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The asymptotic behavior of a logistic SIR epidemic model with stochastic perturbation

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Abstract: A asymptotic behavior of a stochastic logistic SIR epidemic model was studied, whose natural death rates are subject to the environmental white noise. First, it was demonstrated that the model possesses non-negative solutions with probability one. Then, the stochastically asymptotical constancy of the equilibrium was obtained by means of the stochastic Lyapunov functional technique, when $R_0 \leqslant 1$. Additionally, when $R_0 > 1$, some asymptotic outcomes regarding large time behavior were given. When the noise is small and the diseased death rate is limited, the solution will oscillate around the endemic equilibrium of the deterministic model for a long time, and the fluctuation decreases with the decrease of white noise, which reflects the prevalence of the disease.

Key words: stochastic sir model; logistic birth; disease-free equilibrium; endemic equilibrium; stochastic Lyapunov function; asymptotically stable

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一类具有随机扰动的 logistic SIR 传染病模型的渐近行为

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摘要: 考虑了一类自然死亡率受环境噪声随机扰动的 logistic SIR 传染病模型的渐近行为.首先,论证了模型依概率 1 存在正解.然后通过随机 Lyapunov 函数方法证明了当 $R_0<1$ 时无病平衡点的随机稳定性,并给出了当 $R_0>1$ 时的一些考虑长时间状态的渐近结果.当噪声强度很小且因病死亡率满足一定条件时,模型解围绕确定性模型的解长时间随机振荡,振荡幅度随着噪声强度的减小而减小,这说明了疾病将流行.

关键词: 随机 SIR 模型;logistic 出生率;无病平衡点;地方病平衡点;随机 Lyapunov 函数;随机稳定

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0 Introduction

The stochastic epidemic models were advanced in the early 20th century. Early in 1926, M'Kendrick[1] presented a stochastic version regarding the general epidemic model. However, more interest had been in the discrete-time models, and specially in binomial models. More arguments on the continuous time stochastic models have gradually increased. However, most researchers established and analysed stochastic epidemic models through the probabilistic method. The focus was on the likelihood of extinction of disease, likelihood of outbreak of disease, likelihood of distribution of quasistationary, final size distribution as well as anticipated epidemics duration, see, e.g.Refs.[2-5].

Recently, many scholars have studied the dynamics of models of epidemics troubled by the Gaussian white noise. Refs.[6-7] have considered the positive equilibrium of the white noise stochastic pertubation situations.. The approach of parameters perturbation has been utilized by a great number of researchers to build stochastic epidemic models[8-11]. Ref.[8] pointed out that due to environmental fluctuations, the structures entailed the system of the model to display indiscriminate variations. On the other hand, some researchers have examined other kinds of environmental noise, including colour noise[12-13].

The stochastic SIR models have also been examined in the recent studies. Tornatore et al.[14] suggested a stochastic SIR model with or without allocated delay in time. The authors presented a satisfactory situation for the asymptotic constancy of the disease-free equilibrium. Besides, they only illustrated that the introduction of noise customized the inception of the scheme for an epidemic to take place in view of numerical models. Lin et al.[5] referred to a stochastic SIR model that has perturbed disease transmission coefficient. They included sufficient conditions in relation to the disease exponential extinction. Considering persistence, the authors analysed the long-time behaviour of densities of the solutions distributions and affirmed that these solutions densities might join in L^1 to an invariant density within considerable situations. Moreover, they also noted the sustenance of the invariant density. More particularly, where the white noise intensities are comparatively small, they provided a new inception or threshold for the occurrence of an epidemic. Ji et al.[15] further elaborated on a two-group SIR model with the transmission parameter dependent on the white noise, whereas Yu et al.[16] examined a two-group SIR model consisting of a stochastic perturbation about the positive equilibrium. They all achieved an in-depth analysis on asymptotic behaviour of an SIR epidemic model with an invariable size of the population or invariable recruitment. Nevertheless, we believe that there are no outcomes on the stochastic SIR model with logistic growth. Motivated by these, we intend to devise our general SIR epidemic models using the logistic growth through an introduction of the stochasticity into the deterministic model, whose natural death rate and diseased death rate are subject to environmental white noise, respectively.

This article mainly focuses on investigating the environmental variations that are set up in the natural death rate on diseases dynamics by investigating a logistic SIR model. The other sections of this paper intend to present the following: In Section 1, we will develop the stochastic model and demonstrate the main findings. Some helpful preliminaries are presented in Section 2. Then, in Section 3, we will present the assertions of the main findings in depth. In Section 4, the numerical models will be conducted. Lastly, we will present a concise discussion as well as a summary of the main findings.

1 Model and main results

1.1 Model derivations

Zhang et al.[17] mentioned an SIR model with logistic birth:

$$\begin{cases}\dfrac{\mathrm{d}S}{\mathrm{d}t}=(b-r\dfrac{N}{K})N-\beta SI-\mu S,\\ \dfrac{\mathrm{d}I}{\mathrm{d}t}=\beta SI-\delta I,\\ \dfrac{\mathrm{d}R}{\mathrm{d}t}=\gamma I-\mu R\end{cases}\tag{1}$$

where $N(t)=S(t)+I(t)+R(t)$ and $\beta>0$ is the transmission coefficient, b is the birth rate, μ is the natural death rate, $r=b-\mu$ is the intrinsic rate, $\delta=\alpha+\gamma+\mu$ and α is a non-negative constant and represents the diseased death rate, and $\gamma>0$ is the rate constant for recovery. They used N as a variable in place of S, then the SIR model is described by the following system of different equations

$$\begin{cases}\dfrac{\mathrm{d}I}{\mathrm{d}t}=\beta I(N-I-R)-\delta I,\\ \dfrac{\mathrm{d}R}{\mathrm{d}t}=\gamma I-\mu R,\\ \dfrac{\mathrm{d}N}{\mathrm{d}t}=r(1-\dfrac{N}{K})N-\alpha I\end{cases}\tag{2}$$

They showed that the region $H=\{(I,R,N)\in\mathbb{R}^3_+ : I+R\leqslant N\leqslant K\}$ is the positively invariant set with respect to (2). The disease-free $E_0=(0,0,K)$ of System (2) always exists, and if $R_0=\dfrac{\beta K}{\delta}\leqslant 1$, E_0 is globally asymptotically stable in H. If $R_0>1$, E_0 is unstable and there is a unique endemic equilibrium $E^*(I^*, R^*, N^*)$ which is locally asymptotically stable and globally asymptotically stable when $\alpha\leqslant\min\{2\mu,\dfrac{1}{2}r\}$.

Let $(\Omega,\{\mathrm{F}_t\}_{t\geqslant 0},P)$ be a complete probability space with a filtration $\{\mathrm{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i. e. it is right continuous and F_0 contains all P-null sets). Statistically, the natural death rate is not a fixed number, but fluctuates around a fixed number. And studies have shown that the diseased rate was affected by a number of factors, especially medical condition. Hence the diseased rate was always a certain degree of random variability. So in this article, we assume that fluctuations in the environment will manifest themselves mainly as fluctuations in the natural death rate and the diseased death rate, respectively. The natural death rate μ is subject to the environmental white noise, that is

$$-\mu\to-\mu+\sigma\dot{B}(t).$$

Then, $-\mu\mathrm{d}t\to-\mu\mathrm{d}t+\sigma\mathrm{d}B(t)$, where $B(t)$ is a standard Brownian motion, $\sigma^2>0$ is the intensity of environment white noise. Then model (1) becomes

$$\begin{cases}\mathrm{d}S(t)=[(b-r\dfrac{N(t)}{K})N(t)-\beta S(t)I(t)-\\ \qquad\qquad \mu S(t)]\mathrm{d}t+\sigma(S(t)-\dfrac{N^2(t)}{K})\mathrm{d}B(t),\\ \mathrm{d}I(t)=[\beta S(t)I(t)-\delta I(t)]\mathrm{d}t+\sigma I(t)\mathrm{d}B(t),\\ \mathrm{d}R(t)=(\gamma I(t)-\mu R(t))\mathrm{d}t+\sigma R(t)\mathrm{d}B(t)\end{cases}\tag{3}$$

Use N as a variable in place of S, then System (3) becomes

$$\begin{cases}\mathrm{d}I(t)=[\beta(N(t)-I(t)-R(t))I(t)-\\ \qquad\qquad \delta I(t)]\mathrm{d}t+\sigma I(t)\mathrm{d}B(t),\\ \mathrm{d}R(t)=(\gamma I(t)-\mu R(t))\mathrm{d}t+\sigma R(t)\mathrm{d}B(t),\\ \mathrm{d}N(t)=[r(1-\dfrac{N(t)}{K})N(t)-\alpha I(t)]\mathrm{d}t+\\ \qquad\qquad \sigma(1-\dfrac{N(t)}{K})N(t)\mathrm{d}B(t)\end{cases}\tag{4}$$

1.2 Main results

The goal of the paper is to study the disease dynamics of the stochastic model (4). Fist we prove the existence, uniqueness and boundedness of the positive solution to System (3).

Theorem 1.1 There is a unique solution $(S(t),I(t),R(t))$ of System (3) on $t\geqslant 0$ for any initial value $(S(0),I(0),R(0))\in\Omega^*$, and the solution will still belong to Ω^* with probability one, namely $(S(t),I(t),R(t))\in\Omega^*$ for all $t\geqslant 0$ almost surely.

Then we denote a bounded set Ω^* as follow:

$$\Omega^*=\{(I,R,N)\in\mathbb{R}^3_+,\ I+R\leqslant N\leqslant K\}.$$

Theorem 1.2 (i) If $R_0=\dfrac{\beta K}{\delta}\leqslant 1$, $\sigma^2<\min\{\mu, r\}$, then the solution $(0,0,K)$ of System (4) is stochastically asymptotically stable in the large.

(ii) Let $(I(t),R(t),N(t))$ be the solution of System (4) with any initial value $(I(0),R(0),N(0))\in\Omega^*$. If $R_0>1$, $\alpha<\dfrac{r}{2+\left(\dfrac{1}{1+\gamma/\mu}\right)}$ and $\sigma^2<2\mu$, then we have

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t[(I-I^*)^2+(R-\frac{2\mu}{2\mu-\sigma^2}R^*)^2+(N-N^*)^2]\mathrm{d}s\leqslant\frac{K_\sigma}{M},\ \text{a.s.},$$

where

$$K_\sigma=(\frac{1}{2\beta}I^*+\frac{1}{\gamma}\frac{\mu}{2\mu-\sigma^2}R^{*2}+\frac{1}{2\alpha}N^{*2})\sigma^2,$$

$$M=\min\{\frac{1}{\gamma}(\mu-\frac{1}{2}\sigma^2),(\frac{rN^*}{K\alpha}-1)\},$$

and $E^*=(I^*,R^*,N^*)$ is the endemic equilibrium of System (2).

2 Preliminaries

We provide some useful lemmas to prove our main results. Consider the randomized logistic equation

$$\mathrm{d}x(t)=x(t)[1-\frac{x(t)}{K}](r\mathrm{d}t+\sigma\mathrm{d}B(t))\quad(5)$$

where $B(t)$ is 1-dimensional standard Brownian motion. The following result can be found in Ref.[19].

Lemma 2.1[19] For any initial value $0<x_0<K$, there is a unique solution $x(t)$ to Eq.(5) on $t\geqslant0$ and $\phi(t)<x(t)<K$, where

$$\phi(t)=\frac{K}{1+(\frac{K}{x_0}-1)\mathrm{e}^{-rt+\frac{1}{2}\sigma^2t-\sigma B(t)}}.$$

Then we can see the following two lemmas in Ref.[21]. Consider a stochastic different equation

$$\mathrm{d}x(t)=f(x(t),t)\mathrm{d}t+g(x(t),t)\mathrm{d}B(t),\ t\geqslant t_0\quad(6)$$

Assume $f(0,t)=0$ and $g(0,t)=0$ for all $t\geqslant t_0$. So $x(t)\equiv0$ is a solution to Eq.(6), called the trivial solution or equilibrium position.

Lemma 2.2[21] If there exists a positive-definite decrescent radially unbounded function $V(x,t)\in C^{2,1}(\mathbb{R}^d\times[t_0,+\infty);\mathbb{R}_+)$ such that $LV(x,t)$ is negative-definite, then the trivial solution of Eq.(6) is stochastically asymptotically stable in the large.

Lemma 2.3[21] (Strong law of large numbers) Let $M=\{M_t\}_{t\geqslant0}$ be a real-valued continuous local martingale vanishing at $t=0$. Then

$$\lim_{t\to\infty}\langle M,M\rangle_t=\infty\ \text{a.s.}\Rightarrow\lim_{t\to\infty}\frac{M_t}{\langle M,M\rangle_t}=0\ \text{a.s.}$$

and also

$$\limsup_{t\to\infty}\frac{\langle M,M\rangle_t}{t}<\infty\ \text{a.s.}\Rightarrow\lim_{t\to\infty}\frac{M_t}{t}=0\ \text{a.s..}$$

At last, We prove a useful lemma.

Lemma 2.4 Assume the endemic equilibrium $E^*=(I^*,R^*,N^*)$ of System (2) exists and $\alpha<\dfrac{r}{2+\left(\dfrac{1}{1+\gamma/\mu}\right)}$. Then

$$\frac{rN^*}{\alpha K}-1>0\quad(7)$$

Proof $E^*=(I^*,R^*,N^*)$ is the endemic equilibrium of System (2), according to Ref.[17], N^* is the root of the following equation in the interval $(0,K)$:

$$\alpha-\frac{\alpha\delta}{\beta N}-r(1+\frac{\gamma}{\mu})(1-\frac{N}{K})=0,$$

i.e.

$$\frac{r}{K}(1+\frac{\gamma}{\mu})N^2+[\alpha-r(1+\frac{\gamma}{\mu})]N-\frac{\alpha\delta}{\beta}=0.$$

Obviously,

$$N^*>-\frac{\alpha-r(1+\frac{\gamma}{\mu})}{2\frac{r}{K}(1+\frac{\gamma}{\mu})}\quad(8)$$

If $\alpha<\dfrac{r}{2+\left(\dfrac{1}{1+\gamma/\mu}\right)}$, then

$$-\frac{\alpha-r(1+\gamma/\mu)}{2\frac{r}{K}(1+\gamma/\mu)}>\frac{\alpha}{r/K}\quad(9)$$

By (8) and (9), we have

$$N^*>\frac{\alpha}{r/K}.$$

So,

$$\frac{rN^*}{\alpha K}-1>0.$$

The proof is thus complete.

3 Proof of the main results

In this section, by applying results in the previous section, we provide the detailed proofs of the main results illustrated in Section 1.2. First we prove Theorem 1.1.

Proof of Theorem 1.1 Since coefficients of the equation are locally Lipschitz continuous for any initial value $(S(0),I(0),R(0))\in\Omega^*$, there is a unique local solution on $t\in[0,\tau_e)$, where τ_e is the explosion time[21]. To show this solution is global, we need to show that $\tau_e=\infty$ a.s.. Let $k_0>1$ be sufficiently large so that $S(0),I(0),R(0)$ all lie within the interval $[\frac{1}{k_0},k_0]$. For each integer $k\geqslant k_0$ define the stopping time

$$\tau_k=\inf\{t\in[0,\tau_e):S(t)\notin(\frac{1}{k},k)\text{ or }I(t)\notin(\frac{1}{k},k)\text{ or }R(t)\notin(\frac{1}{k},k)\},$$

where throughout this paper, we set $\inf\varnothing=\infty$ (as unual $\varnothing$ denotes the empty set). Clearly, τ_k is increasing as $k\to\infty$. Set $\tau_\infty=\lim\limits_{k\to\infty}\tau_k$, whence $\tau_\infty\leqslant\tau_e$ a.s.. If we can show that $\tau_\infty=\infty$ a.s. then $\tau_e=\infty$ and $(S(t),I(t),R(t))\in\mathbb{R}^3_+$ a.s. for all $t\geqslant0$. In other words, to complete the proof all we need to show is that $\tau_\infty=\infty$ a.s.. For if this statement is false, then there is a pair of constants $T>0$ and $\epsilon\in(0,1)$ such that

$$P\{\tau_\infty\leqslant T\}>\epsilon.$$

Hence there is an integer $k_1\geqslant k_0$ such that,

$$P\{\tau_k\leqslant T\}>\epsilon\text{ for all }k\geqslant k_1 \tag{10}$$

Besides, for $t\leqslant\tau_k$, we can see

$$\mathrm{d}N(t)\leqslant[r(1-\frac{N(t)}{K})N(t)]\mathrm{d}t+\sigma(1-\frac{N(t)}{K})N(t)\mathrm{d}B(t).$$

Consider the following auxiliary equation

$$\begin{cases}\mathrm{d}x(t)=[(1-\frac{x(t)}{K})x(t)](r\mathrm{d}t+\sigma\mathrm{d}B(t)),\\ x(0)=N(0)\end{cases} \tag{11}$$

From Lemma 2.1, we known that there exists a unique continuous positive solution $x(t)$ of System (11) for any positive initial value $0<N(0)<K$ and $x(t)<K$. Consequently, by the comparison theorem of stochastic differential equation we have

$$N(t)\leqslant x(t)\text{ for all }t\geqslant0\text{ a.s..}$$

Therefore, $N(t)<K<\infty$ for all $t\geqslant0$ a.s.. Define a C^2-function $V:\mathbb{R}^3_+\to\overline{\mathbb{R}}_+$ by

$$V(S,I,R)=S-a-a\ln\frac{S}{a}+I-1-\ln I+R-1-\ln R\ .$$

The non-negativity of this function can be seen from $u-1+\ln u\geqslant0\ \ \forall\ \ u>0$. Using It $\hat{\text{o}}$'s formula, we compute

$$\mathrm{d}V=LV\mathrm{d}t+\sigma[(S-a-(1-\frac{a}{S})\frac{N^2}{K}+(I-1)+(R-1)]\mathrm{d}B(t),$$

where

$$\begin{aligned}LV&=(1-\frac{a}{S})[(b-r\frac{N}{K})N-\beta SI-\mu S]+\frac{a\sigma^2}{2S^2}(S-\frac{N^2}{K})^2+\\&(1-\frac{1}{I})(\beta SI-\delta I)+\frac{\sigma^2}{2}+(1-\frac{1}{R})(\gamma I-\mu R)+\frac{\sigma^2}{2}=\\&(b-r\frac{N}{K})N-\beta SI-\mu S-a(b-r\frac{N}{K})\frac{N}{S}+a\beta I+\mu a+\beta SI-\delta I-\beta S+\delta+\gamma I-\mu R-\gamma\frac{I}{R}+\\&\mu+\frac{\sigma^2}{2}(\frac{a(S-\frac{N^2}{K})^2}{S^2}+2)\leqslant\\&bK+\delta+(1+a)\mu+\sigma^2(\frac{a}{2}+1)+[a\beta-(\alpha+\mu)]I.\end{aligned}$$

Choose $a=\frac{\alpha+\mu}{\beta}$, such that $a\beta-(\alpha+\mu)=0$, then

$$LV(S,I,R)\leqslant bK+\delta+(1+a)\mu+\sigma^2(\frac{a}{2}+1):=\overline{K}.$$

Therefore,

$$\int_0^{\tau_k\wedge T}\mathrm{d}V(I(t),R(t),N(t))\leqslant\int_0^{\tau_k\wedge T}\sigma[(S-a-(1-\frac{a}{S})\frac{N^2}{K}+(I-1)+(R-1)]\mathrm{d}B(t)+\int_0^{\tau_k\wedge T}\widetilde{K}\,\mathrm{d}t.$$

This implies that,

$$E[V(I(\tau_k \wedge T),R(\tau_k \wedge T),N(\tau_k \wedge T))] \leqslant V(I(0),R(0),N(0)) + E\int_0^{\tau_k \wedge T} \widetilde{K}\,\mathrm{d}t \leqslant V(I(0),R(0),N(0)) + \widetilde{K}T. \quad (12)$$

Set $\Omega_k = \tau_k \leqslant T$ for $k \geqslant k_1$ and by (10), $P(\Omega_k) \geqslant \epsilon$. Note that for every $\Omega \in \Omega_k$, there is at least one of $S(\tau_k,\Omega), I(\tau_k,\Omega), R(\tau_k,\Omega)$ equals k or $\frac{1}{k}$, and hence $V(S(\tau_k \wedge T), I(\tau_k \wedge T), R(\tau_k \wedge T))$ is no less than $k-1-\ln k$ or $\frac{1}{k}-1-\ln\frac{1}{k}=\frac{1}{k}-1+\ln k$.

Consequently,

$$V(S(\tau_k \wedge T),I(\tau_k \wedge T),R(\tau_k \wedge T)) \geqslant (k-1-\ln k) \wedge (\frac{1}{k}-1+\ln k).$$

Then it follows from (10) and (12) that

$$V(S(0),I(0),R(0)) + \widetilde{K}T \geqslant E[1_{\Omega_k(\Omega)} V(S(\tau_k \wedge T),I(\tau_k \wedge T),R(\tau_k \wedge T))] \geqslant \epsilon\,[(k-1-\ln k) \wedge (\frac{1}{k}-1+\ln k)],$$

where $1_{\Omega_k(\Omega)}$ is the indicator function of Ω_k. Let $k \to \infty$ lead to the contradiction

$$\infty > V(S(0),I(0),R(0)) + \widetilde{K}T = \infty.$$

Therefore we have $\tau_\infty = \infty$ a.s..

Remark 3.1　From Theorem 1.1, we can easily see that the region $H = \{(I,R,N) \in \mathbb{R}^3_+,\ I+R \leqslant N \leqslant K\}$ is a positively invariant set with respect to (4).

3.1　Stochastically asymptotical stability

In this subsection, we prove Theorem 1.2(i) about the stochastically asymptotical stability in the large of the disease-free equilibrium for the stochastic model (4).

Proof of Theorem 1.2(i)　Let $x=I$, $y=R$, $z=N-K$. Then $x \geqslant 0$, $y \geqslant 0$, $z \leqslant 0$ and System (4) becomes

$$\begin{cases} \mathrm{d}x = (\beta(K+z-x-y)x - \delta x)\mathrm{d}t + \sigma x\,\mathrm{d}B(t), \\ \mathrm{d}y = (\gamma x - \mu y)\mathrm{d}t + \sigma y\,\mathrm{d}B(t), \\ \mathrm{d}z = (-\frac{r}{K}z(K+z) - \alpha x)\mathrm{d}t - \frac{\sigma}{K}z(K+z)\mathrm{d}B(t) \end{cases} \quad (13)$$

Define the stochastic Lyapunov function $\mathbb{R}^3 \to \overline{\mathbb{R}}_+$:

$$V(x,y,z) = \frac{1}{\beta}x + \frac{1}{2\gamma}y^2 + \frac{1}{2\alpha}z^2.$$

Obviously, $V(x,y,z)$ is positive-definite, decrescent and radially unbounded. By Itô's formula, we compute

$$LV = Kx + xz - x^2 - xy - \frac{\delta}{\beta}x + xy - \frac{\mu}{\gamma}y^2 + \frac{\sigma^2}{\gamma}y^2 - \frac{r}{\alpha K}z^2(K+z) - xz + \frac{\sigma^2}{\alpha K^2}z^2(K+z)^2 = (K-\frac{\delta}{\beta})x - x^2 - \frac{1}{\gamma}(\mu-\sigma^2)y^2 + \frac{\sigma^2}{\alpha K^2}z^3(K+z) - \frac{1}{\alpha K}z^2(K+z)(r-\sigma^2).$$

Based on the fact that $R_0 = \frac{\beta K}{\delta} \leqslant 1$ and $K+z \geqslant 0$, choose $\sigma^2 < \min\{\mu, r\}$, then LV is negative-definite. By Lemma 2.2, we conclude that under the condition of $R_0 = \frac{\beta K}{\delta} \leqslant 1$, the trivial solution of System (13) is stochastically asymptotically stable in the large, i.e., the solution $(0,0,K)$ of System (4) is stochastically asymptotically stable in the large.

Remark 3.2　From Theorem 1.2(i), If $R_0 = \frac{\beta K}{\delta} \leqslant 1$, $\sigma^2 < \min\{\mu, r\}$, then the solution $(0,0,K)$ of System (4) is stochastically asymptotically stable in the large. It means that the disease would die out when the noise meets certain conditions.

3.2　Asymptotic behavior

In this section, we show the asymptotic behavior around the endemic equilibrium of the deterministic model.

Proof of Theorem 1.2(ii)　Since $E^* = (I^*, R^*, N^*)$ is the endemic equilibrium of System (2), we have

$$\left.\begin{aligned} &\beta(N^* - I^* - R^*) = \delta,\ \gamma I^* - \mu R^* = 0, \\ &r(1-\frac{N^*}{K}) - \alpha\frac{I^*}{N^*} = 0 \end{aligned}\right\} \quad (14)$$

Define

$$V(I,R,N) = \frac{1}{\beta}(I - I^* - I^*\ln\frac{I}{I^*}) + \frac{1}{2\gamma}(R-R^*)^2 +$$

$$\frac{N^*}{\alpha}(N-N^*-N^*\ln\frac{N}{N^*}):=V_1+V_2+V_3.$$

Obviously, V is positively definite. By Itô's formula, we compute

$$\mathrm{d}V:=(LV_1+LV_2+LV_3)\mathrm{d}t+\sigma[\frac{1}{\beta}(I-I^*)+\frac{1}{\gamma}(R-R^*)R+\frac{N^*}{\alpha}(N-N^*)(1-\frac{N}{K})]\mathrm{d}B(t),$$

where

$$LV_1=\frac{1}{\beta}(1-\frac{I^*}{I})(\beta(N-I-R)I-\delta I)+\frac{1}{2\beta}\sigma^2I^*=$$
$$\frac{1}{\beta}(I-I^*)(\beta(N-I-R)-\beta(N^*-I^*-R^*))+\frac{1}{2\beta}\sigma^2I^*=$$
$$-(I-I^*)^2-(I-I^*)(R-R^*)+(I-I^*)(N-N^*)+\frac{1}{2\beta}\sigma^2I^* \quad (15)$$

$$LV_2=\frac{1}{\gamma}(R-R^*)(\gamma I-\mu R)+\frac{1}{2\gamma}\sigma^2R^2=$$
$$\frac{1}{\gamma}(R-R^*)(\gamma I-\mu R-\gamma I^*+\mu R^*)+\frac{1}{2\gamma}\sigma^2R^2=$$
$$(I-I^*)(R-R^*)-\frac{\mu}{\gamma}(R-R^*)^2+\frac{1}{2\gamma}\sigma^2R^2 \quad (16)$$

$$LV_3=\frac{N^*}{\alpha}(1-\frac{N^*}{N})[rN(1-\frac{N}{K})-\alpha I]+\frac{N^{*2}}{2\alpha}\sigma^2(1-\frac{N}{K})^2=$$
$$\frac{N^*}{\alpha}(N-N^*)[r(1-\frac{N}{K})-\frac{\alpha I}{N}]+\frac{N^{*2}}{2\alpha}\sigma^2(1-\frac{N}{K})^2=$$
$$\frac{N^*}{\alpha}(N-N^*)[r(1-\frac{N}{K})-\frac{\alpha I}{N}-r(1-\frac{N^*}{K})+\frac{\alpha I^*}{N^*}]+\frac{N^{*2}}{2\alpha}\sigma^2(1-\frac{N}{K})^2=$$
$$\frac{N^*}{\alpha}(N-N^*)[-\frac{r}{K}(N-N^*)-\alpha(\frac{I}{N}-\frac{I^*}{N^*})]+\frac{N^{*2}}{2\alpha}\sigma^2(1-\frac{N}{K})^2=\frac{N^*}{\alpha}(-\frac{r}{K}+\frac{\alpha I}{NN^*})(N-N^*)^2-$$
$$(N-N^*)(I-I^*)+\frac{N^{*2}}{2\alpha}\sigma^2(1-\frac{N}{K})^2\leqslant$$
$$\frac{N^*}{\alpha}(-\frac{r}{K}+\frac{\alpha}{N^*})(N-N^*)^2-(N-N^*)(I-I^*)+\frac{N^{*2}}{2\alpha}\sigma^2(1-\frac{N}{K})^2 \quad (17)$$

where (14) is used in the above equalities. So,

$$LV=-(I-I^*)^2-\frac{\mu}{\gamma}(R-R^*)^2-(\frac{rN^*}{\alpha K}-1)(N-N^*)^2+\frac{1}{2\beta}\sigma^2I^*+\frac{1}{2\gamma}\sigma^2R^2+\frac{N^{*2}}{2\alpha}\sigma^2(1-\frac{N}{K})^2\leqslant$$
$$-(I-I^*)^2-\frac{\mu}{\gamma}R^2+\frac{2\mu}{\gamma}RR^*-\frac{\mu}{\gamma}R^{*2}-(\frac{rN^*}{\alpha K}-1)(N-N^*)^2+\frac{1}{2\beta}\sigma^2I^*+\frac{1}{2\gamma}\sigma^2R^2+\frac{N^{*2}}{2\alpha}\sigma^2=$$
$$-(I-I^*)^2-\frac{1}{\gamma}(\mu-\frac{1}{2}\sigma^2)(R-\frac{2\mu}{2\mu-\sigma^2}R^*)^2-(\frac{rN^*}{\alpha K}-1)(N-N^*)^2+\frac{1}{2\beta}\sigma^2I^*+\frac{1}{\gamma}\frac{\mu\sigma^2}{2\mu-\sigma^2}R^{*2}+\frac{N^{*2}}{2\alpha}\sigma^2.$$

Note that $\mu-\frac{1}{2}\sigma^2>0$ and from Lemma 2.4, we have $\frac{rN^*}{\alpha K}-1>0$, therefore,

$$\mathrm{d}V\leqslant[-(I-I^*)^2-\frac{1}{\gamma}(\mu-\frac{1}{2}\sigma^2)\cdot(R-\frac{2\mu}{2\mu-\sigma^2}R^*)^2-(\frac{rN^*}{\alpha K}-1)(N-N^*)^2+\frac{1}{2\beta}\sigma^2I^*+\frac{1}{\gamma}\frac{\mu\sigma^2}{2\mu-\sigma^2}R^{*2}+\frac{N^{*2}}{2\alpha}\sigma^2]\mathrm{d}t+\sigma[\frac{1}{\beta}(I-I^*)+\frac{1}{\gamma}(R-R^*)R+\frac{N^*}{\alpha}(N-N^*)(1-\frac{N}{K})]\mathrm{d}B(t).$$

Then

$$\int_0^t[(I-I^*)^2+\frac{1}{\gamma}(\mu-\frac{1}{2}\sigma^2)(R-\frac{2\mu}{2\mu-\sigma^2}R^*)^2+(\frac{rN^*}{\alpha K}-1)(N-N^*)^2]\mathrm{d}s\leqslant V(0)+\frac{1}{2\beta}\sigma^2I^*t+\frac{1}{\gamma}\frac{\mu\sigma^2}{2\mu-\sigma^2}R^{*2}t+\frac{N^{*2}}{2\alpha}\sigma^2t+\int_0^t\sigma[\frac{1}{\beta}(I-I^*)+\frac{1}{\gamma}(R-R^*)R+\frac{N^*}{\alpha}(N-N^*)(1-\frac{N}{K})]\mathrm{d}B(s).$$

Let

$$M_t=\int_0^t\sigma[\frac{1}{\beta}(I-I^*)+\frac{1}{\gamma}(R-R^*)R+$$

$$\frac{N^*}{\alpha}(N-N^*)(1-\frac{N}{K})]\mathrm{d}B(t),$$

which is a real-valued continuous local martingale, $M_0=0$ and

$$\frac{\langle M,M\rangle_t}{t}=\frac{1}{t}\int_0^t\sigma^2[\frac{1}{\beta}(I-I^*)+\frac{1}{\gamma}(R-R^*)R+\frac{N^*}{\alpha}(N-N^*)(1-\frac{N}{K})]^2\mathrm{d}s\leqslant[\frac{1}{\beta}K+\frac{1}{\gamma}K^2+\frac{1}{\alpha}K^2]^2\sigma^2<\infty.$$

Then by Lemma 2.3, we have

$$\lim_{t\to\infty}\frac{M_t}{t}=0 \text{ a.s..}$$

Hence

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t[(I-I^*)^2+\frac{1}{\gamma}(\mu-\frac{1}{2}\sigma^2)\cdot(R-\frac{2\mu}{2\mu-\sigma^2}R^*)^2+(\frac{rN^*}{\alpha K}-1)(N-N^*)^2]\mathrm{d}s\leqslant K_\sigma \text{ a.s..}$$

where $K_\sigma=(\frac{1}{2\beta}I^*+\frac{1}{\gamma}\frac{\mu}{2\mu-\sigma^2}R^{*2}+\frac{1}{2\alpha}N^{*2})\sigma^2$.

Let

$$M=\min\{\frac{1}{\gamma}(\mu-\frac{1}{2}\sigma^2),(-\frac{rN^*}{K\alpha}+1)\},$$

then

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t[(I-I^*)^2+(R-\frac{2\mu}{2\mu-\sigma^2}R^*)^2+(N-N^*)^2]\mathrm{d}s\leqslant\frac{K_\sigma}{M} \text{ a.s..}$$

The proof is thus complete.

Remark 3.3　From Theorem 1.2(ii), If $R_0>1$, $\alpha<\dfrac{r}{2+\left(\dfrac{1}{1+\gamma/\mu}\right)}$ and $\sigma^2<2\mu$, the solution will oscillate around the endemic equilibrium of the deterministic model for a long time, and the fluctuation decreases with the decrease of white noise, which reflects the prevalence of the disease.

4　Numerical simulations

In this section, we make numerical simulations to illustrate our results by using Milstein's higher order method[20]. We get the simulation figures with the initial value $(I(0),R(0),N(0))=(0.2,0.2,0.6)$ and time step $\delta t=\frac{1}{2^5}$, the parameters in (4) are given by

$K=2$, $r=0.2$, $\mu=0.4$, $\alpha=0.05$, $\gamma=0.3$.

First, we take $\beta=0.3$, $\sigma_1=0.3$, $\sigma_2=0.8$, in this case, $R_0=\frac{4}{5}<1$, and $\beta=0.375$ such that $R_0=1$. We find that these lines in Figs.1 and 2 fit very well, which implies that the disease-free equilibrium $E_0=(0,0,2)$ of System (4) is globally asymptotically stable but too large σ affects the stability. This result is consistent with the result of Theorem 1.2(i).

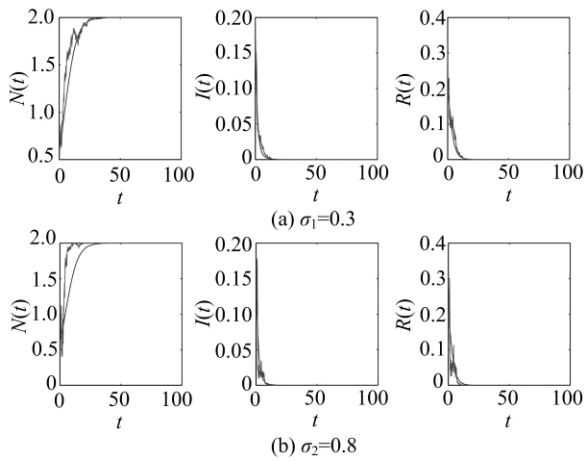

Fig.1　$E_0=(0,0,2)$ of System (4) is globally asymptotically stable, when $R_0<1$

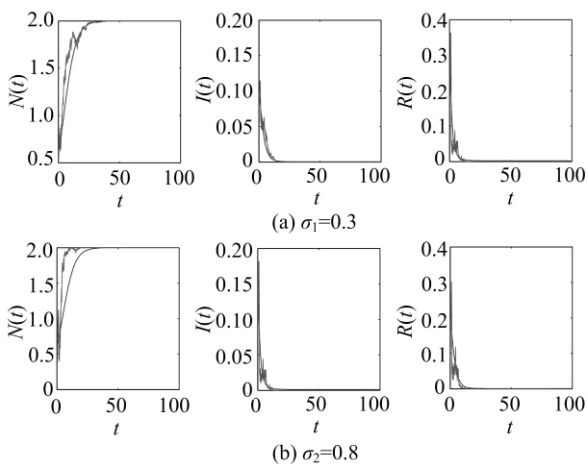

Fig.2　$E_0=(0,0,2)$ of System (4) is globally asymptotically stable, when $R_0=1$

When $\beta=0.5$, it is easy to check that $R_0=\frac{4}{3}>1$. We choose $\sigma_1=0.1$, $\sigma_2=0.05$, and the solution

goes around the endemic equilibrium E^* for a long time, and the fluctuation decreases with the decrease of white noise (see Fig. 3. This result is consistent with the result of Theorem 1.2(ii)).

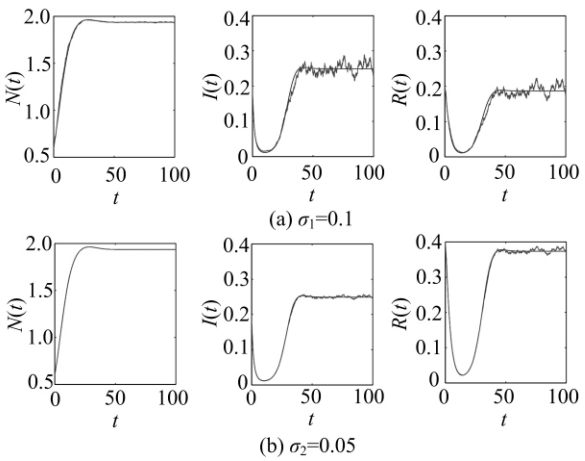

Fig.3 The solution of System (4) is going around the endemic equilibrium of System (2)

5 Conclusion

Environmental noise can be described to have a significant effect on the advancement of an epidemic. For this study, we present the dynamics of a stochastic logistic SIR model with an overall force of infection under the noises of environment. We suppose that the stochastic perturbation is a white noise sort that disturbs the natural death rate μ.

Compared with the deterministic model (2), we find that the intensity of the noise level plays a critical role. We, therefore, summarise our main results as follows:

(Ⅰ) From Theorem 1.2(i), if $R_0 \leqslant 1$ under the small noise intensity case, that is $\sigma^2 < \min\{\mu, r\}$, the solution $(0,0,K)$ is found to be stochastically asymptotically stable. This reveals that the stochastic model (4) has disease extinction with probability one (see Figs. 1 and 2). Particularly, the noise intensity is zero when the models become the deterministic model (2), thereby the disease is extinct too.

(Ⅱ) From Theorem 1.2(ii), if $R_0 > 1$, the solution revolves on the endemic equilibrium E^* for a longer duration, while the variation reduces due to the declining white noise (Fig. 3). This means that the stochastic model (4) has endemic equilibrium. Particularly, the noise intensity is zero when the models become the deterministic model (2), thus the disease is endemic, too.

Nevertheless, only the asymptotic behavior of the model is discussed in this paper. In our upcoming work, we will further discuss other properties of the model, such as ergodic property, the existence of an invariant distribution. And we would build some model with time delay, age composition and control item.

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