文章编号:0253-2778(2019)03-0182-13

# Complexity of one-prey multi-predator system with impulsive effect and incomplete trophic transfer

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**Abstract**: A new one-prey multi-predator system with impulsive effect and incomplete trophic transfer was proposed. This system used a different rate of trophic absorption of predators from the rate of the conversion of consumed prey to predator in Ivlev-type functional responses. The extinction and permanence of the system with impulsive perturbation on the predators at fixed moments was investigated. And the conditions for asymptotically stable and permanence of the system was given by using Floquet theory and comparison theorem. Finally, numerical simulations demonstrated the obtained conclusions.

**Key words**: incomplete trophic transfer; Ivlev-type; impulsive effect; extinction; permanence
**CLC number**: O175 **Document code**: A doi:10.3969/j.issn.0253-2778.2019.03.002
**2010 Mathematics Subject Classification**: 34D20; 34C23; 37N25; 92D25

**Citation**: CHENG Xian, YAN Ping, LIU Liping, et al. Complexity of one-prey multi-predator system with impulsive effect and incomplete trophic transfer[J]. Journal of University of Science and Technology of China, 2019,49(3):182-194.
程娴,闫萍,刘利平,等. 具有不完全营养转换和脉冲效应的单食饵多捕食者系统的复杂性分析[J]. 中国科学技术大学学报,2019,49(3):182-194.

# 具有不完全营养转换和脉冲效应的单食饵多捕食者系统的复杂性分析

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**摘要**:提出了一类具有不完全营养转换和脉冲效应的单食饵多捕食者系统.在该系统的Ivlev型功能反应项中,选取了不同的捕食者营养吸收率与消耗食饵的转化率.在周期性投放捕食者的脉冲效应下,分析了系统的灭绝和持续生存,并利用Floquet乘子理论和比较定理,给出了食饵根除周期解渐近稳定与系统持续生存的条件.最后,通过数值模拟验证了所得结论.

**关键词**:不完全营养转换;Ivlev型;脉冲效应;灭绝;持续生存

**Received**: 2017-07-14; **Revised**: 2018-03-27
**Foundation item**: Supported by the NNSF of China (11201002), Natural Science Research Projects in Anhui Universities (KJ2019A0215).
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## 0 Introduction

The predator-prey system is an important domain theme in ecology. Using mathematical simulations, we can analyze the systems and gain control of some ecological phenomena. Lotka[1] and Volterra[2] have given early examples of the biological system models. And many scholars and researchers have shown interests in the complexity of the condition and application of the universality of the predator-prey model with different functional responses.

A classical predator-prey system can be the form[3] of

$$\left.\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}&=xf(x)-yg(x,y),\\ \frac{\mathrm{d}y}{\mathrm{d}t}&=mg(x,y)y-dy\end{aligned}\right\}\qquad(1)$$

where $x(t)$ and $y(t)$ are prey and predator densities , respectively, $f(x)$ is the prey growth rate in the absence of the predator, $m>0$ is the rate of conversion of consumed prey to predator, $d>0$ is the natural mortality rate of the predator, and $g(x,y)$ is the predator functional response. One of the most widely used functional responses is the Ivlev-type functional response which was proposed by Ivlev[4]. We also consider the Ivlev-type functional response:

$$g(x)=\beta(1-\exp\{-\alpha x(t)\}),$$

where $\alpha,\beta$ are positive parameters.

Many researchers have concentrated on the predator-prey system with the Ivlev-type functional response. In Refs. [5-7], the existence of positive solutions for the predator-prey system were studied. In Ref. [8], a delayed stage-structured Ivlev functional response predator-prey model with impulsive stocking on prey and continuous harvesting on predators was analyzed, and other time delayed models were investigated in Refs. [9-11].

However, it is more real and scientific to describe many natural phenomena or man-made factors using impulsive differential equations. For example, the births of some creatures are seasonal, increasing dramatically in the breeding season, while otherwise increasing quite gently. So the changes of births need to be described more accurately in an implusive way. There are more impulse phenomena in the development and utilization of biological resources. The dropping of baits and harvesting of fish at fixed times can make the numbers of fish increase or decrease rapidly. In agriculture, pesticide is sprayed and natural enemies are released at fixed times to kill pests in pest management. It can be seen that impulsive equations are used in many domains of applied science.

Some investigations have been done on impulsive differential equations in relation to: impulsive birth[12-13], impulsive vaccination[14-15], chemotherapeutic treatment[16] and so on. Especially in impulsive prey-predator systems, many researchers have studied the existence of periodic solution[17-19], the stability of impulsive system[20-22] and dynamical behavior[23-25]. And most of the studies that have been done are about one-prey two-predator impulsive systems with various functional responses[26-28] or one-prey multi-predator impulsive systems with Holling type function response[29-31]. Only a few researches are about the one-prey multi-predator impulsive system with Ivlev function response[32]. In our paper, we investigate the impact of the biological control technique of using multi-predators to kill pests on a system with Ivlev-type functional response. The corresponding one prey multi-predators system with impulsive effect takes the form

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=x(t)(a-bx(t))-\sum_{i=1}^{m}(1-\exp\{-c_ix(t)\})y_i(t), \\ \dfrac{\mathrm{d}y_i}{\mathrm{d}t}=q_i(1-\exp\{-c_ix(t)\})y_i(t)-d_iy_i(t), \end{cases} t\neq nT; \\ \left.\begin{matrix} x(nT^+)=x(nT), \\ y_i(nT^+)=y_i(nT)+p_i, \end{matrix}\right\} t=nT, i=1,2,\cdots,m \tag{2}$$

where $x(t)$ and $y_i(t)$ are prey and predator densities, respectively, $a$ is the intrinsic growth rate of the prey, $b$ is the coefficient of intraspecific competition, $c_i$ is the rate of the conversion of consumed prey to predators, $d_i$ is the natural mortality rate of the predators, $q_i$ is the rate of conversion of consumed prey to predators, $x(nT^+)$ and $y_i(nT^+)$, respectively, denote the numbers of prey and predator $i$ after $n$th release of predator $i$, and $p_i$ is the number of predators released each time. All parameters are positive constants. The trophic transfer between prey and predator is assumed to be equal in most predator-prey systems[6,7,24,25]. However, we should consider some losses, because trophic transfer between species may diminish in a more complex food chain. Then it is more realistic to render the nonlinear trophic transfers with trophic losses. So based on model (2), we propose the following one-prey multi-predator impulsive system with a different rate of trophic absorption of predator $s_i$ from the rate of the conversion of consumed prey to predator $c_i$ in the Ivlev-type functional responses.

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=x(t)(a-bx(t))-\sum_{i=1}^{m}(1-\exp\{-c_ix(t)\})y_i(t), \\ \dfrac{\mathrm{d}y_i}{\mathrm{d}t}=q_i(1-\exp\{-s_ix(t)\})y_i(t)-d_iy_i(t), \end{cases} t\neq nT; \\ \left.\begin{matrix} x(nT^+)=x(nT), \\ y_i(nT^+)=y_i(nT)+p_i, \end{matrix}\right\} t=nT, i=1,2,\cdots,m \tag{3}$$

Because of the trophic loss in the transfer we assume $0<s_i<c_i$.

In this paper, we investigate the extinction, permanence and complexity of system (3). In Section 1, we introduce definitions and state necessary lemmas. In Section 2, in the case of incomplete trophic transfer, we prove that all solutions of system (3) are still uniformly upper bounded. Thus, we propose the conditions for the extinction and permanence of system (3) with comparison theorems. In Section 3, we show numerical simulations to confirm theoretical results obtained in Section 2. Finally, in Section 4 we conclude with a discussion of the studies.

## 1 Preliminaries

An effective method to discuss the stability of the impulsive system is Liapunov function. Because the solution of impulsive differential equation is piecewise continuous, so it is required that its Liapunov function be also piecewise continuous. For this purpose, class $V_0$ is given.

Let $R_+=[0,\infty)$, $R_+^{(m+1)}=\{X\in R^{(m+1)}:X\geqslant 0\}$, $N$ be the set of all non-negative integers. The map $f=(f_1,f_2,\cdots,f_{m+1})^{\mathrm{T}}$ is defined by the right hand of system (3). Let $V:R_+\times R_+^{(m+1)}\to R_+$, then $V$ is said to belong to class $V_0$ if

① $V$ is continuous in $(nT,(n+1)T]\times R_+^{(m+1)}$, and there exists $V(nT^+,X)$ for each $X\in R_+^{(m+1)}$, $n\in N$, such that

$$\lim_{(t,Y)\to(nT^+,X)} V(t,Y)=V(nT^+,X);$$

② $V$ is locally Lipschitzian in $X$.

**Definition 1.1** Let $V\in V_0$, then for $(t,X)\in(nT,(n+1)T]\times R_+^{(m+1)}$, the upper right derivative of $V(t,X)$ with respect to the impulsive differential system (3) is defined as

$D^+V(t,X)=$

$$\lim_{h\to 0^+}\sup\left\{\frac{V(t+h,X+hf(t,X))-V(t,X)}{h}\right\}.$$

The solution of system (3) is a piecewise continuous function $X:R_+\to R_+^{(m+1)}$, $X(t)$ is continuous on $(nT,(n+1)T]$, $n\in N$, and there exists $X(nT^+)=\lim_{t\to nT^+}X(t)$. The smoothness properties of $f$ guarantee the global existence and uniqueness of solutions of system (3) (see Ref. [33] for details on fundamental properties of impulsive systems).

It is easy to prove the following lemma.

**Lemma 1.1** Let $X(t)$ be a solution of system (3) with $X(0^+)\geqslant 0$, then $X(t)\geqslant 0$ for all $t\geqslant 0$ and further $X(t)>0$, $t\geqslant 0$ if $X(0^+)>0$.

For convenience, we will state the result of the important comparison theorem[33] using our notation.

Suppose function $g:R_+\times R_+\to R$ satisfies:

(Ⅰ) Function $g$ is continuous in $(nT,(n+1)T]\times R_+$, and there exists $V(nT^+, X)$ for $X\in R_+^{(m+1)}$, $n\in N$, such that

$$\lim_{(t,Y)\to(nT^+,X)}V(t,Y)=V(nT^+,X).$$

**Lemma 1.2** Suppose $V\in V_0$. Assume that

$$\left.\begin{aligned}&D^+V(t,X)\leqslant g(t,V(t,X)),\ t\neq nT;\\&V(t,X(t^+))\leqslant\psi_k(V(t,X)),\ t=nT\end{aligned}\right\}\quad(4)$$

where $g:R_+\times R_+\to R$ satisfies (I) and $\psi_k:R_+\to R_+$ is nondecreasing. Let $r(t)$ be the maximal solution of the scalar impulsive differential equation

$$\left.\begin{aligned}&\frac{du}{dt}=g(t,u(t)),\ t\neq nT;\\&u(t^+)=\psi_k(u(t)),\ t=nT;\\&u(0^+)=u_0\geqslant 0\end{aligned}\right\}\quad(5)$$

existing on $[0,\infty)$. Then $V(0^+,X_0)\leqslant u_0$ implies that $V(t,X(t))\leqslant r(t)$, $t\geqslant 0$, where $X(t)$ is any solution of (3) existing on $[0,\infty)$.

Finally, we give some basic properties about the following subsystem of system (3):

$$\left.\begin{aligned}&\frac{dy_i}{dt}=-d_iy_i(t),\ t\neq nT;\\&y_i(t^+)=y_i(t)+p_i,\ t=nT;\\&y_i(0^+)=y_{0i}\end{aligned}\right\}\quad(6)$$

Clearly, the system (6) has a positive periodical solution

$$y_i^*(t)=\frac{p_i\exp(-d_i(t-nT))}{1-\exp(-d_iT)},\quad t\in(nT,(n+1)T],$$

and

$$y_i^*((nT)^+)=\frac{p_i}{1-\exp(-d_iT)}.$$

Since the solution of system (6) is

$$y_i(t)=\left(y_i(0^+)-\frac{p_i}{1-\exp(-d_iT)}\right)\exp(-d_it)+y_i^*(t),\quad t\in(nT,(n+1)T].$$

Then we have the following results.

**Lemma 1.3** Let $y_i^*(t)$ be a positive periodic solution of system (6), then each solution $y_i(t)$, $i=1, 2,\cdots, m$, of system (6) satisfies that $|y_i(t)-y_i^*(t)|\to 0$ as $t\to\infty$.

Then we can obtain the complete expression for the prey-eradication periodic solution of system (3)

$$(0,y_1^*(t),\cdots,y_m^*(t))=\left(0,\frac{p_1\exp(-d_1(t-nT))}{1-\exp(-d_1T)},\cdots,\frac{p_m\exp(-d_m(t-nT))}{1-\exp(-d_mT)}\right).$$

## 2 Extinction and permanence

In this section, we study the conditions for the extinction and permanence of system (3).

**Definition 2.1** System (3) is said to be permanent if there exist positive constants $m$, $M$ and $t_0$ such that each positive solution $(x(t), y_1(t),\cdots,y_m(t))$ of the system (3) satisfies $m\leqslant x(t)\leqslant M$, $m\leqslant y_i(t)\leqslant M$, for all $t>t_0, i=1, 2,\cdots,m$.

According to the Floquet theory of impulsive differential equations, we now study the stability of the prey-eradication periodic solution.

**Theorem 2.1** Let $(x(t),y_1(t),\cdots,y_m(t))$ be any solution of system (3), then the solution $(0, y_1^*(t),\cdots,y_m^*(t))$ is asymptotically stable if

$$T<\sum_{i=1}^{m}\frac{c_ip_i}{ad_i}.$$

**Proof** The local stability of periodic solution $(0,y_1^*(t),\cdots,y_m^*(t))$ may be determined by considering the behavior of small amplitude perturbations of the solution. Define

$$u(t)=x(t),\ v_i(t)=y_i(t)-y_i^*(t),\quad i=1,2,\cdots,m,$$

where $y_i(t)$ are the solutions of the system (3) and $y_i^*(t)$ are the periodic solutions. (3) can be expanded in a Taylor series after neglecting higher order terms, the linearized equations can be written as following:

$$\begin{cases}\left.\begin{aligned}&\frac{du}{dt}=(a-\sum_{i=1}^{m}c_iy_i^*(t))\,u(t),\\&\frac{dv_i}{dt}=q_is_iy_i^*(t)u(t)-d_iv_i(t),\end{aligned}\right\}\ t\neq nT;\\ u(nT^+)=u(nT),v_i(nT^+)=v_i(nT).\end{cases}$$

Let $\Phi(t)=(u(t),v_1(t),\cdots,v_m(t))^{\mathrm{T}}$, then $\Phi(t)$ must satisfy

$$\frac{d\Phi(t)}{dt}=\begin{pmatrix}a-\sum_{i=1}^{m}c_iy_i^* & 0 & 0 & \cdots & 0 & 0\\ q_1s_1y_1^* & -d_1 & 0 & \cdots & 0 & 0\\ q_2s_2y_2^* & 0 & -d_2 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\ q_ms_my_m^* & 0 & 0 & \cdots & 0 & -d_m\end{pmatrix}\Phi(t),$$

and $\Phi(0)=I$ is the identity matrix. The pulse conditions of system (3) becomes

$$\begin{pmatrix}u(nT^+)\\ v_1(nT^+)\\ v_2(nT^+)\\ \vdots\\ v_m(nT^+)\end{pmatrix}=\begin{pmatrix}1&0&0&\cdots&0\\0&1&0&\cdots&0\\ \vdots&\vdots&\vdots&\cdots&\vdots\\0&0&0&\cdots&1\end{pmatrix}\begin{pmatrix}u(nT)\\ v_1(nT)\\ v_2(nT)\\ \vdots\\ v_m(nT)\end{pmatrix}$$

Obviously, the stability of solution $(0,y_1^*(t),\cdots,y_m^*(t))$ is determined by eigenvalues of $\Theta$, where

$$\Theta=\begin{pmatrix}1&0&0&\cdots&0\\0&1&0&\cdots&0\\ \vdots&\vdots&\vdots&\cdots&\vdots\\0&0&0&\cdots&1\end{pmatrix}\Phi(t).$$

If all eigenvalues of $\Theta$ have absolute values less than one, then the periodic solution $(0,y_1^*(t),\cdots,y_m^*(t))$ is locally stable. Since all eigenvalues of $\Theta$ are

$$\mu=\exp\left(\int_0^T(a-\sum_{i=1}^{m}c_iy_i^*)dt\right),$$
$$\mu_i=\exp(-d_iT)<1,\ i=1,2,\cdots,m.$$

It is easy to see that $|\mu|<1$ if and only if

$$T<\sum_{i=1}^{m}\frac{c_ip_i}{ad_i}.$$

We complete the proof.

Now we show that all solutions to system (3) are uniformly upper bounded.

**Theorem 2.2** There exists a constant $M>0$ such that $x(t)\leqslant M$, $y_i(t)\leqslant M$, $i=1,2,\cdots,m$, for each solution $(x(t),\ y_1(t),\ \cdots,y_m(t))$ of system (3) with all $t$ large enough.

**Proof** Suppose $(x(t),y_1(t),\cdots,y_m(t))$ is any solution of system (3). Define

$$V(t,X)=x(t)+\sum_{i=1}^{m}\frac{y_i(t)}{q_i}.$$

It is easy to see that $V\in V_0$. And we can get

$$\left.\begin{aligned}&D^+V(t,X)+\lambda V(t,X)=\\&(a+\lambda)x(t)-b(x(t))^2+\\&\sum_{i=1}^{m}(\exp(-c_ix)-\exp(-s_ix))y_i(t)+\\&\sum_{i=1}^{m}\frac{\lambda-d_i}{q_i}y_i(t),\ t\neq nT;\\&V(t,X(t^+))=V(t,X)+\sum_{i=1}^{m}\frac{p_i}{q_i},\ t=nT\end{aligned}\right\}\quad(7)$$

Because of $0<s_i\leqslant c_i$, we can get

$$\exp(-c_ix)-\exp(-s_ix)\leqslant 0.$$

So the right hand of the first equation in (7) is bounded when $0<\lambda<\min(d_1,\cdots,d_m)$. Select such $\lambda_0$ and let $K$ be the bound. Thus

$$\begin{cases}D^+V(t,X)+\lambda_0V(t,X)\leqslant K,\ t\neq nT;\\ V(t,X(t^+))=V(t,X)+\sum_{i=1}^{m}\frac{p_i}{q_i},\ t=nT.\end{cases}$$

Then by Ref. [34, Lemma 2.2], we can get

$$V(t)\leqslant V(0^+)\exp(-\lambda_0t)+\frac{K}{\lambda_0}(1-\exp(-\lambda_0t))+\left(\sum_{i=1}^{m}\frac{p_i}{q_i}\right)\frac{1-\exp(-n\lambda_0T)}{1-\exp(-\lambda_0T)}\exp(-\lambda_0(t-nT)).$$

So

$$\lim_{t\to+\infty}V(t)\leqslant\frac{K}{\lambda_0}+\left(\sum_{i=1}^{m}\frac{p_i}{q_i}\right)\frac{\exp(\lambda_0T)}{\exp(\lambda_0T)-1}.$$

Therefore $V(t)$ is ultimately bounded by a constant and there exists a constant $M > 0$ such that $x(t) \leqslant M, y_i(t) \leqslant M$, $i = 1, 2, \cdots, m$, for each solution $(x(t), y_1(t), \cdots, y_m(t))$ of (3) with all $t$ large enough. This completes the proof.

**Theorem 2.3** System (3) is permanent if $T > \sum_{i=1}^{m} \frac{c_i p_i}{a d_i}$.

**Proof** Let $X(t)$ be any solution of (3) with $X(0^+) > 0$. From Theorem 2.2, we know that $x(t) \leqslant M$, $y_i(t) \leqslant M$, $i = 1, 2, \cdots, m$, with all $t$ large enough. So suppose $x(t) \leqslant M$, $y_i(t) \leqslant M$, and $M > \frac{a}{b}$, $t \geqslant 0$.

From system (3), we know that $\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} \geqslant -d_i y_i(t)$. So we consider the following system

$$\begin{cases} \frac{\mathrm{d}u_i}{\mathrm{d}t} = -d_i u_i(t), \ t \neq nT; \\ u_i(t^+) = u_i(t) + p_i, \ t = nT; \\ u_i(0^+) = y_{0i} \geqslant 0. \end{cases}$$

From Lemmas 1.2 and 1.3, we can easily obtain $y_i(t) \geqslant u_i(t)$ and $u_i(t) \to u_i^*(t)$. So there exists $\varepsilon_i > 0$, when $t$ is large enough, $y_i(t) \geqslant u_i(t) > u_i^*(t) - \varepsilon_i$. Let

$$m_i = \frac{p_i \exp(-d_i T)}{1 - \exp(-d_i T)} - \varepsilon_i > 0, \ \varepsilon_i > 0, \quad i = 1, 2, \cdots, m,$$

we can get $y_i(t) > m_i$ for all $t$ large enough. We shall find an $m'_0 > 0$ such that $x(t) \geqslant m'_0$ for $t$ large enough. We will prove this in the following two steps.

**Step 1** Since $T > \sum_{i=1}^{m} \frac{c_i p_i}{a d_i}$, we can select

$$0 < m_0 < \min\left(\frac{a}{b}, \frac{\ln \frac{q_1}{q_1 - d_1}}{s_1}, \cdots, \frac{\ln \frac{q_m}{q_m - d_m}}{s_m}\right)$$ and $\varepsilon'_i > 0$, $i = 1, 2, \cdots, m$, small enough such that

$$\delta_i = q_i(1 - \exp(-s_i m_0)) < d_i,$$

$$\sigma = \exp\left[(a - bm_0)T - \sum_{i=1}^{m} \frac{c_i p_i}{(d_i - \delta_i)} - \sum_{i=1}^{m} \varepsilon'_i T\right] = \exp\left[aT - \sum_{i=1}^{m} \frac{c_i p_i}{d_i} - \left(\sum_{i=1}^{m} \frac{c_i p_i}{(d_i - \delta_i)} - \sum_{i=1}^{m} \frac{c_i p_i}{d_i}\right) - bm_0 T - \sum_{i=1}^{m} \varepsilon'_i T\right] > 1.$$

We will prove there exists $t_1 \in (0, \infty)$ such that $x(t_1) \geqslant m_0$. Otherwise,

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} \leqslant (-d_i + \delta_i) y_i(t) \tag{8}$$

Consider the following system

$$\left.\begin{aligned} &\frac{\mathrm{d}v_i}{\mathrm{d}t} = (-d_i + \delta_i) v_i(t), \ t \neq nT; \\ &v_i(t^+) = v_i(t) + p_i, \ t = nT; \\ &v_i(0^+) = y_i(0^+) > 0 \end{aligned}\right\} \tag{9}$$

We can obtain $y_i(t) \leqslant v_i(t)$ and $v_i(t) \to v_i^*(t)$, where

$$v_i^*(t) = \frac{p_i \exp((-d_i + \delta_i)(t - nT))}{1 - \exp((-d_i + \delta_i)T)}, \quad t \in (nT, (n+1)T].$$

So there exists $T_i > 0$, when $t \geqslant T_i$,

$$y_i(t) \leqslant v_i(t) < v_i^*(t) + \varepsilon'_i \tag{10}$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}t} \geqslant x(t)(a - bm_0 - \sum_{i=1}^{m} (v_i^*(t) + \varepsilon'_i)) \tag{11}$$

for all $t > \max(T_1, T_2, \cdots, T_m)$. Select constant $N_i \in Z^+$ to make $N_i T \geqslant T_i$. Integrating (12) on $(nT, (n+1)T]$, $n \geqslant \max(N_1, N_2, \cdots, N_m)$, we can obtain the following result:

$$x((n+1)T) \geqslant x(nT)\exp\left(\int_{nT}^{(n+1)T} [a - bm_0 - \sum_{i=1}^{m} (v_i^*(t) + \varepsilon'_i)] \mathrm{d}t\right) = x(nT)\sigma \tag{13}$$

Then, $x((n+k)T) \geqslant x(nT)\sigma^k \to +\infty$ when $k \to +\infty$, which is a contradiction. Hence there exists $t_1 \in (0, \infty)$ such that $x(t_1) \geqslant m_0$.

**Step 2** If $x(t) \geqslant m_0$ for all $t > t_1$, then our aim is achieved. Otherwise, if $x(t) < m_0$ for some $t > t_1$, let $t^* = \inf_{t \geqslant t_1}\{x(t) < m_0\}$. Then $t^*$ is an impulsive point or a non-impulsive point.

(Ⅰ) If $t^*$ is an impulsive point. Let $t^* = n_0 T$, $n_0 \in N$. Then we choose $\varepsilon_0 > 0$, small enough, which implies $t_0 = t^* - \varepsilon_0$, $t_0$ is a non-impulsive point, such that $x(t_0) \geqslant m_0$.

(Ⅱ) If $t^*$ is a non-impulsive point, then we have $x(t) \geqslant m_0$ for $t \in [t_1, t^*)$ and $x(t^*) = m_0$, since $x(t)$ is continuous. Suppose $t^* \in (n_1 T,$

$(n_1+1)T)$, $n_1 \in N$. Choose $n_{2i}$, $n_3 \in N$ such that

$$n_{2i}T > \frac{\ln(\frac{\varepsilon'_i}{M+p_i})}{-d_i+\delta_i},$$

$$\exp(\sigma_1(n_2+1)T)\sigma^{n_3} > 1,$$

where $\sigma_1 = a - bm_0 - mM < 0$, $n_2 = \max\{n_{21}, \cdots, n_{2m}\}$. Let $T' = n_2 T + n_3 T$, then there exists $t_2 \in ((n_1+1)T, (n_1+1)T+T']$ such that $x(t_2) \geqslant m_0$. Otherwise $x(t) < m_0$, $t \in ((n_1+1)T, (n_1+1)T+T']$. Considering (9) with $v_i((n_1+1)T^+) = y_i((n_1+1)T^+)$, we have

$$v_i(t) = \left(v_i((n_1+1)T^+) - \frac{p_i}{1-\exp(-d_i+\delta_i)T}\right)\cdot$$
$$\exp((-d_i+\delta_i)(t-(n_1+1)T)) + v_i^*(t),$$

for $t \in (nT, (n+1)T]$, $n_1+1 \leqslant n \leqslant n_1+1+n_2+n_3$.

Then we can get

$$|v_i(t) - v_i^*(t)| <$$
$$(M+p_i)\exp((-d_i+\delta_i)n_{2i}T) < \varepsilon'_i \quad (14)$$

and $y_i(t) \leqslant v_i(t) \leqslant v_i^*(t) + \varepsilon'_i$ for $t \in ((n_1+n_2+1)T, (n_1+1)T+T']$, which implies (11) holds on $t \in ((n_1+n_2+1)T, (n_1+1)T+T']$. From the first step, we have

$$x((n_1+1+n_2+n_3)T) \geqslant x((n_1+1+n_2)T)\sigma^{n_3}.$$

From the system (3), we can get

$$\frac{dx}{dt} \geqslant x(t)(a - bm_0 - mM) = \sigma_1 x(t) \quad (15)$$

Integrating (15) on $[t^*, (n_1+1+n_2)T]$, we can get

$$x((n_1+1+n_2)T) \geqslant m_0\exp(\sigma_1(n_2+1)T).$$

Thus

$$x((n_1+1+n_2+n_3)T) \geqslant$$
$$m_0\exp(\sigma_1(n_2+1)T)\sigma^{n_3} > m_0 \quad (16)$$

which is a contradiction.

Let $\bar{t} = \inf_{t>t^*}\{x(t) \geqslant m_0\}$, then $x(\bar{t}) = m_0$ and (15) holds for $t \in [t^*, \bar{t})$. Integrating (15) on $[t^*, \bar{t})$, we have

$$x(t) \geqslant x(t^*)\exp(\sigma_1(t-t^*)) \geqslant$$
$$m_0\exp(\sigma_1(n_2+n_3+1)T) \triangleq m'_0.$$

For $t > \bar{t}$, the same arguments can be continued since $x(\bar{t}) \geqslant m_0$. Hence $x(t) \geqslant m'_0$ for all $t > t_1$. The proof is completed.

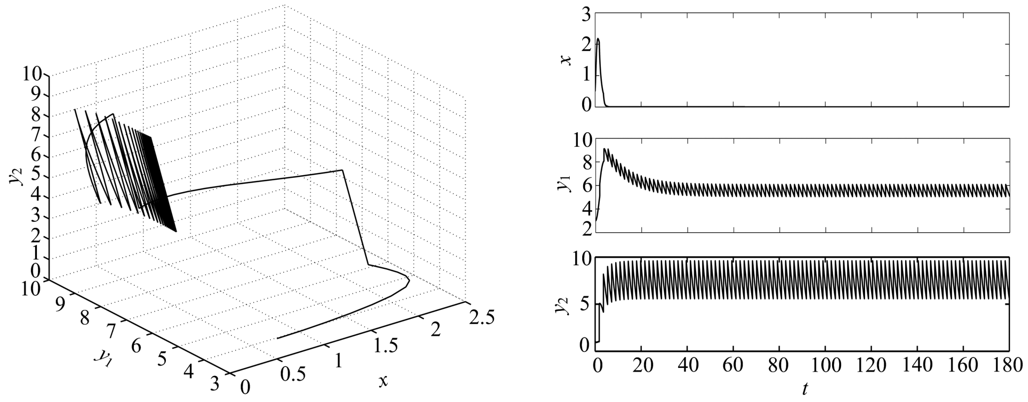
## 3 Numerical simulations

In this section, we show numerical simulations to confirm the theoretical results we obtained in Section 2. Using MATLAB, we performed numerical simulations for $m=2$, $a=6$, $b=2$, $c_1=0.59$, $c_2=0.4$, $s_1=0.5$, $s_2=0.38$, $d_1=0.1$, $d_2=0.3$, $q_1=0.7$, $q_2=0.67$ and different values of $p_1$, $p_2$, $T$, $x_0$, $y_{01}$, $y_{02}$. When $p_1=1$, $p_2=4$, we have $\frac{c_1p_1}{ad_1}+\frac{c_2p_2}{ad_2}=1.8722$ and by Theorem 2.1, there should exist an asymptotically stable pest-eradication periodic solution when the impulsive period $T$ is smaller than a threshold $\sum_{i=1}^{m}\frac{c_ip_i}{ad_i}$.

In Figs. 1 and 2, the phase portrait and time series of $x$, $y_1$, $y_2$ for system (3) with different $T$ and initial value are shown. From Fig. 1 we see that when we choose $T=1.8$ and the initial values $(x_0, y_{01}, y_{02}) = (0.5, 3, 1)$, the solution $(0, y_1^*(t), \cdots, y_m^*(t))$ is asymptotically stable. This confirms Theorem 2.1. However, when $T$ is larger than the threshold, system (3) should be permanent. When we choose $T=4$ and the initial values $(x_0, y_{01}, y_{02}) = (1, 4, 1)$, we can see from Fig. 2 that system (3) is permanent. This confirms Theorem 2.3.

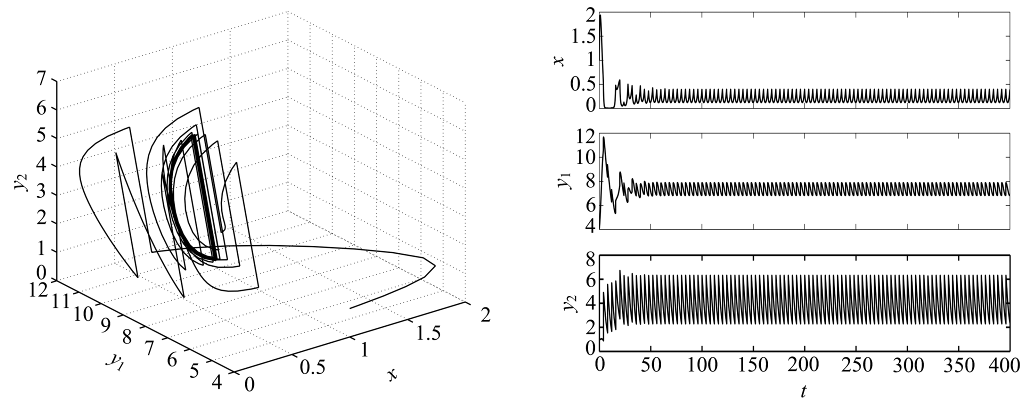
Moreover, let us rewrite the condition of Theorem 2.1 as $a < \frac{1}{T}\sum_{i=1}^{m}\frac{c_ip_i}{d_i}$. Then when we fix the birth rate $a$ of pests, the death rates $d_i$ of natural enemies and the conversion rate $c_i$, we can eradicate the pest by choosing a suitable impulsive period $T$ and appropriate impulse and by releasing the number of predator $p_i$ such that $\frac{1}{T}\sum_{i=1}^{m}\frac{c_ip_i}{d_i}$ is larger than $a$.

In the following, we show the numerical simulations we performed for the effect of impulsive perturbations of natural enemies on system (3). When system (3) is free from impulsive effects, then (3) becomes




(a) phase portrait　　(b) time series

**Fig. 1　The numerical simulations of Theorem 2. 1 when $p_1=1$, $p_2=4$, $T=1.8$, $x_0=0.5$, $y_{01}=3$, $y_{02}=1$**




(a) phase portrait　　(b) time series

**Fig. 2　The numerical simulation of Theorem 2. 3 when $p_1=1$, $p_2=4$, $T=4$, $x_0=1$, $y_{01}=4$, $y_{02}=1$**

$$
\begin{cases}
\dfrac{\mathrm{d}x}{\mathrm{d}t}=x(t)(a-bx(t))- \\
\qquad \displaystyle\sum_{i=1}^{m}(1-\exp\{-c_ix(t)\})y_i(t), \\
\dfrac{\mathrm{d}y_i}{\mathrm{d}t}=q_i(1-\exp\{-s_ix(t)\})y_i(t)-d_iy_i(t), \\
i=1,2,\cdots,m
\end{cases}
\tag{17}
$$

For the same values of $m$, $a$, $b$, $c_1$, $c_2$, $s_1$, $s_2$, $d_1$, $d_2$, $q_1$, $q_2$ and ($x_0$, $y_{01}$, $y_{02}$) chosen above, we show that system (17) has a stable equilibrium ($x^*$, $y_1^*$, 0). This indicates that predator $y_2$ is extinct (see Fig. 3).

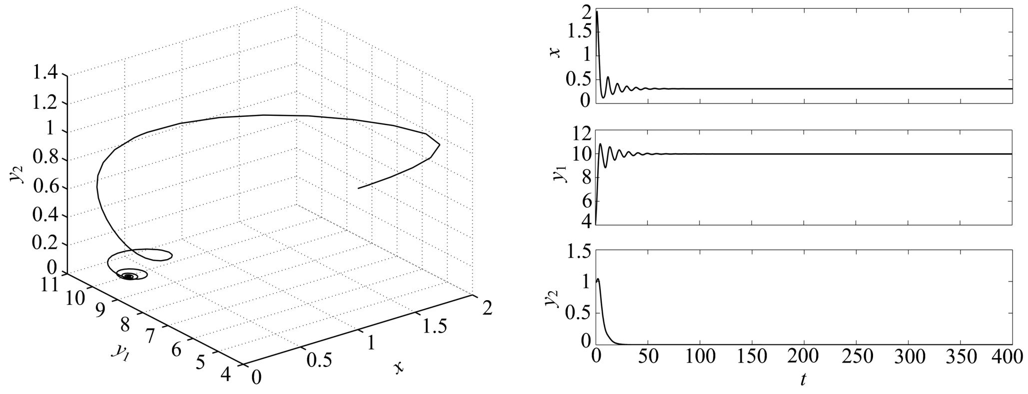
When we choose a suitable impulsive period $T$ and release the number of predator $p_i$, for example $p_1=0$, $p_2=1.5$, $T=2$, we see that system (3) is permanent (see Fig. 4).

By choosing $p_2=7$, we see that the population $y_1$ is extinct, and populations $x$ and $y_2$ are permanent (see Fig. 5).

By choosing $p_2=12$, we show the populations $x$ and $y_1$ are extinct, but $y_2$ is permanent (see Fig. 6).

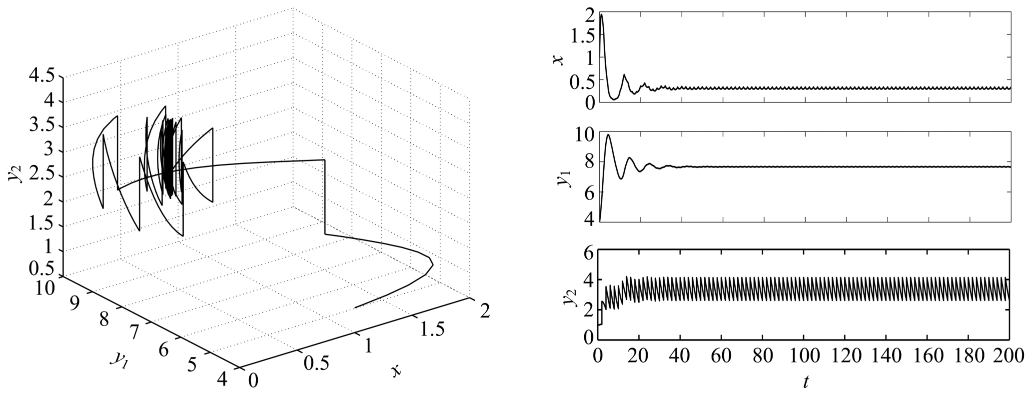
In the following, we investigate the dynamical behavior of system (3). We consider the following set of parameters for our analysis: $m=2$, $a=4.9$, $b=0.31$, $c_1=0.26$, $c_2=0.25$, $s_1=0.23$, $s_2=0.22$, $d_1=0.25$, $d_2=0.24$, $q_1=0.38$, $q_2=0.36$, $p_1=1$, $p_2=4$. Then we can get $\dfrac{c_1p_1}{ad_1}+\dfrac{c_2p_2}{ad_2}=1.0626$. We have got bifurcation diagrams (see Fig. 7) of system (3) as $T$ increases from 1.0626 to 8.5 with initial values $(x_0, y_{01}, y_{02})=(1,1,1)$. As $T$ increases, the resulting bifurcation diagrams clearly show that system (3) has rich dynamic behaviors.

From Fig. 7, we can observe that when $1.0626<T<1.201$, the $T$-period solution of



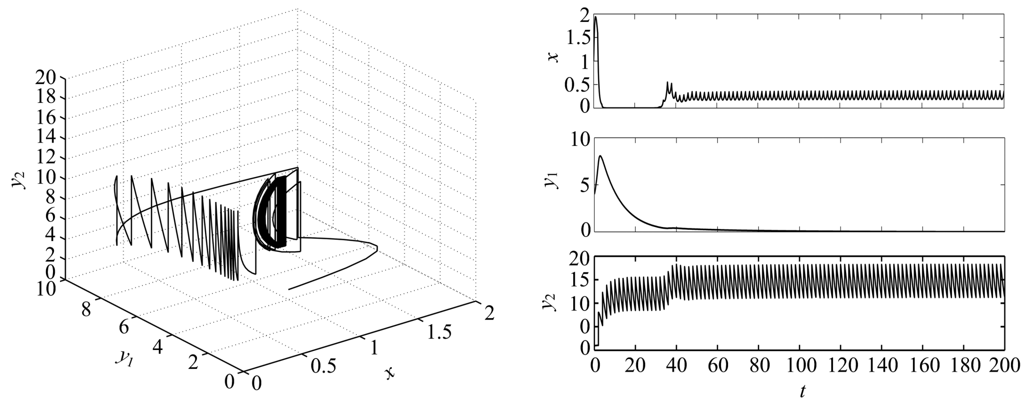

(a) phase portrait (b) time series

**Fig. 3** (a) and (b) are the phase portrait and time series of $x$, $y_1$, $y_2$ for system (17) when $x_0=1$, $y_{01}=4$, $y_{02}=1$, which is free from impulse, respectively




(a) phase portrait (b) time series

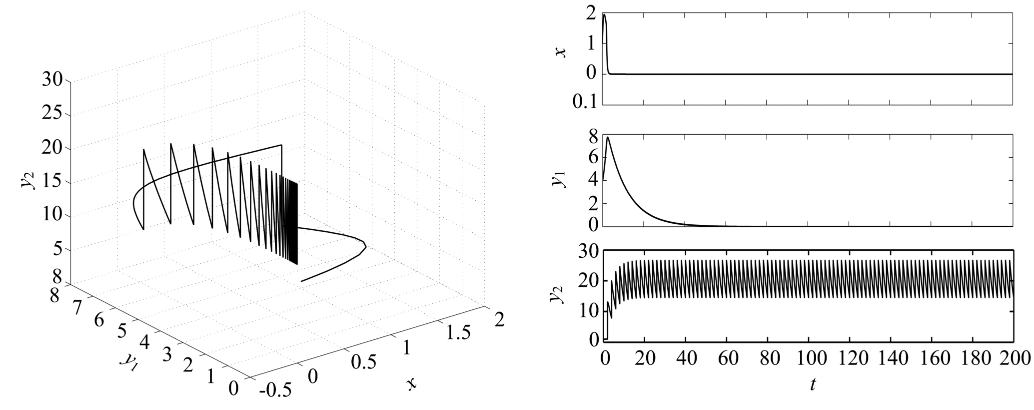
**Fig. 4** The effects of the impulsive perturbations on the system (3), where $p_1=0$, $p_2=1.5$, $T=2$, $x_0=1$, $y_{01}=4$, $y_{02}=1$




(a) phase portrait (b) time series

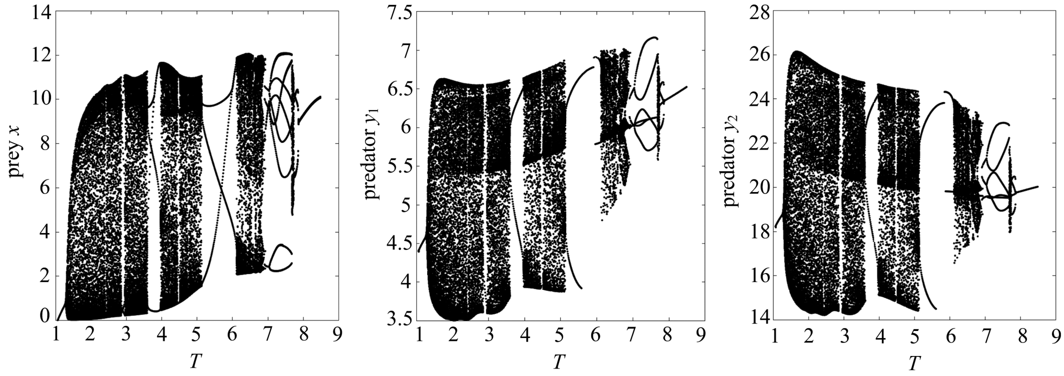
**Fig. 5** The effects of the impulsive perturbations on the system (3), where $p_1=0$, $p_2=7$, $T=2$, $x_0=1$, $y_{01}=4$, $y_{02}=1$

system (3) is still stable (see Fig. 8(a)). When the parameter $T$ is increased beyond $T\approx 1.201$, the dynamic behavior of system (3) is complicated and shows chaos (see Fig. 8(b)). In particular, when $T$ slightly increases beyond $T\approx 2.860$, the chaos suddenly disappears and suddenly appears again (when $T$ slightly increases beyond $T\approx 2.963$) (see Fig. 9). This phenomenon is called crisis. Then the solution of system (3) is periodic again for $3.596 < T < 3.973$ (see Fig. 10). When $3.974 < T < 7.74$ chaotic bands with period windows can be seen (see Fig. 11) and chaos is observed in some




(a) phase portrait　　(b) time series

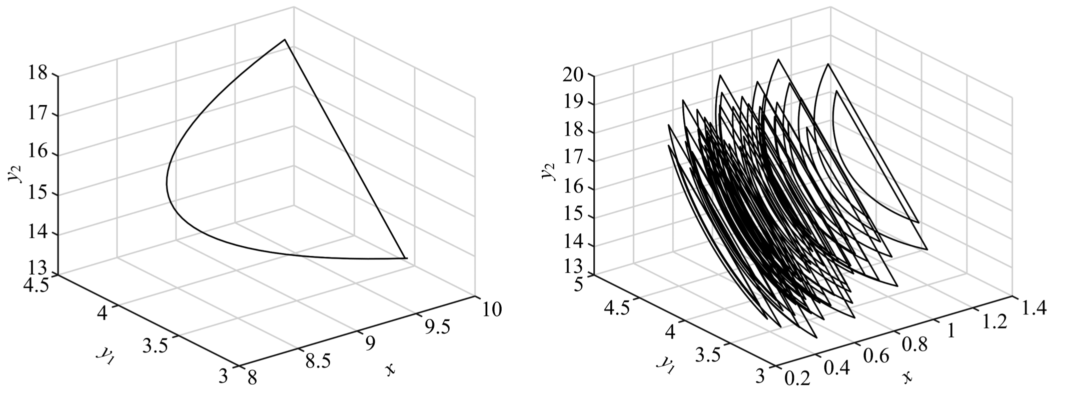
**Fig. 6　The effects of the impulsive perturbations on the system (3), where $p_1=0$, $p_2=12$, $T=2$, $x_0=1$, $y_{01}=4$, $y_{02}=1$**




(a)　(b)　(c)

(a) prey population $x$, (b) predator population $y_1$ and (c) predator population $y_2$ are plotted for $T$ over [1.0626, 8.5].

**Fig. 7　Bifurcation diagrams of system (3) showing the effect of $T$ when $m=2$, $a=4.9$, $b=0.31$, $c_1=0.26$, $c_2=0.25$, $s_1=0.23$, $s_2=0.22$, $d_1=0.25$, $d_2=0.24$, $q_1=0.38$, $q_2=0.36$, $p_1=1$, $p_2=4$, $(x_0, y_{01}, y_{02})=(1,1,1)$**



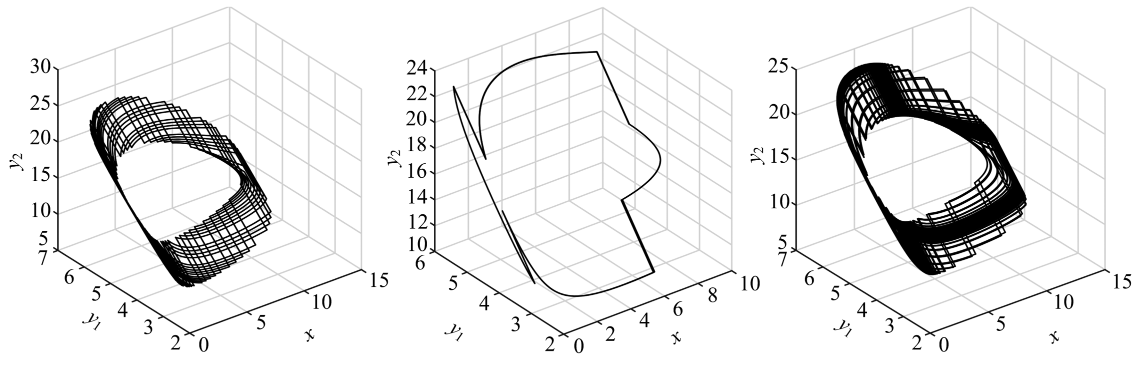

(a)phase portrait of $T$-period solution for $T=1.063$　　(b)phase portrait of the solution enters chaos for $T=1.25$

**Fig. 8　Dynamical behavior of system (3)**

regions (see Fig. 11(a),(b),(d),(f)). After these chaotic areas, when the parameter $T$ is increased beyond $T \approx 7.93$, the $T$-period solution gradually appears again (see Fig. 12).

## 4　Conclusion

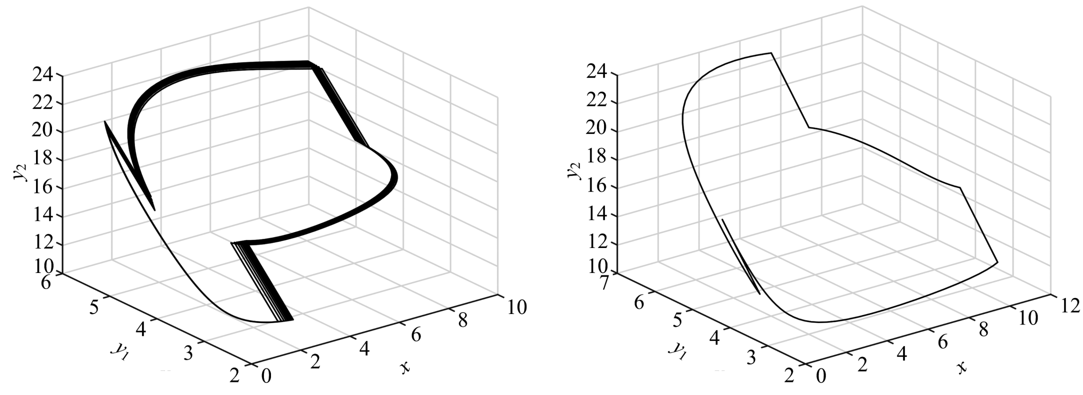
In this paper, we have proposed a one-prey multi-predator system model (3) with Ivlev functional response and impulsive effect. In our model, the rate of the trophic absorption of the




(a) $T$=2.20　(b) $T$=2.862　(c) $T$=2.965

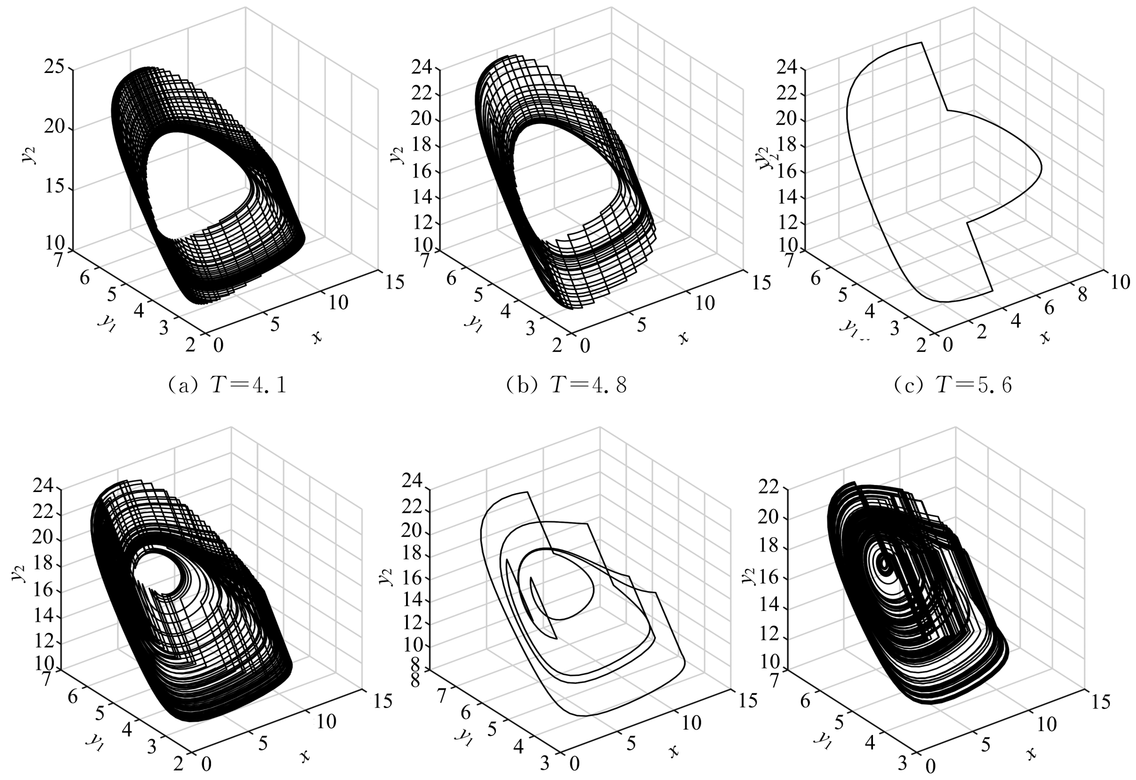
From (a) to (b) there is a crisis during which the chaos suddenly disappears, and from (b) to (c) there is a crisis during which the chaos suddenly appears

**Fig. 9. Dynamical behavior of system (3). Crises are shown with different $T$**




(a) $T$=3.597　(b) $T$=3.88

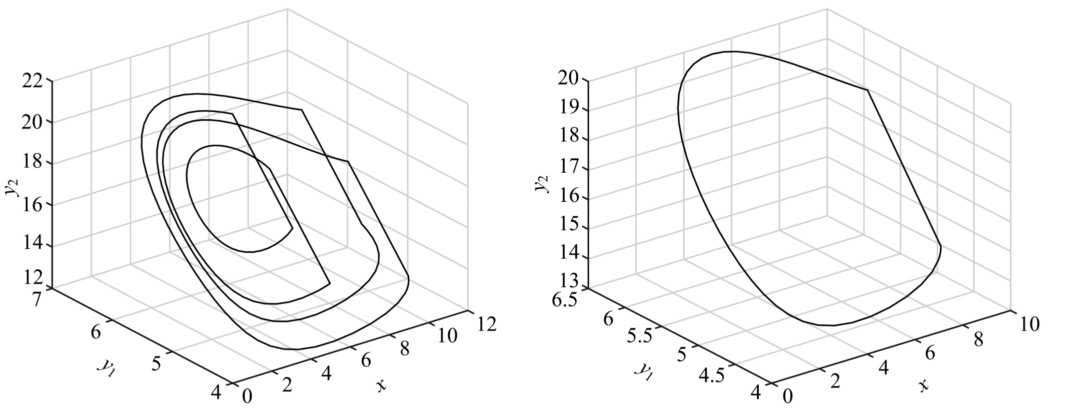
**Fig. 10 Phase portraits of system (3) with initial values $(x_0, y_{01}, y_{02})=(1,1,1)$, the solution being periodic**




(a) $T$=4.1　(b) $T$=4.8　(c) $T$=5.6

(d) $T$=6.51　(e) $T$=7.43　(f) $T$=7.72

**Fig. 11 Dynamical behavior of system (3). Chaotic bands with period windows can be seen**




(a) phase portrait of $4T$-periodic solution for $T=7.82$ (b) phase portrait of $T$-periodic solution for $T=8.11$

**Fig. 12 Dynamical behavior of system (3). The *T*-period solution gradually appears again**

predator is smaller than that of the conversion of consumed prey to predator in the Ivlev-type functional responses. We have investigated the extinction, permanence and complexity of system (3). The conditions for the extinction and permanence of system (3) have been given. We have found that if the impulsive period $T$ is less than $\sum_{i=1}^{m}\frac{c_i p_i}{ad_i}$, the pest-eradication periodic solution $(0, y_1^*, \cdots, y_m^*)$ is asymptotically stable; if $T$ is larger than $\sum_{i=1}^{m}\frac{c_i p_i}{ad_i}$, system (3) is permanent. Our numerical simulations have demonstrated the above conclusions. We also find that when we fix the other parameters, we can eradicate a pest by choosing an appropriate impulsive period $T$ and impulsively releasing the number of predator $p_i$ such that $\frac{1}{T}\sum_{i=1}^{m}\frac{c_i p_i}{d_i}$ is larger than $a$. All these results show that the multi-predator impulsive control strategy is more effective than the classical one and makes the behavior dynamics of the system more complex. For future studies, we would like to consider adding time delays to the existing system (3), and to research the stability and dynamics of that system.

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