第49卷第9期 中国科学技术大学学报 Vol. 49, No. 9
2019年9月 JOURNAL OF UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA Sep. 2019

文章编号:0253-2778(2019)09-0689-10

On the Sparre-Andersen dual model perturbed by diffusion

CHEN Yu, ZHANG Qi

(*Department of Statistics and Finance, School of Management, University of Science and Technology of China, Hefei* 230026, *China*)

Abstract: A diffusion perturbed Sparre-Andersen dual risk model was studied, in which the times between gains are independent and identically distributed random variables with a generalized Erlang(n) distribution. An integro-differential equation with certain boundary for the Laplace transform of the ruin time was derived and then its explicit expression was obtained. In particular, an explicit form of the Laplace transform of the time to ruin were studied when jump sizes were exponential. Finally, by studying the expected discounted dividends with the threshold-dividend strategy in the diffusion perturbed Sparre-Andersen dual risk model, an integro-differential equation with certain boundary for the expected discounted dividends was derived.

Key words: Sparre-Andersen dual model; generalized Erlang(n) innovation times; ruin time; discounted dividend payments

CLC number: O211.67 **Document code**: A doi:10.3969/j.issn.0253-2778.2019.09.001

2010 Mathematics Subject Classification: 62P05

Citation: CHEN Yu, ZHANG Qi. On the Sparre-Andersen dual model perturbed by diffusion[J]. Journal of University of Science and Technology of China, 2019,49(9):689-698.
陈昱,张棋. 带干扰的 Sparre-Andersen 对偶风险模型[J]. 中国科学技术大学学报,2019,49(9):689-698.

带干扰的 Sparre-Andersen 对偶风险模型

陈 昱,张 棋

(中国科学技术大学管理学院统计与金融系,合肥 230026)

摘要:研究了带干扰的对偶风险模型,其中收入时间间隔是服从于广义 Erlang(n)分布的独立同分布的随机变量.推导出了破产时间 Laplace 变换满足的积分-微分方程和边界条件,并且得到了其精确表表达式.特别地,以收入变量服从指数分布为例,给出了破产时间 Laplace 变换的具体解.最后,考虑了阈值分红下的带干扰的对偶风险模型,得到了期望折现分红满足的积分-微分方程和边界条件.

关键词:Sparre-Andersen 对偶模型;广义 Erlang(n)更新时间;破产时间;折现分红支付

0 Introduction

The dual model to the classical risk model describes the surplus process $U=\{U(t), t\geqslant 0\}$ of a portfolio of insurance contracts as

Received: 2017-12-02; **Revised**: 2018-06-01

Foundation item: Supported by the National Key Research and Development Plan(2016YFC0800104), the National Natural Science Foundation of China(71771203).

Biography: CHEN Yu (corresponding author), female, born in 1978, PhD/ associate Prof. Research field: Limit theorem in risk theory. E-mail: cyu@ustc.edu.cn

$$U(t)=u-ct+\sum_{i=1}^{N(t)}X_i,t\geqslant 0 \quad (1)$$

where $u(>0)$ is the initial capital and $c(>0)$ is the expense rate. The revenue number process $\{N(t),t\geqslant 0\}$, defined by $N(t)=\max\{n:T_1+T_2+\cdots+T_n\leqslant t\}$, denoting the number of revenues up to time t, is a renewal process, where T_i for $i\geqslant 2$ denotes the inter-income time between the $(i-1)$th and the ith income arrival and T_1 is the time until the first income arrival. $\{X_n,n\geqslant 1\}$ (representing the individual revenue amounts and independent of $\{N(t),t\geqslant 0\}$) is a sequence of independent and identically distributed strictly positive random variables with a common distribution function $P(x)$ that satisfies $P(0)=0$ and has a positive mean μ. We assume that $P(x)$ is differentiable and $p(x)=P'(x)$ is the individual revenue amount probability density function. Further assume $cE(T_i)<\mu$, providing a positive safety loading.

Recently, dual risk models have drawn lots of attention in ruin theory. For the compound Poisson dual risk model, Avanzi et al.[1] have studied the expected total discounted dividends until ruin with barrier and Albrecher et al.[2] have considered the tax payment problem when the surplus is at a running maximum. And for expected total discounted dividends before ruin for exponentially distributed profits, see also Ref. [3]. Besides, Landriault and Sendova[4] generalized the Sparre-Andersen dual risk model by adding a budget-restriction strategy. Ji and Zhang[5] have shown the roots to the Lundberg's equation of the Sparre-Andersen dual risk model are distinct, Rodríguez et al.[6] derived an explicit form of the Laplace transform of the ruin time under the Sparre-Andersen dual risk model. Moreover, Yang and Sendova[7] derived an explicit expression for the Laplace transform of the ruin time, which involves multiple roots and also obtains the expected discounted dividends for the threshold-dividend strategy in the Sparre-Andersen dual risk model. In this paper, we study a diffusion perturbed Sparre-Andersen dual risk model, whose inter-gain times are generalized Erlang(n) distributed. We derive an integro-differential equation with certain boundary for the Laplace transform of the ruin time and then we obtain its explicit expression. In particular, we derive an explicit form of the Laplace transform of the time to ruin when jump sizes are exponential. Finally, we consider a threshold dividend payment strategy in the model, and derive an integro-differential equation with certain boundary for the expected discounted dividends.

In this paper, we consider the diffusion perturbed Sparre-Anderson dual risk model, which can be described as:

$$U(t)=u-ct+\sum_{i=1}^{N(t)}X_i+\sigma B(t),t\geqslant 0 \quad (2)$$

where $c>0$ represents the constant expense rate, $S(t)=\sum_{i=1}^{N(t)}X_i$ is the aggregate revenue from time 0 up to time t, $\{B(t),t\geqslant 0\}$ is a standard Wiener process (that is independent of $S(t)$) and $\sigma\geqslant 0$ is the dispersion parameter. $\sum_{j=1}^{n}\frac{1}{\lambda_j}<\frac{\mu}{c}$ is the relative security loading. The time of ruin is defined as $\tau:=\inf\{t\geqslant 0:U(t)=0\}$. And probability of ruin can be defined as

$$\psi(u)=P(\tau<\infty\mid U(0)=u)=$$
$$E[I(\tau<\infty)\mid U(0)=u](u>0),$$

in which $I(A)$ is the indicator function of an event A. When the initial capital is u, Laplace transform of the ruin time can be defined as:

$$\psi_\delta(u)=E[e^{-\delta\tau}I(\tau<\infty)\mid U(0)=u]\ (u>0).$$

The rest of the paper is organized as follows. In Section 1, we derive an integro-differential equation with certain boundary for the Laplace transform of the ruin time. In Section 2, we solve the integro-differential equation deduced in Section 1 and obtain an explicit form of Laplace transform of the ruin time. In Section 3, we obtain an integro-differential equation with certain boundary for the expected discounted dividends when the threshold dividend strategy is discussed.

1 An integro-differential equation for Laplace transform of the ruin time

In this section, we focus on the Laplace transform of the ruin time of the risk process (2). A generalized Erlang(n) random variable can be expressed as an independent sum of n exponential random variables, i. e., Z is a generalized Erlang(n) random variable, Z can be denoted as

$$Z \stackrel{d}{=} W_1 + W_2 + \cdots + W_n,$$

where $W_i (i = 1, 2, \cdots, n)$ are independent exponential random variables with parameters $\lambda_i (i=1, 2, \cdots, n)$. Then we will derive an integro-differential equation with certain boundary for the Laplace transform of the ruin time.

Theorem 1. 1 When $u > 0$, the Laplace transform of the time to ruin $\psi_\delta(u)$ satisfies

$$\left\{\prod_{i=1}^{n}\left(I + \frac{\delta}{\lambda_i} I + \frac{c}{\lambda_i}\frac{\partial}{\partial u} - \frac{\sigma^2}{2\lambda_i}\frac{\partial^2}{\partial u^2}\right)\right\}\psi_\delta(u) = \int_0^\infty \psi_\delta(u+y)p(y)\mathrm{d}y \tag{3}$$

and boundary conditions

$$\left.\left\{\prod_{i=1}^{k}\left(I + \frac{\delta}{\lambda_i} I + \frac{c}{\lambda_i}\frac{\partial}{\partial u} - \frac{\sigma^2}{2\lambda_i}\frac{\partial^2}{\partial u^2}\right)\right\}\psi_\delta(u)\right|_{u=0} = 1, \quad k = 0,1,2,\cdots,n-1 \tag{4}$$

where I, $\frac{\partial}{\partial u}$ and $\frac{\partial^2}{\partial u^2}$ denote the identity operator, differential operator and second-difference operator, respectively.

Proof We consider the first time interval $[0, T_1]$. Since T_1 follows a generalized Erlang(n) distribution, we can consider n states of the risk process by decomposing the inter-income time into the independent sum of exponential random variables with parameters $\lambda_1, \lambda_2, \cdots, \lambda_n$, i. e.,

$$T_1 = W_1 + W_2 + \cdots + W_n.$$

Suppose that there are $n-1$ incomes that arrive at $S_i = \sum_{j=1}^{i} W_j, i = 1,2,\cdots, n-1$, but every income amount is zero. Let $T_1^{(i)} = \sum_{j=i}^{n} W_j, i = 1,2,\cdots, n-1$. Define

$$N_i(t) = \max\{n: T_1^{(i)} + T_2 + \cdots + T_n \leqslant t\} \tag{5}$$

thus $\{N_i(t), t \geqslant 0\}$ becomes a delayed update process and $N_1(t)$ is the same process as $N(t)$. Now we introduce some notations in order to prove this theorem. Denote by

$$U_i(t) = u - ct + \sum_{i=1}^{N_i(t)} X_i + \sigma B(t), \ t \geqslant 0;$$

and

$\psi_{\delta,i}(u) = E[\mathrm{e}^{-\delta\tau_i} I(\tau_i < \infty) \mid U_i(0) = u](u > 0)$ for $i = 1,2,\cdots,n$, where τ_i is the time of ruin which defined as $\tau_i := \inf\{t \geqslant 0: U_i(t) = 0\}$. It is clear that $\psi_\delta(u)$ can be obtained from $\psi_{\delta,i}(u)$ by letting $i=1$. When $j = 1,\cdots,n-1$, by considering whether or not W_j is greater than the infinitesimal time $\Delta t (\leqslant \frac{u}{c})$ and using the total probability theorem, we have

$$\begin{aligned}\psi_{\delta,j}(u) = {} & P(W_j > \Delta t)\mathrm{e}^{-\delta\Delta t} \cdot E[\psi_{\delta,j}(u - c\Delta t + \sigma B(\Delta t))] + \\ & P(W_j \leqslant \Delta t)\mathrm{e}^{-\delta\Delta t} \cdot E[\psi_{\delta,j+1}(u - c\Delta t + \sigma B(\Delta t))] + o(\Delta t)\end{aligned} \tag{6}$$

For

$$\left.\begin{aligned} P(W_j > \Delta t)\mathrm{e}^{-\delta\Delta t} &= \mathrm{e}^{-(\lambda_j+\delta)\Delta t} = 1 - (\lambda_j + \delta)\Delta t + o(\Delta t), \\ P(W_j \leqslant \Delta t)\mathrm{e}^{-\delta\Delta t} &= (1 - \mathrm{e}^{-\lambda_j \Delta t})\mathrm{e}^{-\delta\Delta t} = \lambda_j \Delta t + o(\Delta t)\end{aligned}\right\} \tag{7}$$

Taylor expansion yields

$$\psi_{\delta,j}(u - c\Delta t + \sigma B(\Delta t)) = \sum_{k=0}^{2}\frac{1}{k!}\psi_{\delta,j}^{(k)}(u)[-c\Delta t + \sigma B(\Delta t)]^k + \frac{1}{3!}\psi_{\delta,j}^{(3)}(u^*)[-c\Delta t + \sigma B(\Delta t)]^3,$$

where u^* is during the time interval from $u - c\Delta t + \sigma B(\Delta t)$ to u. In addition, $E[B(\Delta t)] = E[B^3(\Delta t)] = 0$ and $E[B^2(\Delta t)] = V[B(\Delta t)] = \Delta t$, we have

$$E[\psi_{\delta,j}(u - c\Delta t + \sigma B(\Delta t))] = \psi_{\delta,j}(u) - c\psi'_{\delta,j}(u)\Delta t + \frac{\sigma^2}{2}\psi''_{\delta,j}(u)\Delta t + o(\Delta t) \tag{8}$$

Thus,

$$\begin{aligned}\psi_{\delta,j}(u) = {} & [1 - (\lambda_j + \delta)\Delta t] \cdot [\psi_{\delta,j}(u) - c\psi'_{\delta,j}(u)\Delta t + \frac{\sigma^2}{2}\psi''_{\delta,j}(u)\Delta t] + \\ & \lambda_j \Delta t \cdot [\psi_{j+1}(u) - c\psi'_{\delta,j+1}(u)\Delta t +\end{aligned}$$

$$\frac{\sigma^2}{2}\psi''_{\delta,j+1}(u)\Delta t]+o(t) \tag{9}$$

After sorting, dividing two sides of the equation by Δt and letting $\Delta t \to 0$, we have

$$\psi_{\delta,j+1}(u)=(1+\frac{\delta}{\lambda_i})\psi_{\delta,j}(u)+ \frac{c}{\lambda_j}\psi'_{\delta,j}(u)-\frac{\sigma^2}{2\lambda_j}\psi''_{\delta,j}(u)= \left[(1+\frac{\delta}{\lambda_i})I+\frac{c}{\lambda_j}\frac{\partial}{\partial u}-\frac{\sigma^2}{2\lambda_j}\frac{\partial^2}{\partial u^2}\right]\psi_{\delta,j}(u) \tag{10}$$

When $j=n$, if $W_n \leqslant \Delta t$, then there is a revenue during the time interval $[0,\Delta t]$. Hence,

$$\psi_{\delta,n}(u)=P(W_n>\Delta t)e^{-\delta\Delta t}\cdot E\left[\psi_{\delta,n}(u-c\Delta t+\sigma B(\Delta t))\right]+ P(W_n\leqslant \Delta t)e^{-\delta\Delta t}\cdot E[\psi_{\delta,1}(u-c\Delta t+\sigma B(\Delta t)+X)]+o(\Delta t) \tag{11}$$

Again, we expand $\psi_{\delta,n}(u-c\Delta t+\sigma B(\Delta t))$ and $\psi_{\delta,1}(u-c\Delta t+\sigma B(\Delta t)+X)$ in a Taylor's series u, $u+X$ respectively, to the term of $\psi'_{\delta,n}$, $\psi''_{\delta,n}$ to get,

$$\left[(1+\frac{\delta}{\lambda_n})I+\frac{c}{\lambda_n}\frac{\partial}{\partial u}-\frac{\sigma^2}{2\lambda_n}\frac{\partial^2}{\partial u^2}\right]\psi_{\delta,n}(u)= \int_0^\infty \psi_{\delta,1}(u+y)p(y)dy \tag{12}$$

Applying Eq. (10), we have

$$\left.\begin{aligned}&\psi_{\delta,k}(u)=\\&\left\{\prod_{i=1}^{k-1}\left[(1+\frac{\delta}{\lambda_i})I+\frac{c}{\lambda_i}\frac{\partial}{\partial u}-\frac{\sigma^2}{2\lambda_i}\frac{\partial^2}{\partial u^2}\right]\right\}\psi_\delta(u),\\&k=2,\cdots,n\end{aligned}\right\} \tag{13}$$

Because $\psi_{\delta,k}(0)=1$, we arrive at the boundary conditions.

According to $\psi_\delta(u)=\psi_{\delta,1}(u)$ together with Eqs. (12) and (13), we get

$$\left\{\prod_{i=1}^{n}\left((1+\frac{\delta}{\lambda_i})I+\frac{c}{\lambda_i}\frac{\partial}{\partial u}-\frac{\sigma^2}{2\lambda_i}\frac{\partial^2}{\partial u^2}\right)\right\}\psi_\delta(u)= \int_0^\infty \psi_\delta(u+y)p(y)dy.$$

We arrive at Eq. (3).

In particular, the ruin probability for the process $\{U(t),t\geqslant 0\}$, denoted $\psi(u)$, is obtained from $\psi_\delta(u)$ by letting $\delta=0$.

Theorem 1.2 When $u>0$, $\psi(u)$ is the solution of the following integro-differential equation:

$$\left\{\prod_{i=1}^{n}\left(I+\frac{c}{\lambda_i}\frac{\partial}{\partial u}-\frac{\sigma^2}{2\lambda_i}\frac{\partial^2}{\partial u^2}\right)\right\}\psi(u)= \int_0^\infty \psi(u+y)p(y)dy \tag{14}$$

and with boundary conditions

$$\left.\begin{aligned}&\left\{\prod_{i=1}^{k}\left(I+\frac{c}{\lambda_i}\frac{\partial}{\partial u}-\frac{\sigma^2}{2\lambda_i}\frac{\partial^2}{\partial u^2}\right)\right\}\psi(u)\Big|_{u=0}=1,\\&k=0,1,2,\cdots,n-1\end{aligned}\right\} \tag{15}$$

2 The explicit expression of the Laplace transform of the ruin time

In order to solve Eq. (3), we first derive the Lundberg's equation. Then, applying Rouch Theorem we prove that the generalized Lundberg's equation has n roots in the right half of the complex plane and provide the explicit expression of the Laplace transform of the time to ruin.

Lemma 2.1 Let $(\Omega,\mathscr{F},\mathbb{P})$ be the complete probability space and $\{\tau_k\}_{k=0}^n$ be a series of stop times satisfying the condition $\tau_0\equiv 0$. If $\{\tau_k-\tau_{k-1}\}_{k=1}^n$ are independent and identically distributed (i. i. d), so are $\{B(\tau_k)-B(\tau_{k-1})\}_{k=1}^n$.

The Lemma has been proved by Ref. [8].

Let $M_0=0$ and $M_k=\sum_{j=1}^k T_j$ for $k\in\mathbb{N}_+$, denote the occurrence time of the kth income event. Let $U(0)=u$ and $U_k(k\in\mathbb{N}_+)$, denote the instantaneous surplus after the kth income event. We have

$$U_k=U(M_k)=u-cM_k+\sum_{i=1}^k X_i+\sigma B(M_k)= u-c\sum_{j=1}^k T_j+\sigma\sum_{j=1}^k[B(M_j)-B(M_{j-1})]+ \sum_{j=1}^k X_i \overset{d}{=} u-c\sum_{j=1}^k T_j+\sigma\sum_{j=1}^k B(T_j)+\sum_{j=1}^k X_i \tag{16}$$

Hence, $\{U_k\}_{k\geqslant 0}$ is a discrete time Markov chain with stationary and independent increments. Also, for all $\delta>0$, the random variable sequence $\{e^{-\delta M_k-sU_k}\}_{k\in\mathbb{N}_+}$ has stationary independent increments.

Lemma 2.2 If $R(t)$ is a stochastic process with stationary and independent increments, the sufficient and necessary condition for $\{e^{-R(t)}:t\geqslant 0\}$ being a martingale is: for all $t\geqslant 0$,

$$E[e^{-R(t)}\mid R(0)=u]=e^{-u}.$$

For further detail on martingale, see Ref. [9]. Assume that the generalized Erlang(n) distribution with parameters $\lambda_1,\lambda_2,\cdots,\lambda_n>0$ with probability density function $f(t)$, the Laplace form of $f(t)$ can be expressed as

$$\tilde{f}(s)=\int_0^\infty e^{-st}f(t)\mathrm{d}t=\prod_{j=1}^{n}\frac{\lambda_j}{\lambda_j+s},\mathrm{Re}(s)\geqslant 0 \tag{17}$$

where $\mathrm{Re}(s)$ is the real part of s. Hence, to find $s\in\mathbb{C}$ make $\{e^{-\delta M_k-sU_k}\}_{k\in\mathbb{N}_+}$ is a martingale, we should have for s,

$$E[e^{-\delta T_1-s(X_1-cT_1+\sigma B(T_1))}]=1 \tag{18}$$

which means that

$$\int e^{-\delta t+sct-\frac{s^2\sigma^2}{2}}f(t)\mathrm{d}t\cdot\int e^{-sy}p(y)\mathrm{d}y=1.$$

Thus we get

$$\tilde{f}\left(\delta-sc-\frac{s^2\sigma^2}{2}\right)\tilde{p}(s)=1 \tag{19}$$

which is the generalized Lundberg's equation for the generalized Erlang(n) dual risk process, in which $\tilde{f}(s),\tilde{p}(s)$ are the Laplace form of $f(s)$ and $p(s)$, respectively.

Because a sequence of i.i.d random vectors T_i have common generalized Erlang(n) distribution, according to (17), the generalized Lundberg equation can be written as

$$\prod_{j=1}^{n}[1+\frac{\delta}{\lambda_j}-\frac{c}{\lambda_j}s-\frac{\sigma^2}{2\lambda_j}s^2]=\tilde{p}(s) \tag{20}$$

In order to simplify Eq. (20), letting

$$r(s)=\prod_{j=1}^{n}[1+\frac{\delta}{\lambda_j}-\frac{c}{\lambda_j}s-\frac{\sigma^2}{2\lambda_j}s^2],$$

Eq. (20) becomes

$$r(s)=\tilde{p}(s) \tag{21}$$

Now we obtained the generalized Lundberg equation and then we give a lemma which indicated that Eq. (20) has exactly n roots in the right part of the complex plane.

Lemma 2.3 When $\delta>0$, the generalized Lundberg Eq. (20) has exactly n roots $\rho_1(\delta)$, $\rho_2(\delta),\cdots,\rho_n(\delta)$ in the right part of the complex plane, for $j=1,2,\cdots,n$, $\mathrm{Res}(\rho_j)>0$.

For the proof please refer to Ref. [8].

To obtain the explicit expression of the Laplace transform of the ruin time, we need to refer to the lemma mentioned in Ref. [7].

Lemma 2.4 Suppose that $h(x)$ is an arbitrary polynomial of degree $2n\in\mathbb{N}$, namely,

$$h(x)=h_0+h_1x+\cdots+h_{2n}x^{2n},$$
$$h_0,h_1,\cdots,h_{2n}\in\mathscr{C},\ h_n\neq 0.$$

Define

$$h\left(\frac{\partial}{\partial u}\right)f(u)=h_0f(u)+h_1f'(u)+\cdots+h_{2n}f^{(2n)}(u),\ f\in\mathscr{C}^n(-\infty,+\infty),$$

then for any $i\in\mathbb{N},\xi\in\mathbb{R}$,

$$h\left(\frac{\partial}{\partial u}\right)(u^ie^{\xi u})=e^{\xi u}\sum_{j=0}^{2n}\frac{h^{(j)}(\xi)}{j!}\frac{\partial^j}{\partial u^j}(u^i).$$

Then, we formally give the explicit expression of the Laplace transform of the ruin time.

Theorem 2.1 Suppose the generalized Lundberg Eq. (20) has m different roots in the right part of the complex plane: $\rho_1(\delta),\rho_2(\delta),\cdots,\rho_m(\delta)$, that are roots with multiplicity $\upsilon_i(i=1,2,\cdots,m)$ respectively. For any polynomial of degree $\upsilon_i-1\in\mathbb{N}$, $\pi_{\upsilon_i}(u)=r_{i,0}+r_{i,1}u+\cdots+r_{i,\upsilon-1}u^{\upsilon_i-1}(r_{i,\upsilon-1}\neq 0)$, the explicit expression of Eq. (3) can be written as

$$\psi_\delta(u)=\sum_{j=1}^{m}(\sum_{k=0}^{\upsilon_j-1}r_{j,k}u^k)e^{-\rho_ju} \tag{22}$$

Proof First of all, we prove that if $\rho\in\mathbb{C}$, $\mathrm{Res}(\rho)>0$ is a root with multiplicity υ of the generalized Lundberg Eq. (20), then for any polynomial $\pi_\upsilon(u)$ of degree $\upsilon-1$, $\pi_\upsilon(u),\psi^*(u)=\pi_\upsilon(u)e^{-\rho u}$, is a solution of Eq. (3).

Let

$$h(x)=\prod_{j=1}^{n}(\lambda_j+\delta+cx-\frac{\sigma^2}{2}x^2),$$

using Lemma 2.4, we have

$$\left[\prod_{j=1}^{n}(\lambda_j+\delta+c\frac{\partial}{\partial u}-\frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}\right]\psi^*(u)=$$
$$h\left(\frac{\partial}{\partial u}\right)(\sum_{i=0}^{\upsilon-1}r_iu^ie^{-\rho u})=\sum_{i=0}^{\upsilon-1}r_ih\left(\frac{\partial}{\partial u}\right)(u^ie^{-\rho u})=$$

$$\sum_{i=0}^{\upsilon-1} r_i e^{-\rho u} \cdot \sum_{j=0}^{2n} \frac{h^{(j)}(-\rho)}{j!} \frac{\partial^j}{\partial u^j}(u^i).$$

Substituting the above back into Eq. (20) yields

$$h(-s) - (\prod_{j=1}^{n} \lambda_j)\tilde{p}(s) = 0 \quad (23)$$

Since ρ is a root of Eq. (20) with multiplicity υ, we may write

$$h(-s) - (\prod_{j=1}^{n} \lambda_j)\tilde{p}(s) = (s-\rho)^{\upsilon}\eta(s) \quad (24)$$

where $\eta(s)$ is an analytical function with $\eta(\rho) \neq 0$. Then for $j = 1, 2, \cdots, \upsilon - 1$, differentiation j times of (24) yields

$$(-1)^j h^{(j)}(-\rho) - (\prod_{j=1}^{n} \lambda_j)\tilde{p}^{(j)}(\rho) =$$

$$\sum_{k=0}^{j} \left[\binom{j}{k} \frac{\upsilon!}{(\upsilon-k)!}(s-\rho)^{\upsilon-k}\eta^{(j-k)}(s)\right]\Big|_{s=\rho} = 0,$$

i. e.,

$$\left.\begin{aligned} &h^{(j)}(-\rho) = (-1)^j (\prod_{j=1}^{n} \lambda_j)\tilde{p}^{(j)}(\rho), \\ &j = 0, 1, \cdots, \upsilon - 1 \end{aligned}\right\} \quad (25)$$

In addition, for $i \leqslant \upsilon - 1 < n$, when $i < j \leqslant n$, $\frac{d^j}{du^j}(u^i) = 0$. Hence, we have

$$\sum_{j=0}^{2n} \frac{h^{(j)}(-\rho)}{j!} \frac{\partial^j}{\partial u^j}(u^i) =$$

$$(\prod_{j=1}^{n} \lambda_j) \sum_{j=0}^{i} \frac{(-1)^j \tilde{p}^{(j)}(\rho)}{j!} \frac{\partial^j}{\partial u^j}(u^i) =$$

$$(\prod_{j=1}^{n} \lambda_j) \sum_{j=0}^{i} \binom{i}{j} (-1)^j \tilde{p}^{(j)}(\rho) u^{i-j}.$$

Thus,

$$\sum_{i=0}^{\upsilon-1} r_i h\left(\frac{\partial}{\partial u}\right)(u^i e^{-\rho u}) =$$

$$(\prod_{j=1}^{n} \lambda_j) \sum_{i=0}^{\upsilon-1} r_i e^{-\rho u} \sum_{j=0}^{i} \binom{i}{j} (-1)^j \tilde{p}^{(j)}(\rho) u^{i-j} \quad (26)$$

Since

$$(-1)^j \tilde{p}^{(j)}(\rho) = \int_0^{\infty} y^i e^{-\rho y} p(y) dy,$$

we have

$$\sum_{i=0}^{\upsilon-1} r_i h\left(\frac{\partial}{\partial u}\right)(u^i e^{-\rho u}) = (\prod_{j=1}^{n} \lambda_j) \sum_{i=0}^{\upsilon-1} r_i e^{-\rho u} \cdot$$

$$\int_0^{\infty} (u+y)^i e^{-\rho y} p(y) dy =$$

$$(\prod_{j=1}^{n} \lambda_j) \int_0^{\infty} \psi^*(u+y) p(y) dy,$$

i. e.,

$$\left[\prod_{j=1}^{n} (\lambda_j + \delta + c\frac{\partial}{\partial u} - \frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2})\right] \psi^*(u) =$$

$$(\prod_{j=1}^{n} \lambda_j) \int_0^{\infty} \psi^*(u+y) dP(y),$$

so $\psi^*(u)$ is a solution of Eq. (3). Then, $\psi_\delta(u)$ has the solution of the following form:

$$\psi_\delta(u) = \sum_{j=1}^{m} (\sum_{k=0}^{\upsilon_j - 1} r_{j,k} u^k) e^{-\rho_j u} + rI(\delta = 0),$$

where ρ_j is a root of Eq. (20) with multiplicity υ_j in the right part of the complex plane.

When $u \to \infty$, we have $\psi_\delta(u) \to 0$, hence $r = 0$. We may satisfy Eq. (27) as

$$\psi_\delta(u) = \sum_{j=1}^{m} (\sum_{k=0}^{\upsilon_j - 1} r_{j,k} u^k) e^{-\rho_j u} \quad (27)$$

That is the result of this section.

The following example provides the Laplace transform of the ruin time for some special cases.

Example 2. 1 Suppose a Sparre-Anderson dual model perturbed by diffusion with exponentially distributed revenue with parameter $\lambda = 1$, while the inter-innovation times are i. i. d generalized Erlang(2) random variables with parameters $\lambda_1 = 4$, $\lambda_2 = 9$. Now, let the expense rate $c = 1$. Then the generalized Lundberg's equation is

$$(4 + \delta - s - \frac{\sigma^2}{2}s^2)(9 + \delta - s - \frac{\sigma^2}{2}s^2) = \frac{36}{s+1}.$$

When $\delta = 0, 0.2$ and $\sigma^2 = 0.25$, 0.5, the generalized Lundberg's equation has 2 positive real part roots (ρ_1, ρ_2) and its corresponding coefficients are $r_{1,0}$, $r_{2,0}$, respectively. We calculate the values of ρ_1, ρ_2, $\rho_{1,0}$ and $\rho_{2,0}$ for $\delta = 0$, 0.2 and $\sigma^2 = 0.25$, 0.5. The results are given in Tab. 1.

Tab. 1 Positive real part roots and its corresponding coefficients

	$\delta=0,\sigma^2=0.25$	$\delta=0,\sigma^2=0.5$	$\delta=0.1,\sigma^2=0.25$	$\delta=0.1,\sigma^2=0.5$
ρ_1,ρ_2	1.7154, 5.7581	1.4002, 4.6492	1.9290, 5.8361	1.5809, 4.7064
$r_{1,0},r_{2,0}$	1.2664, −0.2664	1.2316, −0.2316	1.2850, −0.2850	1.2495, −0.2495

Fig. 1 displays the graph of $\psi_\delta(u)$ for $\delta=0,0.2$ and $\sigma^2=0.25,0.5$. It's not hard to find out that if δ is fixed, $\psi_\delta(u)$ increases as σ^2 increases and if σ^2 is fixed, $\psi_\delta(u)$ decreases as δ increases.

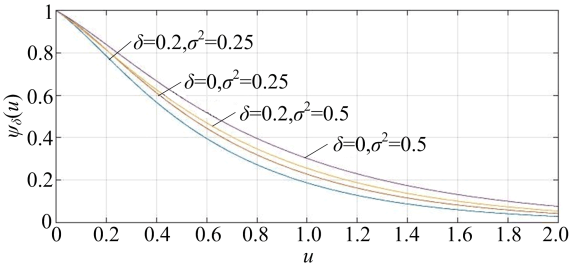

Fig. 1 Graphics of $\psi_\delta(u)$, $\delta=0$, 0.2 and $\sigma^2=0.25,0.5$

3 Expected discounted dividends under a model with a threshold strategy

In this section, we consider a diffusion perturbed Sparre-Anderson dual risk model with generalized Erlang(n) inter-event times' distribution and a threshold strategy b. When the surplus reaches a threshold b, dividends at a constant rate $\alpha>0$. An expense rate without dividend payments, c_1 is assumed to satisfy the security loading condition of model (2) with c replaced by c_1. Let $c_2=\alpha+c_1$. Thus, the dynamics of the surplus process $\{U_b(t),t\geqslant 0\}$ is

$$\mathrm{d}U_b(t)=\begin{cases}-c_1\mathrm{d}t+\mathrm{d}S(t)+\sigma\mathrm{d}B(t),\ 0<U_b(t)\leqslant b;\\ -c_2\mathrm{d}t+\mathrm{d}S(t)+\sigma\mathrm{d}B(t),\ U_b(t)>b.\end{cases}$$

Define the time of ruin $\tau_b:=\inf\{t\geqslant 0:U_b(t)=0\}$. Let $D(t)$ denote the total dividend payments from time 0 to time t. Assume $\delta\geqslant 0$ is an interest force for the calculation of the present value of dividends, then the present value of total dividends until ruin is defined as

$$D_{u,b}=(c_2-c_1)\int_0^{\tau_b}\mathrm{e}^{-\delta t}I(U_b(t)>b)\mathrm{d}t.$$

The expected total dividends paid until ruin are

$$V(u;b)=E[D_{u,b}\mid U_b(0)=u].$$

Write $V(u;b)=V_1(u;b)$ for $U_b(t)\leqslant b$ and $V(u;b)=V_2(u;b)$ for $U(t)\geqslant b$. We derive integro-differential equations for $V_1(u;b)$ and $V_2(u;b)$ in the following part.

Firstly, we give the definition of stopping time and strong Markov property (for further detail on stopping time and strong Markov property, see Ref. [10].

Definition 3.1 Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space, and let $\{\mathscr{F}_t\}_{t\geqslant 0}$ be a σ-filtration. A non-negative random variable T is called a stopping time (with respect to the σ-filtration $\mathscr{F}_t$) if $\{T\leqslant t\}\in\{\mathscr{F}_t\}$ for all $t\geqslant 0$.

Definition 3.2 Suppose the Markov process $\{X(t),t\geqslant 0\}$ and state space $S=\{0,1,2,\cdots\}$, $0\leqslant\rho_1\leqslant\rho_2\leqslant\cdots\leqslant\rho_n(n\geqslant 1)$ are stopping times with respect to $X(t)$. The process $\{X(t),t\geqslant 0\}$ with respect to $\{\rho_k,0\leqslant k\leqslant n\}(n\geqslant 1)$ satisfies strong Markov property, if

$$P\{X(\rho_n+t)=j\mid X(0)=i_0,\ X(\rho_k)=i_k,\cdots,X(\rho_n)=i_n\}=P\{X(\rho_n+t)=j\mid X(\rho_n)=i_n\},$$

for all $j,i_k\in S(0\leqslant k\leqslant n)$.

Strong Markov property has a variety of equivalent forms, one of which is

$[E[f(X_t)\mid\mathscr{F}_s]=E[f(X_t)\mid X_s]\ (0\leqslant s<t)$.

This will be used in the proof of Theorem 3.1.

Theorem 3.1 $V(u;b)$ satisfies the following integro-differential equations.

When $0<u\leqslant b$,

$$\left[\prod_{j=1}^{n}\left(\lambda_j+\delta+c_1\frac{\partial}{\partial u}-\frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}\right)\right]V_1(u;b)=\left(\prod_{j=1}^{n}\lambda_j\right)\left[\int_0^{b-u}V_1(u+y;b)\mathrm{d}P(y)+\int_{b-u}^{\infty}V_2(u+y;b)\mathrm{d}P(y)\right]\tag{28}$$

When $u>b$,

$$\left[\prod_{j=1}^{n}\left(\lambda_j+\delta+c_2\frac{\partial}{\partial u}-\frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}\right)\right]\cdot$$

$$\left[V_2(u;b)-\frac{c_2-c_1}{\delta}\right]=$$

$$\left(\prod_{j=1}^{n}\lambda_j\right)\int_0^{\infty}\left[V_2(u+y;b)-\frac{c_2-c_1}{\delta}\right]\mathrm{d}P(y) \tag{29}$$

and boundary conditions

$$V_1(0;b)=0,$$

$$V_1(b;b)=V_2(b+;b),$$

$$\lim_{u\to\infty}V_2(u;b)=\frac{c_2-c_1}{\delta},$$

$$\frac{\partial^i V_2(u;b)}{\partial u^i}\bigg|_{u=b+}=$$

$$\left(\frac{c_1}{c_2}\right)^i\frac{\partial^i V_1(u;b)}{\partial u^i}\bigg|_{u=b}-\frac{c_2-c_1}{\delta}\left(-\frac{\delta}{c_2}\right).$$

Proof Similar to the proof of Theorem 1.1, let $S_i=\sum_{j=i}^{n}W_j$, in which W_i are assumed independent exponential random variables with parameters λ_i. If $T_1\overset{\mathrm{d}}{=}S_i$ and $T_i\overset{\mathrm{d}}{=}S_1(i\geqslant 2)$, $\{N_i(t),\ t\geqslant 0\}$ (see Eq. (5)) becomes a delayed renewal process, and for $i=1,2,\cdots,n$, we define $U_{i,b}(t)$ as follows

$$\mathrm{d}U_{i,b}(t)=$$

$$\begin{cases}-c_1\mathrm{d}t+\mathrm{d}S_i(t)+\sigma\mathrm{d}B(t),\ 0<U_{i,b}(t)\leqslant b;\\ -c_2\mathrm{d}t+\mathrm{d}S_i(t)+\sigma\mathrm{d}B(t),\ U_{i,b}(t)>b\end{cases}$$

where $S_i(t)=\sum_{j=1}^{N_i(t)}X_j$. For $i=1,2,\cdots,n$, we denote by

$$V_{i,1}(u;b)=E[D_{u,b}\mid U_{i,b}(0)=u],u\leqslant b;$$

$$V_{i,2}(u;b)=E[D_{u,b}\mid U_{i,b}(0)=u],\ u>b.$$

It is clear that $V_j(u;b)$ can be obtained from $V_{i,j}(u;b)$ by letting $i=1$, where $j=1,2$.

Let $h_t=u-c_1t+\sigma B_t$, we have $\mathrm{d}h_t=-c_1\mathrm{d}t+\sigma\mathrm{d}B(t)$ and $(\mathrm{d}h_t)^2=\sigma^2\mathrm{d}t$, .

For $j=1,\cdots,n-1$ and $0<u<b$, let $\epsilon,\ t>0$ be such that $\epsilon<u<b$. Considering W_j, define $\tau^j=T_t^\epsilon\wedge W_j$ and $T_t^\epsilon=\inf\{s>0:h_s\notin(\epsilon,b)\}\wedge t$. It is clear that $P(\tau^j<\infty)=1,\forall s\in(0,\tau^j)$. According to the strong Markov property, we have

$$\begin{aligned}V_{j,1}(u;b)=E[D_{u,b}\mid U_{j,b}(0)=u]=E\{E[D_{u,b}\mid \mathscr{F}_{\tau^j}]\mid U_{j,b}(0)=u\}=\\ E\{E[\mathrm{e}^{-\delta\tau^j}D_{U_{j,b}(\tau^j),b}\mid U_{j,b}(\tau^j)]\mid U_{j,b}(0)=u\}=\\ E[I(W_j>t)\mathrm{e}^{-\delta T_t^\epsilon}V_{j,1}(U_{j,b}(T_t^\epsilon);b)\mid U_{j,b}(0)=u]+\\ E[I(W_j\leqslant t)\mathrm{e}^{-\delta T_{W_j}^\epsilon}V_{j+1,1}(U_{j+1,b}(T_{W_j}^\epsilon);b)\mid U_{j,b}(0)=u]:=K_1(t)+K_2(t)\end{aligned} \tag{30}$$

By Ito integral formula, we have

$$\mathrm{d}V_{j,1}(h_{T_t^\epsilon};b)=$$

$$\left[-c_1V'_{j,1}(h_{T_t^\epsilon};b)+\frac{\sigma^2}{2}V''_{j,1}(h_{T_t^\epsilon};b)\right]\mathrm{d}T_t^\epsilon+$$

$$\sigma V'_{j,1}(h_{T_t^\epsilon};b)\mathrm{d}B(T_t^\epsilon)+o(\mathrm{d}T_t^\epsilon).$$

Changing the above equation to the form of integral, we obtain

$$V_{j,1}(h_{T_t^\epsilon};b)=V_{j,1}(u;b)-c_1\int_0^{T_t^\epsilon}V'_{j,1}(h_{T_t^\epsilon};b)\mathrm{d}T_t^\epsilon+$$

$$\frac{1}{2}\sigma^2\int_0^{T_t^\epsilon}V''_{j,1}(h_{T_t^\epsilon};b)\mathrm{d}T_t^\epsilon+$$

$$\sigma\int_0^{T_t^\epsilon}V'_1(h_{T_t^\epsilon};b)\mathrm{d}B(T_t^\epsilon)+o(T_t^\epsilon).$$

For $\lim_{t\to 0}P(T_t^\epsilon=t)=1$, $\lim_{t\to 0}P(T_t^\epsilon<t)=0$, a. s. and $t\to 0,\mathrm{e}^{-\lambda_t}=1-\lambda t+o(t)$.

Dividing the two sides of Eq. (30) by t and letting $t\to 0$, we first deal with $K_1(t)$.

$$K_1(t)=P(W_j>t)\cdot$$

$$E[\mathrm{e}^{-\delta T_t^\epsilon}V_{j,1}(U_{j,b}(T_t^\epsilon);b)\mid U_{j,b}(0)=u]=$$

$$\mathrm{e}^{-\lambda_j t}E[\mathrm{e}^{-\delta T_t^\epsilon}V_{j,1}(U_{j,b}(T_t^\epsilon);b)\mid U_{j,b}(0)=u].$$

Calculating $K_1(t)$ by Ito integral formula, we have

$$\lim_{t\to 0}\frac{K_1(t)-V_{j,1}(u;b)}{t}=\frac{\sigma^2}{2}\frac{\partial^2V_{j,1}(u;b)}{\partial u^2}-$$

$$(\lambda_j+\delta)V_{j,1}(u;b)-c_1\frac{\partial V_{j,1}(u;b)}{\partial u} \tag{31}$$

Now we will deal with $K_2(t)$ in the same way as $K_1(t)$,

$$K_2(t)=\int_0^t\lambda_j\ \mathrm{e}^{-\lambda_j s}E[\mathrm{e}^{-\delta s}V_{j+1,1}(u-cs+\sigma B_s)\cdot$$

$$I(T_s^\epsilon=s)\mid U_{j,b}(0)=u]\mathrm{d}s+$$

$$\int_0^t\lambda_j\ \mathrm{e}^{-\lambda_j s}E[\mathrm{e}^{-\delta T_s^\epsilon}V_{j+1,1}(U_{j+1,b}(T_s^\epsilon);\ b)\cdot$$

$$I(T_s^\epsilon<s)\mid U_{j,b}(0)=u]\mathrm{d}s.$$

By Ito integral formula, we have for $j=1,\cdots,n-1$

$$\lim_{t\to 0}\frac{K_2(t)}{t}=\lambda_j V_{j+1,1}(u;b) \tag{32}$$

Using Eqs. (20)～(32), we know

$$(\delta+\lambda_j)V_{j,1}(u;b)+c_1\frac{\partial V_{j,1}(u;b)}{\partial u}-\frac{\sigma^2}{2}\frac{\partial^2 V_{j,1}(u;b)}{\partial u^2}-\lambda_j V_{j+1,1}(u;b)=0.$$

I. e., for $j=1,\cdots,n-1$

$$V_{j+1,1}(u;b)=\frac{\delta+c_1\frac{\partial}{\partial u}-\frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}+\lambda_j}{\lambda_j}V_{j,1}(u;b) \tag{33}$$

Hence

$$\left.\begin{array}{l} V_{k,1}(u;b)= \\ \left\{\prod_{j=1}^{k-1}\frac{\delta+c_1\frac{\partial}{\partial u}-\frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}+\lambda_j}{\lambda_j}\right\}V_1(u;b), \\ k=2,\cdots,n \end{array}\right\} \tag{34}$$

When $j=n$, by the same approach we have

$$\frac{(\delta+\lambda_n)+c_1\frac{\partial}{\partial u}-\frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}}{\lambda_n}V_{n,1}(u;b)-\int_0^{b-u}V_1(u+y;b)p(y)\mathrm{d}y-\int_{b-u}^{\infty}V_2(u+y;b)p(y)\mathrm{d}y=0.$$

Hence

$$\prod_{j=1}^{n}\frac{(\delta+\lambda_j)+c_1\frac{\partial}{\partial u}-\frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}}{\lambda_j}V_1(u;b)-\int_0^{b-u}V_1(u+y;b)p(y)\mathrm{d}y-\int_{b-u}^{\infty}V_2(u+y;b)p(y)\mathrm{d}y=0.$$

We can derive from Eq. (28) for $V_1(u;b)$ on $0<u\leqslant b$.

Now we turn to the case $u>b$. For $j=1,\cdots,n-1$, considering W_j, let ϵ, $t>0$ be satisfied such that $b+\epsilon<u$. Define $\tau^j=T_t^\epsilon\wedge W_j$ and $T_t^\epsilon=\inf\{s>0:h_s\notin(\epsilon,b)\}\wedge t$. Clearly $P(\tau^j<\infty)=1$, $U_{j,b}(s)>b$, $\forall s\in(0,\tau^j)$, .

For $j=1,\cdots,n-1$,

$$V_{j,2}(u;b)=E\left[I(W_j>t)(\alpha T_t^\epsilon+\mathrm{e}^{-\delta T_t^\epsilon}V_{j,2}(U_{j,b}(T_t^\epsilon);b))\mid U_{j,b}(0)=u\right]+E\left[I(W_j\leqslant t)(\alpha T_{W_j}^\epsilon+\mathrm{e}^{-\delta T_{W_j}^\epsilon}V_{j+1,2}(U_{j+1,b}(T_{W_j}^\epsilon);b))\mid U_{j,b}(0)=u\right]:=L_1(t)+L_2(t) \tag{35}$$

For $\lim_{t\to 0}P(T_t^\epsilon=t)=1$, $\lim_{t\to 0}P(T_t^\epsilon<t)=0$, a. s., using the Ito formula we have come to

$$\lim_{t\to 0}\frac{L_1(t)-V_{j,2}(u;b)}{t}=\frac{\sigma^2}{2}\frac{\partial^2 V_{j,2}(u;b)}{\partial u^2}-c_2\frac{\partial V_{j,2}(u;b)}{\partial u}-(\lambda_j+\delta)V_{j,2}(u;b)+\alpha,$$

and

$$\lim_{t\to 0}\frac{L_2(t)}{t}=\lambda_j V_{j+1,2}(u;b) \tag{36}$$

i. e, for $j=1,2,\cdots,n-1$,

$$V_{j+1,2}(u;b)-\frac{\alpha}{\delta}=\left\{\frac{\delta+c_2\frac{\partial}{\partial u}-\frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}+\lambda_j}{\lambda_j}\right\}\left(V_{j,2}(u;b)-\frac{\alpha}{\delta}\right) \tag{37}$$

Hence

$$\left.\begin{array}{l} V_{k,2}(u;b)-\frac{\alpha}{\delta}= \\ \left\{\prod_{j=1}^{k-1}\frac{\delta+c_2\frac{\partial}{\partial u}-\frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}+\lambda_j}{\lambda_j}\right\}\left(V_2(u;b)-\frac{\alpha}{\delta}\right), \\ k=2,\cdots,n-1 \end{array}\right\} \tag{38}$$

For $j=n$, we have

$$\frac{(\delta+\lambda_n)+c_2\frac{\partial}{\partial u}-\frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}}{\lambda_n}\left(V_{n,2}(u;b)-\frac{\alpha}{\delta}\right)=\int_0^{\infty}\left[V_2(u+y;b)-\frac{\alpha}{\delta}\right]\mathrm{d}P(y).$$

Therefore

$$\left[\prod_{j=1}^{n}\left(\lambda_j+\delta+c_2\frac{\partial}{\partial u}-\frac{1}{2}\sigma^2\frac{\partial^2}{\partial u^2}\right)\right]\cdot\left[V_2(u;b)-\frac{c_2-c_1}{\delta}\right]=$$

$$\left(\prod_{j=1}^{n}\lambda_j\right)\int_0^{\infty}\left[V_2(u+y;b)-\frac{c_2-c_1}{\delta}\right]\mathrm{d}P(y) \tag{39}$$

for all $u>b$.

Next, we show the boundary conditions. Clearly $V_1(0;b)=0$ and for all y, we have

$$\lim_{u\to\infty}V_2(u;b)=\lim_{u\to\infty}V_2(u+y;b).$$

Plugging the equation into Eq. (29), we arrive at $V_2(\infty;b)=\frac{c_2-c_1}{\delta}$.

Now we borrow the idea from Ref. [11], define a new process

$$\mathrm{d}U_{\epsilon,b}(t)=\begin{cases}-(c_1+c_\epsilon)\mathrm{d}t+\mathrm{d}S(t)+\epsilon\mathrm{d}N_\epsilon(t),\\ \quad 0<U_{\epsilon,b}(t)\leqslant b;\\ -(c_2+c_\epsilon)\mathrm{d}t+\mathrm{d}S(t)+\epsilon\mathrm{d}N_\epsilon(t),\\ \quad U_{\epsilon,b}(t)>b;\end{cases}$$

where $N_\epsilon(t)$ is a Poisson process with parameter λ_ϵ, $\lambda_\epsilon=\frac{\sigma^2}{\epsilon^2}$ and $c_\epsilon=\frac{\sigma^2}{\epsilon}$. It is easy to prove that the process $\{\epsilon N_\epsilon(t)-c_\epsilon t, t\geqslant 0\}$ converges weakly to $\{\sigma B(t), t\geqslant 0\}$, therefore, the surplus $\{U_{\epsilon,b}(t), t\geqslant 0\}$ converges weakly to $\{U_\epsilon(t), t\geqslant 0\}$. According to Ref. [7, Theorem 6.1], we can obtain the boundary conditions.

References

[1] AVANZI B, GERBER H U, SHIU E S. Optimal dividends in the dual model [J]. Insurance: Mathematics and Economics, 2007, 41(1): 111-123.

[2] ALBRECHER H, BADESCU A, LANDRIAULT D. On the dual risk model with tax payments [J]. Insurance: Mathematics and Economics, 2008, 42(3): 1086-1094.

[3] NG A C Y. On a dual model with a dividend threshold [J]. Insurance: Mathematics and Economics, 2009, 44(2): 315-324.

[4] LANDRIAULT D, SENDOVA K P. A direct approach to a first-passage problem with applications in risk theory[J]. Stochastic Models, 2011, 27(3): 388-406.

[5] JI L, ZHANG C. Analysis of the multiple roots of the Lundberg fundamental equation in the PH(n) risk model[J]. Applied Stochastic Models in Business and Industry, 2012, 28(1): 73-90.

[6] RODRÍGUEZ-MARTÍNEZ E V, CARDOSO R M, DOS REIS A D E. Some advances on the Erlang(n) dual risk model[J]. Astin Bulletin, 2015, 45(1): 127-150.

[7] YANG C, SENDOVA K P. The ruin time under the Sparre-Andersen dual model [J]. Insurance: Mathematics and Economics, 2014, 54: 28-40.

[8] ZHANG Z Z, ZOU J Z, LIU Y Y. The maximum surplus distribution before ruin in an Erlang(n) risk process perturbed by diffusion[J]. Acta Mathematica Sinica, English Series, 2011, 27(9): 1869-1880.

[9] GERBER H U. An Introduction to Mathematical Risk Theory [M]. Homewood, IL: Richard D Irwin Inc, 1979.

[10] GRIMMETT G, STIRZAKER D. Probability and Random Processes[M]. Oxford: Oxford University Press, 2001: 487-490.

[11] LI S. The distribution of the dividend payments in the compound Poisson risk model perturbed by diffusion [J]. Scandinavian Actuarial Journal, 2006(2): 73-85.