right\}+\frac{\widehat{\delta_C}(r;n-1)}{n(n-1)}-\frac{npr(n^2-n+qr-r)\|H\|^2}{(n-1)\{(n^2-n-r+qr)^2+p^2r^2\}}\geqslant\rho \tag{9}$$

for any real number $r>n(n-1)$.

Equalities hold in (8) and (9) if and only if the shape operators for the suitable tangent and normal orthonormal frames are given by Eq. (7).

Moreover, let $p=0, q=n$ and $f=1$. Then we have

Corollary 1.2. Let $\phi: M^n\to\widetilde{M}^m(a)$ be an isometric immersion of an n-dimensional warped product submanifold into $\widetilde{M}^m(a)$. We have

(i) for any real number r such that $0<r<n(n-1)$,

$$\delta_C(r;n-1)+n(n-1)a\geq n(n-1)\rho \tag{10}$$

(ii) for any real number r such that $r>n(n-1)$,

$$\widehat{\delta_C}(r;n-1)+n(n-1)a\geq n(n-1)\rho \tag{11}$$

Equalities hold in (10) and (11) if and only if M^n is an invariantly quasi-umbilical submanifold.

Remark: Corollary 1.2 is Theorem 2.1, and Corollary 3.1 in Ref. [5].

2 Preliminaries

Let M^n be an n-dimensional warped product submanifold of an m-dimensional Riemannian manifold of quasi-constant curvature $\widetilde{M}^m$. Let ∇ and $\widetilde{\nabla}$ be the Levi–Civita connection on M^n and $\widetilde{M}^m$, respectively. Then, the Gauss and Weingarten formulas are given respectively by

$$\left.\begin{array}{l}\widetilde{\nabla}_XY=\nabla_XY+h(X,Y)\\ \widetilde{\nabla}_XN=-A_NX+\nabla_X^{\perp}N\end{array}\right\} \tag{12}$$

for vector fields X, Y tangent to M^n, and vector field N normal to M^n. Here h denotes the second fundamental form, $\nabla^{\perp}$ is the normal connection and A is the shape operator. The second fundamental form and shape operator are related by

$$\tilde{g}(h(X,Y),N)=g(A_NX,Y) \tag{13}$$

where $\tilde{g}$ and g denote the metric on $\widetilde{M}^m$ and M^n respectively. If R and $\widetilde{R}$ are the curvature tensors of M^n and $\widetilde{M}^m$, respectively, then the Gauss equation is given by

$$R(X,Y,Z,W)=\widetilde{R}(X,Y,Z,W)+\tilde{g}(h(X,Z),h(Y,W))-\tilde{g}(h(X,W),h(Y,Z)) \tag{14}$$

for any vector field X, Y, Z, and W tangent to M^n.

Let $\{e_1,\cdots,e_n\}$ be an orthonormal basis of the tangent space T_xM^n and let $\{e_{n+1},\cdots,e_m\}$ be an orthonormal basis of normal space $T_x^{\perp}M^n$. The mean curvature vector H at x is

$$H(x)=\frac{1}{n}\sum_{\alpha=n+1}^{m}\left(\sum_{i=1}^{n}h_{ii}^{\alpha}\right)e_{\alpha} \tag{15}$$

The squared mean curvature of the submanifold M^n in $\widetilde{M}^m$ is defined as

$$\|H\|^2=\frac{1}{n^2}\sum_{\alpha=n+1}^{m}\left(\sum_{i=1}^{n}h_{ii}^{\alpha}\right)^2 \tag{16}$$

Also, we set

$$\left.\begin{array}{l}h_{ij}^{\alpha}=\tilde{g}(h(e_i,e_j),e_{\alpha})\\ \|h\|^2=\sum\limits_{\alpha=n+1}^{m}\sum\limits_{i,j=1}^{n}\tilde{g}(h(e_i,e_j),e_{\alpha})^2\end{array}\right\} \tag{17}$$

Let $K(e_i\wedge e_j)$ be the sectional curvature of the plane section spanning e_i and e_j at $x\in M^n$. Subsequently, the scalar curvature $\tau(x)$ of M^n is given by

$$\tau(x)=\sum_{1\leq i<j\leq n}K(e_i\wedge e_j) \tag{18}$$

DOI: 10.52396/JUSTC-2021-0217
JUSTC, **2022**, 52(3): 6

and the normalized scalar curvature ρ of M^n at x is defined as

$$\rho(x)=\frac{2\tau(x)}{n(n-1)} \tag{19}$$

The Casorati curvature C of the submanifold M^n is the squared norm of the second fundamental form h over dimension n and is given by

$$C=\frac{1}{n}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n}(h_{ij}^{\alpha})^2 \tag{20}$$

If L is an l-dimensional subspace of T_xM^n, where $l\geqslant 2$ and $\{e_1,\cdots,e_l\}$ is an orthonormal basis of L, the scalar curvature $\tau(L)$ of the l-plane section L is defined as

$$\tau(L)=\sum_{1\leq i<j\leq l}K(e_i\wedge e_j) \tag{21}$$

and the Casorati curvature of the subspace L, denoted by $C(L)$, is given by

$$C(L)=\frac{1}{l}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{l}(h_{ij}^{\alpha})^2 \tag{22}$$

The generalized normalized δ–Casorati curvatures $\delta_C(r;n-1)$ and $\widehat{\delta_C}(r;n-1)$ of the submanifold M^n are defined for a positive real number $r\neq n(n-1)$ as

$$[\delta_C(r;n-1)]_x=rC_x+\frac{(n-1)(n+r)(n^2-n-r)}{rn}\inf\{C(L)|L:\text{a hyperplane of }T_xM^n\} \tag{23}$$

if $0<r<n^2-n$; and

$$[\widehat{\delta_C}(r;n-1)]_x=rC_x-\frac{(n-1)(n+r)(r-n^2+n)}{rn}\sup\{C(L)|L:\text{a hyperplane of }T_xM^n\} \tag{24}$$

if $r>n^2-n$.

By Gauss equation, we get

$$K(e_i\wedge e_j)=\widetilde{K}(e_i\wedge e_j)+\sum_{\alpha=n+1}^{m}(h_{ii}^{\alpha}h_{jj}^{\alpha}-(h_{ij}^{\alpha})^2) \tag{25}$$

where $K(e_i\wedge e_j)$ and $\widetilde{K}(e_i\wedge e_j)$ denote the sectional curvatures of the plane section spanned by e_i and e_j at x in the submanifold M^n and in the ambient manifold $\widetilde{M}^m$, respectively. By Eqs. (2) and (25), we have

$$\tau(N_1^p)=\sum_{\alpha=n+1}^{m}\sum_{1\leq i<j\leq p}(h_{ii}^{\alpha}h_{jj}^{\alpha}-(h_{ij}^{\alpha})^2)+\tilde{\tau}(N_1^p)=\frac{p(p-1)a}{2}+b(p-1)\|P^T\|^2_{N_1^p}+\sum_{\alpha=n+1}^{m}\sum_{1\leq i<j\leq p}(h_{ii}^{\alpha}h_{jj}^{\alpha}-(h_{ij}^{\alpha})^2) \tag{26}$$

$$\tau(N_2^q)=\sum_{\alpha=n+1}^{m}\sum_{p+1\leq s<t\leq n}(h_{ss}^{\alpha}h_{tt}^{\alpha}-(h_{st}^{\alpha})^2)+\tilde{\tau}(N_2^q)=\frac{q(q-1)a}{2}+b(q-1)\|P^T\|^2_{N_2^q}+\sum_{\alpha=n+1}^{m}\sum_{p+1\leq s<t\leq n}(h_{ss}^{\alpha}h_{tt}^{\alpha}-(h_{st}^{\alpha})^2) \tag{27}$$

where $\|P^T\|^2_{N_1^p}=\sum_{i=1}^{p}g(P^T,e_i)^2$, $\|P^T\|^2_{N_2^q}=\sum_{s=p+1}^{n}g(P^T,e_s)^2$.

Definition 2.1. For the differential function f on M^n, the Laplacian Δf and gradient ∇f of f are defined by

$$g(\nabla f,X)=X(f) \tag{28}$$

$$\Delta f=\sum_{i=1}^{n}((\nabla_{e_i}e_i)f-e_ie_if) \tag{29}$$

for any vector field X that is tangent to M^n.

Lemma 2.1.[9] Let $M^n=N_1^p\times_f N_2^q$ be a warped product submanifold of $\widetilde{M}^m$. The relation between the sectional curvature and Laplacian Δf of f is

$$\sum_{l=1}^{p}\sum_{k=p+1}^{n}K(e_l\wedge e_k)=\frac{q\Delta f}{f}=q(\Delta\ln f-\|\nabla\ln f\|^2) \tag{30}$$

Lemma 2.2.[10] Let N_1 be a Riemannian submanifold of a Riemannian manifold $(N_2,\bar{g})$, $\varphi:N_2\to\mathbb{R}$ be a differentiable function and consider the constrained extremum problem

$$\min_{x\in N_1}\varphi(x) \tag{31}$$

If $x_0\in N_1$ is a solution of the problem (31), then

(i) $(\operatorname{grad}\varphi)(x_0)\in T^{\perp}_{x_0}N_1$;

(ii) the bilinear form $\Lambda:T_{x_0}N_1\times T_{x_0}N_1\to\mathbb{R}$ defined by

$$\Lambda(X,Y)=\operatorname{Hess}_{\varphi}(X,Y)+\bar{g}(h_1(X,Y),(\operatorname{grad}\varphi)(x_0)) \tag{32}$$

is positive semi-definite, where h_1 is the second fundamental form of N_1 in N_2 and grad φ is the gradient of φ.

3 Proof of the theorem

Proof of Theorem 1.1 From Eqs. (26), (27), (30), and the Gauss equation, we obtain

$$\tau(x)=\sum_{l=1}^{p}\sum_{k=p+1}^{n}K(e_l\wedge e_k)+\sum_{1\leq i<j\leq p}K(e_i\wedge e_j)+\sum_{p+1\leq s<t\leq n}K(e_s\wedge e_t)=\frac{q\Delta f}{f}+\frac{p(p-1)a}{2}+b(p-1)\|P^T\|^2_{N_1^p}+\frac{q(q-1)a}{2}+b(q-1)\|P^T\|^2_{N_2^q}+\sum_{\alpha=n+1}^{m}\sum_{1\leq i<j\leq p}(h_{ii}^{\alpha}h_{jj}^{\alpha}-(h_{ij}^{\alpha})^2)+\sum_{\alpha=n+1}^{m}\sum_{p+1\leq s<t\leq n}(h_{ss}^{\alpha}h_{tt}^{\alpha}-(h_{st}^{\alpha})^2) \tag{33}$$

We define the following quadratic polynomial $\mathcal{P}$ in the components of the second fundamental form as

$$\mathcal{P}=rC+\frac{(n-1)(n+r)(n^2-n-r)}{nr}C(L)-2\tau+2\times\Big\{\frac{q\Delta f}{f}+\frac{p(p-1)a}{2}+b(p-1)\|P^T\|^2_{N_1^p}+\frac{q(q-1)a}{2}+b(q-1)\|P^T\|^2_{N_2^q}\Big\} \tag{34}$$

where L denotes the hyperplane of T_xM^n. Without loss of generality, we can suppose that L is spanned by $e_1,e_2,\cdots,e_{n-1}$. From Eqs. (33) and (34), we have

$$\mathcal{P}=\frac{r}{n}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n}(h_{ij}^{\alpha})^2+\frac{(n+r)(n^2-n-r)}{nr}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n-1}(h_{ij}^{\alpha})^2-2\sum_{\alpha=n+1}^{m}\sum_{1\leqslant i<j\leqslant p}(h_{ii}^{\alpha}h_{jj}^{\alpha}-(h_{ij}^{\alpha})^2)-2\sum_{\alpha=n+1}^{m}\sum_{p+1\leqslant s<t\leqslant n}(h_{ss}^{\alpha}h_{tt}^{\alpha}-(h_{st}^{\alpha})^2)\geqslant\frac{n^2-n+nr-2r}{r}\sum_{\alpha=n+1}^{m}\sum_{i=1}^{n-1}(h_{ii}^{\alpha})^2+\frac{r}{n}\sum_{\alpha=n+1}^{m}(h_{nn}^{\alpha})^2-2\sum_{\alpha=n+1}^{m}\sum_{1\leqslant i<j\leqslant p}h_{ii}^{\alpha}h_{jj}^{\alpha}-2\sum_{\alpha=n+1}^{m}\sum_{p+1\leqslant s<t\leqslant n}h_{ss}^{\alpha}h_{tt}^{\alpha} \tag{35}$$

We consider the quadratic forms

$$\varphi_{\alpha}:\mathbb{R}^n\to\mathbb{R},\qquad \alpha=n+1,n+2,\cdots,m \tag{36}$$

DOI: 10.52396/JUSTC-2021-0217
JUSTC, 2022, 52(3): 6

defined by

$$\varphi_\alpha(h_{11}^\alpha,\cdots,h_{nn}^\alpha)=\frac{n^2-n+nr-2r}{r}\sum_{i=1}^{n-1}(h_{ii}^\alpha)^2+\frac{r}{n}(h_{nn}^\alpha)^2-2\sum_{1\leqslant i<j\leqslant p}h_{ii}^\alpha h_{jj}^\alpha-2\sum_{p+1\leqslant s<t\leqslant n}h_{ss}^\alpha h_{tt}^\alpha \tag{37}$$

Then by Eqs. (35) and (37), we derive

$$\mathcal{P}\geqslant\sum_{\alpha=n+1}^{m}\varphi_\alpha \tag{38}$$

Next, for α, we consider the extremum problem

$$\min\varphi_\alpha,\qquad \text{subject to}\qquad \Gamma: h_{11}^\alpha+h_{22}^\alpha+\cdots+h_{nn}^\alpha=K^\alpha \tag{39}$$

where K^α is a real constant(see Ref. [11]). The partial derivatives of function φ_α are

$$\left.\begin{aligned}
\frac{\partial\varphi_\alpha}{\partial h_{11}^\alpha}&=\frac{2(n+r)(n-1)}{r}h_{11}^\alpha-2\textstyle\sum_{i=1}^{p}h_{ii}^\alpha,\\
&\ \vdots\\
\frac{\partial\varphi_\alpha}{\partial h_{pp}^\alpha}&=\frac{2(n+r)(n-1)}{r}h_{pp}^\alpha-2\textstyle\sum_{i=1}^{p}h_{ii}^\alpha\\
\frac{\partial\varphi_\alpha}{\partial h_{p+1p+1}^\alpha}&=\frac{2(n+r)(n-1)}{r}h_{p+1p+1}^\alpha-2\textstyle\sum_{s=p+1}^{n}h_{ss}^\alpha\\
&\ \vdots\\
\frac{\partial\varphi_\alpha}{\partial h_{n-1n-1}^\alpha}&=\frac{2(n+r)(n-1)}{r}h_{n-1n-1}^\alpha-2\textstyle\sum_{s=p+1}^{n}h_{ss}^\alpha\\
\frac{\partial\varphi_\alpha}{\partial h_{nn}^\alpha}&=\frac{2(n+r)}{n}h_{nn}^\alpha-2\textstyle\sum_{s=p+1}^{n}h_{ss}^\alpha
\end{aligned}\right\} \tag{40}$$

Applying Lemma 2.2, for an optimal solution $(h_{11}^\alpha,h_{22}^\alpha,\cdots,h_{nn}^\alpha)$ of the minimum problem, vector *grad* φ_α is normal at Γ and collinear with the vector $(1,1,\cdots,1)$.

From Eq. (40) and Lemma 2.2, we derive that a critical point of the problem has the following form:

$$\left.\begin{aligned}
h_{11}^\alpha=\cdots=h_{pp}^\alpha&=\frac{pr^2}{(n^2-n-r+qr)^2+p^2r^2}K^\alpha\\
h_{p+1p+1}^\alpha=\cdots=h_{n-1n-1}^\alpha&=\frac{(n^2-n+qr-r)r}{(n^2-n-r+qr)^2+p^2r^2}K^\alpha\\
h_{nn}^\alpha&=\frac{n(n-1)(n^2-n+qr-r)}{(n^2-n-r+qr)^2+p^2r^2}K^\alpha
\end{aligned}\right\} \tag{41}$$

We fixed an arbitrary point $x\in\Gamma$. According to Lemma 2.2, we deduce that the corresponding bilinear form $\Lambda: T_x\Gamma\times T_x\Gamma\to\mathbb{R}$ is given by

$$\Lambda(X,Y)=\mathrm{Hess}_{\varphi_\alpha}(X,Y)+g(h'(X,Y),(\mathrm{grad}\ \varphi_\alpha)(x)) \tag{42}$$

where h' is the second fundamental form of Γ in $\mathbb{R}^n$ and g is the inner product on $\mathbb{R}^n$.

By Eq. (40), for $i,\ j\in\{1,\cdots,p\}$, $i\neq j$ and $s,\ t\in\{p+1,\cdots,n\}$, $s\neq t$, we get

$$\left.\begin{aligned}
\frac{\partial^2\varphi_\alpha}{\partial(h_{ii}^\alpha)^2}&=\frac{2(n+r)(n-1)-2r}{r}\\
\frac{\partial^2\varphi_\alpha}{\partial h_{ii}^\alpha\partial h_{jj}^\alpha}&=-2\\
\frac{\partial^2\varphi_\alpha}{\partial h_{ii}^\alpha\partial h_{tt}^\alpha}&=0\\
\frac{\partial^2\varphi_\alpha}{\partial(h_{ss}^\alpha)^2}&=\frac{2(n+r)(n-1)-2r}{r}\\
\frac{\partial^2\varphi_\alpha}{\partial h_{ss}^\alpha\partial h_{tt}^\alpha}&=-2\\
\frac{\partial^2\varphi_\alpha}{\partial(h_{nn}^\alpha)^2}&=\frac{2r}{n}
\end{aligned}\right\} \tag{43}$$

Note that

$$(\mathrm{Hess}_{\varphi_\alpha})_{ij}=(\varphi_\alpha)_{i,j}=\frac{\partial^2\varphi_\alpha}{\partial h_{ii}^\alpha\partial h_{jj}^\alpha}-\frac{\partial\varphi_\alpha}{\partial h_{kk}^\alpha}\Gamma_{ij}^k \tag{44}$$

where

$$\Gamma_{ij}^k=\frac{1}{2}g^{kl}\left(\frac{\partial g_{il}}{\partial h_{jj}^\alpha}+\frac{\partial g_{lj}}{\partial h_{ii}^\alpha}-\frac{\partial g_{ij}}{\partial h_{ll}^\alpha}\right) \tag{45}$$

Since g is the inner product on $\mathbb{R}^n$, g_{ij} is constant, and Γ_{ij}^k is 0. Then, we have

$$(\mathrm{Hess}_{\varphi_\alpha})_{ij}=(\varphi_\alpha)_{i,j}=\frac{\partial^2\varphi_\alpha}{\partial h_{ii}^\alpha\partial h_{jj}^\alpha} \tag{46}$$

The Hessian matrix of φ_α is

$$\mathrm{Hess}_{\varphi_\alpha}=\begin{pmatrix}
\frac{2(n+r)(n-1)-2r}{r} & \cdots & -2 & 0 & 0 & \cdots & 0\\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\
-2 & \cdots & \frac{2(n+r)(n-1)-2r}{r} & 0 & \cdots & 0 & 0\\
0 & \cdots & 0 & \frac{2(n+r)(n-1)-2r}{r} & \cdots & -2 & -2\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \cdots & 0 & -2 & \cdots & \frac{2(n+r)(n-1)-2r}{r} & -2\\
0 & \cdots & 0 & -2 & \cdots & -2 & \frac{2r}{n}
\end{pmatrix} \tag{47}$$

DOI: 10.52396/JUSTC-2021-0217
JUSTC, 2022, 52(3): 6

As Γ is totally geodesic in $\mathbb{R}^n$, we consider a vector $X=(X_1,X_2,\cdots,X_n)$ tangent to Γ at an arbitrary point x on Γ, that is, we verify the relation $\sum_{i=1}^n X_i=0$(see Ref. [11]). Next, we prove $\Lambda(X,X)\geqslant 0$.

(i) For $n=1$, we have two possibilities

$$\text{(a)}\,\text{Hess}_{\varphi_\alpha}=\begin{pmatrix}2r\end{pmatrix},\quad \text{(b)}\,\text{Hess}_{\varphi_\alpha}=\begin{pmatrix}-2\end{pmatrix} \tag{48}$$

Since $X_1=0$, in this two cases

$$\text{Hess}_{\varphi_\alpha}(X,X)=0 \tag{49}$$

(ii) For $n=2$, we have three possibilities

$$\text{(a)}\,\text{Hess}_{\varphi_\alpha}=\begin{pmatrix}\frac{4}{r} & -2\\ -2 & \frac{4}{r}\end{pmatrix},\quad \text{(b)}\,\text{Hess}_{\varphi_\alpha}=\begin{pmatrix}\frac{4}{r} & 0\\ 0 & r\end{pmatrix},\quad \text{(c)}\,\text{Hess}_{\varphi_\alpha}=\begin{pmatrix}\frac{4}{r} & -2-2r\end{pmatrix} \tag{50}$$

For (a),

$$\text{Hess}_{\varphi_\alpha}(X,X)=\frac{4+2r}{r}(X_1^2+X_2^2)-2(X_1+X_2)^2=\frac{4+2r}{r}(X_1^2+X_2^2)\geqslant 0 \tag{51}$$

For (b), $\text{Hess}_{\varphi_\alpha}$ is positive definite, i.e., $\text{Hess}_{\varphi_\alpha}(X,X)>0$. For (c), $\text{Hess}_{\varphi_\alpha}$ is positive semi-definite, i.e., $\text{Hess}_{\varphi_\alpha}(X,X)\geqslant 0$.

(iii) For $n\geqslant 3$, when $n=p$, we have

$$\text{Hess}_{\varphi_\alpha}(X,X)=\frac{2(n+r)(n-1)}{r}\Big(\sum_{i=1}^n X_i^2\Big)-2\Big(\sum_{i=1}^n X_i\Big)^2=\frac{2(n+r)(n-1)}{r}\Big(\sum_{i=1}^n X_i^2\Big)\geqslant 0 \tag{52}$$

When $n>p$, let

$$A=\begin{pmatrix}\frac{2(n+r)(n-1)-2r}{r} & -2 & \cdots & -2\\ -2 & \frac{2(n+r)(n-1)-2r}{r} & \cdots & -2\\ \vdots & \vdots & \ddots & \vdots\\ -2 & -2 & \cdots & \frac{2(n+r)(n-1)-2r}{r}\end{pmatrix},$$
$$B=\begin{pmatrix}\frac{2(n+r)(n-1)-2r}{r} & -2 & \cdots & -2 & -2\\ -2 & \frac{2(n+r)(n-1)-2r}{r} & \cdots & -2 & -2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ -2 & -2 & \cdots & \frac{2(n+r)(n-1)-2r}{r} & -2\\ -2 & -2 & \cdots & -2 & \frac{2r}{n}\end{pmatrix} \tag{53}$$

Note that

$$0|\lambda E-A|=\left[\lambda-\frac{2(n+r)(n-1)}{r}\right]^{p-1}\left(\lambda-\frac{2(n+r)(n-1)-2pr}{r}\right)=\left[\lambda-\frac{2(n+r)(n-1)}{r}\right]^{p-1}\left(\lambda-\frac{2n(n-1)+2(n-1-p)r}{r}\right) \tag{54}$$

Thus, all eigenvalues of A are greater than 0, i.e., A is positive definite.

Since $n>p$, when $n-p=q=1$, we have

$$B=\begin{pmatrix}\frac{2r}{n}\end{pmatrix} \tag{55}$$

B is positive definite. When $n-p=q\geq 2$, we have

$$\begin{aligned}0=|\lambda E-B|=&\left[\lambda-\frac{2(n+r)(n-1)}{r}\right]^{q-2}\left[\left(\lambda-\frac{2(n+r)}{n}\right)\left(\lambda-\frac{2(n+r)(n-1)-2(q-1)r}{r}\right)-2\left(-\lambda+\frac{2(n+r)(n-1)}{r}\right)\right]=\\&\left[\lambda-\frac{2(n+r)(n-1)}{r}\right]^{q-2}\left[\lambda^2-\left(\frac{2(n+r)}{n}+\frac{2(n+r)(n-1)-2(q-1)r}{r}-2\right)\lambda+\right.\\&\left.\frac{2(n+r)}{n}\times\frac{2(n+r)(n-1)-2(q-1)r}{r}-\frac{4(n+r)(n-1)}{r}\right]=\\&\left[\lambda-\frac{2(n+r)(n-1)}{r}\right]^{q-2}\left[\lambda^2-\left(\frac{2r}{n}+\frac{2(n+r)(n-1)-2(q-1)r}{r}\right)\lambda+\frac{4(n+r)}{r}\times\frac{rp}{n}\right]\end{aligned} \tag{56}$$

Since

DOI: 10.52396/JUSTC-2021-0217
JUSTC, **2022**, 52(3): 6

$$\lambda^2-(\frac{2r}{n}+\frac{2(n+r)(n-1)-2(q-1)r}{r})\lambda+\frac{4(n+r)}{r}\times\frac{rp}{n}=0 \quad (57)$$

we have

$$\lambda_{n-1}\lambda_n=\frac{4(n+r)}{r}\times\frac{rp}{n}\geqslant 0, \lambda_{n-1}+\lambda_n=$$
$$\frac{2r}{n}+\frac{2(n+r)(n-1)-2(q-1)r}{r}=\frac{2r}{n}+\frac{2n(n-1)+2pr}{r}>0 \quad (58)$$

that is,

$$\lambda_{n-1}\geqslant 0,\quad \lambda_n>0 \quad \text{or} \quad \lambda_{n-1}>0,\quad \lambda_n\geqslant 0 \quad (59)$$

We prove that all eigenvalues of B are greater than or equal to 0, i.e., B is positive semi-definite. Thus, we prove that $\mathrm{Hess}_{\varphi_\alpha}(X,X)\geqslant 0$.

Combining (i), (ii) and (iii), we have

$$\Lambda(X,X)\geqslant \mathrm{Hess}_{\varphi_\alpha}(X,X)\geqslant 0 \quad (60)$$

Hence, by Eq. (41), the point $(h^\alpha_{11},h^\alpha_{22},\cdots,h^\alpha_{nn})$ is a global minimum point. From Eqs. (37) and (41), we have

$$\varphi_\alpha\geqslant\frac{(n^2-n+nr-2r)[p^3r^3+(q-1)r(n^2-n+qr-r)^2](K^\alpha)^2}{\{(n^2-n-r+qr)^2+p^2r^2\}^2}+$$
$$\frac{rn(n-1)^2(n^2-n+qr-r)^2(K^\alpha)^2}{\{(n^2-n-r+qr)^2+p^2r^2\}^2}-\frac{p(p-1)(pr^2)^2(K^\alpha)^2}{\{(n^2-n-r+qr)^2+p^2r^2\}^2}-$$
$$\frac{\{(q-1)(q-2)(n^2-n+qr-r)^2r^2+2(n^2-n+qr-r)^2rn(n-1)(q-1)\}(K^\alpha)^2}{\{(n^2-n-r+qr)^2+p^2r^2\}^2}=$$
$$\frac{pr(n^2-n+qr-r)^3(K^\alpha)^2}{\{(n^2-n-r+qr)^2+p^2r^2\}^2}+\frac{p^3r^3(n^2-n+qr-r)(K^\alpha)^2}{\{(n^2-n-r+qr)^2+p^2r^2\}^2}=$$
$$\frac{pr(n^2-n+qr-r)\{(n^2-n-r+qr)^2+p^2r^2\}(K^\alpha)^2}{\{(n^2-n-r+qr)^2+p^2r^2\}^2}=$$
$$\frac{pr(n^2-n+qr-r)(K^\alpha)^2}{(n^2-n-r+qr)^2+p^2r^2} \quad (61)$$

We divide the proof of Theorem 1.1 into two main cases, according to $0<r<n(n-1)$ or $r>n(n-1)$.

Case 1: $0<r<n(n-1)$. In this case, using (38) and (61), we derive

$$\mathcal{P}\geqslant\sum_{\alpha=n+1}^{m}\frac{pr(n^2-n+qr-r)(K^\alpha)^2}{(n^2-n-r+qr)^2+p^2r^2}=\frac{pr(n^2-n+qr-r)n^2\|H\|^2}{(n^2-n-r+qr)^2+p^2r^2} \quad (62)$$

By Eqs. (34) and (62), we derive

$$2\times\left\{\frac{q\Delta f}{f}+\frac{p(p-1)a}{2}+b(p-1)\|P^T\|^2_{N_1^p}+\frac{q(q-1)a}{2}+b(q-1)\|P^T\|^2_{N_2^q}\right\}+rC+$$
$$\frac{(n-1)(n+r)(n^2-n-r)}{nr}C(L)\geqslant 2\tau+\frac{pr(n^2-n+qr-r)n^2\|H\|^2}{(n^2-n-r+qr)^2+p^2r^2} \quad (63)$$

Taking the infimum over all tangent hyperplanes L of T_xM^n in (63), we obtain

$$\frac{2}{n(n-1)}\times\left\{\frac{q\Delta f}{f}+\frac{p(p-1)a}{2}+b(p-1)\|P^T\|^2_{N_1^p}+\frac{q(q-1)a}{2}+b(q-1)\|P^T\|^2_{N_2^q}\right\}+$$
$$\frac{\delta_C(r;n-1)}{n(n-1)}-\frac{npr(n^2-n+qr-r)\|H\|^2}{(n-1)\left\{(n^2-n-r+qr)^2+p^2r^2\right\}}\geqslant\rho \quad (64)$$

Case 2: $r>n(n-1)$. Similarly, using the same method, we obtain

$$\frac{2}{n(n-1)}\times\left\{\frac{q\Delta f}{f}+\frac{p(p-1)a}{2}+b(p-1)\|P^T\|^2_{N_1^p}+\frac{q(q-1)a}{2}+b(q-1)\|P^T\|^2_{N_2^q}\right\}+$$
$$\frac{\widehat{\delta_C}(r;n-1)}{n(n-1)}-\frac{npr(n^2-n+qr-r)\|H\|^2}{(n-1)\left\{(n^2-n-r+qr)^2+p^2r^2\right\}}\geqslant\rho \quad (65)$$

Equalities hold in (64) and (65) at a point $x\in M^n$ if and only if inequalities (35) and (61) become equalities. Thus, we have

DOI: 10.52396/JUSTC-2021-0217
JUSTC, 2022, 52(3): 6

$$
\left.\begin{aligned}
h_{11}^{\alpha} = \cdots = h_{pp}^{\alpha} &= \frac{pr^2}{(n^2-n-r+qr)^2+p^2r^2}K^{\alpha} \\
h_{p+1p+1}^{\alpha} = \cdots = h_{n-1n-1}^{\alpha} &= \frac{(n^2-n+qr-r)r}{(n^2-n-r+qr)^2+p^2r^2}K^{\alpha} \\
h_{nn}^{\alpha} &= \frac{n(n-1)(n^2-n+qr-r)}{(n^2-n-r+qr)^2+p^2r^2}K^{\alpha} \\
h_{ij}^{\alpha} &= 0, \qquad i \neq j
\end{aligned}\right\} \tag{66}
$$

By choosing an orthonormal basis such that e_{n+1} is in the direction of the mean curvature vector, we have

$$
\left.\begin{aligned}
h(e_1,e_1) = \cdots = h(e_p,e_p) &= pr^2 f_1 e_{n+1}, \\
h(e_{p+1},e_{p+1}) = \cdots = h(e_{n-1},e_{n-1}) &= (n^2-n+qr-r)rf_1 e_{n+1}, \\
h(e_n,e_n) &= n(n-1)(n^2-n+qr-r)f_1 e_{n+1}, \\
h(e_i,e_j) &= 0, \qquad i \neq j
\end{aligned}\right\} \tag{67}
$$

where $f_1 = \dfrac{K^{n+1}}{(n^2-n+qr-r)^2+p^2r^2}$ is a function on M^n.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (12026262).

Conflict of interest

The authors declare that they have no conflict of interest.

Biographies

Jiahui Wang is currently a graduate student at the Anhui University of Technology. Her research interests mainly focus on warped product submanifolds and isoparametric hypersurfaces.

Yecheng Zhu received his PhD from the University of Science and Technology of China. He is currently an associate professor at the Anhui University of Technology. He is mainly engaged in differential geometry.

References

[1] Chen B Y. Some pinching and classification theorems for minimal submanifolds. *Arch. Math.*, **1993**, *60*: 568–578.

[2] Chen B Y. A Riemannian invariant and its applications to submanifold theory. *Results in Mathematics*, **1995**, *27*: 17–26.

[3] Casorati F. Mesure de la courbure des surfaces suivant l'idée commune.: Ses rapports avec les mesures de courbure gaussienne et moyenne. *Acta. Math.*, **1890**, *14*: 95–110.

[4] Decu S, Haesen S, Verstraelen L. Optimal inequalities involving Casorati curvatures. *Bull.Transilv. Univ. Brasov Ser. B*, **2007**, *14*: 85–93.

[5] Decu S, Haesen S, Verstraelen L. Optimal inequalities characterising quasi-umbilical submanifolds. *J. Inequal. Pure and Appl. Math.*, **2008**, *9*: 79.

[6] Park K S. Inequalities for the Casorati curvatures of real hypersurfaces in some Grassmannians. *Taiwanese J. Math.*, **2018**, *22*: 63–77.

[7] Choudhary M A, Blaga A M. Inequalities for generalized normalized δ-Casorati curvatures of slant submanifolds in metallic Riemannian space forms. *J. Geom.*, **2020**, *111*: 39.

[8] Chen B Y, Yano K. Hypersurfaces of a conformally flat space. *Tensor, N. S.*, **1972**, *26*: 318–322.

[9] Chen B Y. Another general inequality for *CR*-warped products in complex space forms. *Hokkaido Math. J.*, **2003**, *32*: 415–444.

[10] Oprea T. Chen's inequality in the Lagrangian case. *Colloq. Math.*, **2007**, *108*: 163–169.

[11] Vîlcu G E. An optimal inequality for Lagrangian submanifolds in complex space forms involving Casorati curvature. *J. Math. Anal. Appl.*, **2018**, *465*: 1209–1222.