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MacWilliams identities of linear codes with respect to RT metric over $M_{n\times s}(\mathbb{F}_l+v\mathbb{F}_l+\cdots+v^{k-1}\mathbb{F}_l)$

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Abstract: A new Gray map over the commutative ring $R=\mathbb{F}_l+v\mathbb{F}_l+\cdots+v^{k-1}\mathbb{F}_l$ was defined , where $v^k=v$. Under this new Gray map, the definitions of the Lee complete ρ weight enumerator and the exact complete ρ weight enumerator over $M_{n\times s}(R)$ were given. Then, the MacWilliams identities with respect to the RT metric for these two weight enumerators of linear codes over $M_{n\times s}(\mathbb{F}_l+v\mathbb{F}_l+\cdots+v^{k-1}\mathbb{F}_l)$ were obtained, respectively. In addition, some examples were presented to illustrate the obtained results.

Key words: weight enumerator; MacWilliams identity; linear codes; Gray map

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矩阵环 $M_{n\times s}(\mathbb{F}_l+v\mathbb{F}_l+\cdots+v^{k-1}\mathbb{F}_l)$上线性码关于$RT$重量的MacWillims恒等式

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摘要:首先在交换环$R=\mathbb{F}_l+v\mathbb{F}_l+\cdots+v^{k-1}\mathbb{F}_l$上定义了一个新的Gray映射,在这个映射的基础下,定义了此矩阵环上完全ρ重量计数器和精确完全ρ重量计数器.然后给出了矩阵环$M_{n\times s}(R)$上线性码关于这两类计数器的MacWillims恒等式.此外,给出了几个例子说明了所得结论的正确性.

关键词:重量计数器;MacWillims恒等式;线性码;Gray映射

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0 Introduction

After just several decades of development, coding theory has become a thriving research area. It has found its widespread application in areas ranging from communication systems to storage technology. Especially, the MacWilliams identity is an important tool in error-correcting coding theory. More than a decade ago, Rosenbloom and Tsfasman proposed a non-Hamming distance over linear space $\mathbb{F}_l^n$ in Ref. [1]. This distance is now called the RT metric. Since then, many scholars have been interested in MacWilliams identity with respect to RT metric over various rings[2-4,11]. A lot of contributions about RT metric over rings and fields can be found in Refs. [5, 6, 7]. Ref. [9] studied cyclic codes and the weight enumerators of linear codes over $\mathbb{F}_2+v\mathbb{F}_2+v^2\mathbb{F}_2$. Later, Ref. [8] determined MacWilliams identities of linear codes with respect to RT metric over that ring.

This paper is devoted to generalizing the results in Ref. [8] to the most general case. In Section 1, we first introduce the concept of RT metric, next we give the definition of the complete ρ weight enumerator over $M_{n\times s}(R)$. In Section 2, we start with the new Gray map, and then the definition of the Lee complete ρ weight enumerator over $M_{n\times s}(R)$ is given. Furthermore, we prove a MacWilliams identity with respect to Lee complete ρ weight enumerator over $M_{n\times s}(R)$, and finally we give an example to illustrate the obtained result. In Section 3, we first define the exact weight enumerators and the exact complete ρ weight enumerators over $M_{n\times s}(R)$, then we prove a MacWilliams identity with respect to exact complete ρ weight enumerators over $M_{n\times s}(R)$. Numerical examples are presented to illustrate the obtained results.

1 Preliminary

Let R denote the commutative ring $\mathbb{F}_l+v\mathbb{F}_l+\cdots+v^{k-1}\mathbb{F}_l=\{a_0+a_1v+\cdots+a_{k-1}v^{k-1}\mid a_0,a_1,\cdots,a_{k-1}\in\mathbb{F}_l\}$, where l is a prime and k is a positive integer, and we denote the set of all $n\times s$ matrices over R by $M_{n\times s}(R)$. Now we define the RT weight of p as follows:

$$\omega_N(p)=\begin{cases}\max\{i: p_i\neq 0\}+1,\ p\neq 0;\\ 0,\ p=0;\end{cases}$$

where $p=(p_0,p_1,\cdots,p_{s-1})\in M_{1\times s}(R)$. Let p, $q\in M_{1\times s}(R)$. The RT distance between p and q is defined as $\rho(p,q)=\omega_N(p-q)$. The RT weight is then extended to $P=(P_1,P_2,\cdots,P_n)^{\mathrm{T}}\in M_{n\times s}(R)$ as $w_N(P)=\sum_{i=1}^{n}w_N(P_i)$, where $P_i=(p_{i,0},p_{i,1},\cdots,p_{i,s-1})\in M_{1\times s}(R)$, $1\leqslant i\leqslant n$. The RT distance between P and Q is $\rho(P,Q)=w_N(P-Q)$, where $P,Q\in M_{n\times s}(R)$. It is obvious that the RT distance is a metric on $M_{n\times s}(R)$. In addition, when $s=1$, there is no difference between RT metric and Hamming metric.

A linear code C over $M_{n\times s}(R)$ is an R-submodule of $M_{n\times s}(R)$. Let $\omega_r(C)$ denote the weight spectrum of C, i.e., $\omega_r(C)=|\{P\in C\mid \omega_N(P)=r\}|$, where $0\leqslant r\leqslant ns$, and the ρ weight enumerator of C is defined as $W_C(z)=\sum_{r=0}^{ns}\omega_r(C)z^r=\sum_{P\in C}z^{\omega_N(P)}$. Let $p=(p_0,p_1,\cdots,p_{s-1})$ and $q=(q_0,q_1,\cdots,q_{s-1})$, where $p,q\in M_{1\times s}(R)$. Then the inner product of p and q is defined as $\langle p,q\rangle=\sum_{i=0}^{s-1}p_iq_{s-1-i}$, and this concept can be extended to the inner product of P and Q as $\langle P,Q\rangle=\sum_{i=1}^{n}\langle P_i,Q_i\rangle$, where $P=(P_1,P_2,\cdots,P_n)^{\mathrm{T}}$, $Q=(Q_1,Q_2,\cdots,Q_n)^{\mathrm{T}}\in M_{n\times s}(R)$, $P_i=(p_{i,0},p_{i,1},\cdots,p_{i,s-1})$, and $Q_i=(q_{i,0},q_{i,1},\cdots,q_{i,s-1})\in M_{1\times s}(R)$, $1\leqslant i\leqslant n$. For $s=1$, by properly interchanging the subscripts, we have $\langle P,Q\rangle=\sum_{i=1}^{n}p_iq_i$, i.e., the usual Euclidean inner product.

The dual of C is defined as $C^{\perp}=\{Q\in M_{n\times s}(R)\mid\langle P,Q\rangle=0,\ \forall P\in C\}$. Then $C^{\perp}$ is also a linear code over $M_{n\times s}(R)$. The ring of $n\times s$ matrices over R can be identified with the ring of $n\times 1$ matrices with polynomial entries. We identify the set of all polynomials of degree at most

$s-1$ over R with $R[x]/(x^s)$. Define a map

$$\Psi: M_{n\times s}(R)\to M_{n\times 1}(R[x]/(x^s)),$$
$$P=(P_1,\cdots,P_n)^{\mathrm T}\mapsto(P_1(x),\cdots,P_n(x))^{\mathrm T},$$

where $P_i=(p_{i,0},p_{i,1},\cdots,p_{i,s-1})\in M_{1\times s}(R)$, $P_i(x)=p_{i,0}+p_{i,1}x+\cdots+p_{i,s-1}x^{s-1}\in R[x]/(x^s)$, $1\leqslant i\leqslant n$ and $P^{\mathrm T}$ is the transpose of P.

Let $p(x)=p_0+p_1x+\cdots+p_{s-1}x^{s-1}\in R[x]/(x^s)$. Let the $e^{\text{th}}(0\leqslant e\leqslant s-1)$ coefficient of $p(x)$ be denoted by $c_e(p(x))$. Then the inner product $\langle p(x),q(x)\rangle$ becomes $\langle p(x),q(x)\rangle=c_{s-1}(p(x)q(x))$. Similarly, suppose $P,Q\in M_{n\times 1}(R[x]/(x^s))$. We define

$$\langle P(x),Q(x)\rangle=\sum_{i=1}^{n}\langle P_i(x),Q_i(x)\rangle=\sum_{i=1}^{n}c_{s-1}(P_i(x)Q_i(x)),$$

where $P(x)=(P_1(x),P_2(x),\cdots,P_n(x))^{\mathrm T}$, $Q(x)=(Q_1(x),Q_2(x),\cdots,Q_n(x))^{\mathrm T}$ and $P_i(x)=p_{i,0}+p_{i,1}x+\cdots+p_{i,s-1}x^{s-1}$, $Q_i(x)=q_{i,0}+q_{i,1}x+\cdots+q_{i,s-1}x^{s-1}\in R[x]/(x^s)$, $1\leqslant i\leqslant n$. When $s=1$, by properly interchanging the subscripts, we have $\langle P(x),Q(x)\rangle=\sum_{i=1}^{n}p_iq_i$. For $0\in R$, the Hamming weight $w(0)$ of the zero element is defined as 0, otherwise 1.

Definition 1.1 Let $Y_{ns}=(y_{1,0},\cdots,y_{1,s-1},\cdots,y_{n,0},\cdots,y_{n,s-1})$, $P=(p_{i,j})_{n\times s}\in M_{n\times s}(R)$, where $1\leqslant i\leqslant n$, $0\leqslant j\leqslant s-1$. Define the complete ρ weight enumerator of C over $M_{n\times s}(R)$ as

$$W_C(Y_{ns})=\sum_{P\in C}y_{1,0}^{w(p_{1,0})}\cdots y_{1,s-1}^{w(p_{1,s-1})}\cdots y_{n,0}^{w(p_{n,0})}\cdots y_{n,s-1}^{w(p_{n,s-1})}.$$

When $n=1$, by properly interchanging the subscripts, we get the definition of complete ρ weight enumerator of C over R^s as

$$W_C(Y)=\sum_{P\in C}y_1^{w(p_0)}y_2^{w(p_1)}\cdots y_s^{w(p_{s-1})},$$

where

$p=(p_0,p_1,\cdots,p_{s-1})\in R^s$, $Y=(y_1,y_2,\cdots,y_s)$.

2 Lee complete ρ weight enumerator

Definition 2.1 Define the Gray map $\Phi: R^n\to\mathbb{F}_l^{kn}$ by $\Phi(a_0+a_1v+\cdots+a_{k-1}v^{k-1})=(a_0,a_1,\cdots,a_{k-2},a_0+a_{k-1})$, $\forall a_0+a_1v+\cdots+a_{k-1}v^{k-1}\in R^n$, where $a_0,a_1,\cdots,a_{k-1}\in\mathbb{F}_l^n$.

Definition 2.2 According to Definition 2.1, for $\alpha\in R$, we have the Lee weight of α as

$$W_L(\alpha)=\begin{cases}0, & \text{if } \alpha=0;\\ 1, & \text{if } \alpha\in U;\\ \vdots & \\ i, & \text{if } \alpha\in V;\\ \vdots & \\ k, & \text{if } \alpha\in W;\end{cases}$$

where $U=U_0\cup U_1$, $U_0=\{a_0+(l-a_0)v^{k-1}\mid a_0\in\mathbb{F}_l^*\}$, $U_1=\{a_jv^j\mid a_j\in\mathbb{F}_l^*,1\leqslant j\leqslant k-1\}$, $V=V_0\cup V_1\cup V_2\cup V_3$, $V_0=\{\sum_{j=1}^{i}a_{k_j}v^{k_j}\mid a_{k_j}\in\mathbb{F}_l^*,1\leqslant k_j\leqslant k-1\}$, $V_1=\{a_0+\sum_{j=1}^{i-2}a_{k_j}v^{k_j}\mid a_0,a_{k_j}\in\mathbb{F}_l^*,1\leqslant k_j\leqslant k-2\}$, $V_2=\{a_0+\sum_{j=1}^{i-2}a_{k_j}v^{k_j}+a_{k-1}v^{k-1}\mid a_0,a_{k-1},a_{k_j}\in\mathbb{F}_l^*,1\leqslant k_j\leqslant k-2\}$, $V_3=\{a_0+\sum_{j=1}^{i-1}a_{k_j}v^{k_j}+(l-a_0)v^{k-1}\mid a_0,a_{k_j}\in\mathbb{F}_l^*,1\leqslant k_j\leqslant k-2\}$, $W=W_0\cup W_1$, $W_0=\{\sum_{j=0}^{k-2}a_jv^j\mid a_j\in\mathbb{F}_l^*\}$, $W_1=\{\sum_{j=0}^{k-1}a_jv^j\mid a_j\in\mathbb{F}_l^*\}$, and $a_0+a_{k-1}\neq 0\pmod l$.

Definition 2.3 Let $Y_{ns}=(y_{1,0},\cdots,y_{1,s-1},\cdots,y_{n,0},\cdots,y_{n,s-1})$, $P=(p_{i,j})_{n\times s}\in M_{n\times s}(R)$, where $1\leqslant i\leqslant n$ and $0\leqslant j\leqslant s-1$. Define the Lee complete ρ weight enumerator of C over $M_{n\times s}(R)$ as

$$\mathrm{Lee}(Y_{ns})=\sum_{P\in C}y_{1,0}^{W_L(p_{1,0})}\cdots y_{1,s-1}^{W_L(p_{1,s-1})}\cdots y_{n,0}^{W_L(p_{n,0})}\cdots y_{n,s-1}^{W_L(p_{n,s-1})}.$$

In particular, when $n=1$, by properly interchanging the subscripts, we have the Lee complete ρ weight enumerator of C over R^s as

$$\mathrm{Lee}(Y)=\sum_{P\in C}y_1^{W_L(p_0)}y_2^{W_L(p_1)}\cdots y_s^{W_L(p_{s-1})}.$$

Definition 2.4 Define a map $\chi: R\to\mathbb{C}^*$, $\chi(a_0+a_1v+\cdots+a_{k-1}v^{k-1})=\xi^{a_{k-1}}$, $\forall a_0+a_1v+\cdots+a_{k-1}v^{k-1}\in R$, where $\xi=\mathrm{e}^{\frac{2\pi i}{l}}$. Then χ is a character of the ring R. The character χ plays a crucial role in the following lemmas.

Similar to the proof of Lemma 2 in Ref.[11], we have the following lemma.

Lemma 2.1 Let C be a linear code over $M_{n\times s}(R)$ and $P(x)$, $Q(x)\in M_{n\times 1}(R[x]/(x^s))$. Then we have

$$\sum_{P(x)\in C}\chi(\langle P(x),Q(x)\rangle)=\begin{cases}0, \text{ if } Q\notin C^{\perp};\\ |C|, \text{ if } Q\in C^{\perp}.\end{cases}$$

Lemma 2.2[10] Let χ be a nontrivial character of G, where G is a finite Abelian group. Then

$$\sum_{b\in G}\chi(b)=0.$$

The following lemma plays an important role in obtaining our main results.

Lemma 2.3 Let β be a fixed element of R. Then

$$\sum_{a\in R}\chi(\beta a)y^{W_L(a)}=[1+(l-1)y]^{k-W_L(\beta)}(1-y)^{W_L(\beta)}.$$

Proof Let $a=a_0+a_1v+\cdots+a_{k-1}v^{k-1}$ and $\beta=\beta_0+\beta_1v+\cdots+\beta_{k-1}v^{k-1}\in R$, where a_i, $\beta_i\in\mathbb{F}_l$ and $0\leqslant i\leqslant k-1$. Moreover, let $W_H(a)$ be the Hamming weight of the element a of $\mathbb{F}_l$, and $(\beta a)_i$ denote the coefficient of v^i of the product βa. Then we have

$$\sum_{a\in R}\chi(\beta a)y^{W_L(a)}=\sum_{a\in R}\prod_{i=0}^{k-1}\chi((\beta a)_iv^i)y^{W_L(a)}=$$
$$\sum_{a\in R}\prod_{i=0}^{k-2}y^{W_H(a_i)}\chi((\beta a)_{k-1}v^{k-1})y^{W_H(a_0+a_{k-1})}=$$
$$\sum_{a_i\in\boldsymbol{F}_l}\prod_{i=0}^{k-2}y^{W_H(a_i)}\prod_{i=0}^{k-1}\xi^{\beta_{k-1-i}a_i+a_{k-1}\beta_{k-1}}y^{W_H(a_0+a_{k-1})}=$$
$$\sum_{a_i\in\boldsymbol{F}_l}\prod_{i=1}^{k-2}y^{W_H(a_i)}\xi^{\beta_{k-1-i}a_i}\sum_{a_i\in\boldsymbol{F}_l}\xi^{a_{k-1}(\beta_0+\beta_{k-1})+a_0\beta_{k-1}}y^{W_H(a_0+a_{k-1})}y^{W_H(a_0)}.$$

① Denote $A=\sum_{a_i\in\boldsymbol{F}_l}\prod_{i=1}^{k-2}\xi^{\beta_{k-1-i}a_i}y^{W_H(a_i)}$, and $\beta'=\beta_1v+\cdots+\beta_{k-2}v^{k-2}$.

When $\beta_{k-1-i}=0$, if $a_i=0$, then $\xi^{\beta_{k-1-i}a_i}y^{W_H(a_i)}=1$, otherwise $\xi^{\beta_{k-1-i}a_i}y^{W_H(a_i)}=y$, so $\sum_{a_i\in\boldsymbol{F}_l}\xi^{\beta_{k-1-i}a_i}y^{W_H(a_i)}=1+(l-1)y$.

When $\beta_{k-1-i}\neq 0$, if $a_i=0$, then $\xi^{\beta_{k-1-i}a_i}y^{W_H(a_i)}=1$, otherwise according to Lemma 2.2, we have $\sum_{a_i\in\boldsymbol{F}_l^*}\chi(a_i)=-1$, so $\sum_{a_i\in\boldsymbol{F}_l}\xi^{\beta_{k-1-i}a_i}y^{W_H(a_i)}=1-y$.

From the definition of the Gray map defined above, we can write A as follows:

$$A=[1+(l-1)y]^{k-2-W_L(\beta')}[1-y]^{W_L(\beta')}.$$

② Denote

$B=\sum_{a_0,a_{k-1}\in\boldsymbol{F}_l}\xi^{a_{k-1}(\beta_0+\beta_{k-1})+a_0\beta_{k-1}}y^{W_H(a_0+a_{k-1})}y^{W_H(a_0)}$ and $\beta''=\beta_0+\beta_{k-1}v^{k-1}$.

When $\beta_0=\beta_{k-1}=0$, i.e., $W_L(\beta'')=0$, we have

$$\sum_{a_0,a_{k-1}\in\boldsymbol{F}_l}y^{WH(a_0+a_{k-1})}y^{W_H(a_0)}=[1+(l-1)y]^2.$$

When $\beta_0=0$, $\beta_{k-1}\neq 0$, or $\beta_0\neq 0$, $\beta_0+\beta_{k-1}=0$, i.e., $W_L(\beta'')=1$, we only consider the first case here, then

$$\sum_{a_0,a_{k-1}\in\boldsymbol{F}_l}\xi^{(a_0+a_{k-1})\beta_{k-1}}y^{WH(a_0+a_{k-1})}y^{WH(a_0)}=$$
$$1-y-y^2+y\cdot\sum_{a_0,a_{k-1}\in\boldsymbol{F}_l^*}\xi^{(a_0+a_{k-1})\beta_{k-1}}y^{WH(a_0+a_{k-1})}=$$
$$1-y-y^2+y[l-1-(l-2)y]=$$
$$[1+(l-1)y](1-y).$$

When $\beta_0\neq 0$, $\beta_{k-1}=0$, or $\beta_0\neq 0$, $\beta_0+\beta_{k-1}\neq 0$, i.e., $W_L(\beta'')=2$, we only consider the first case here, then

$$\sum_{a_0,a_{k-1}\in\boldsymbol{F}_l}\xi^{a_{k-1}\beta_0}y^{W_H(a_0+a_{k-1})}y^{W_H(a_0)}=$$
$$1+(l-1)y^2-y+y\cdot\sum_{a_0,a_{k-1}\in\boldsymbol{F}_l^*}\xi^{a_{k-1}\beta_0}y^{W_H(a_0+a_{k-1})}=$$
$$(1-y)^2.$$

Similarly, we can write B as follows:

$$B=[1+(l-1)y]^{2-W_L(\beta'')}(1-y)^{W_L(\beta'')}.$$

From the above discussion, we have

$$\sum_{a\in R}\chi(\beta a)y^{W_L(a)}=A\cdot B=$$
$$[1+(l-1)y]^{k-(W_L(\beta')+W_L(\beta''))}(1-y)^{W_L(\beta')+W_L(\beta'')}=$$
$$[1+(l-1)y]^{k-W_L(\beta)}(1-y)^{W_L(\beta)}.$$

Thus we complete the proof.

Remark 2.1 There is an alternative form of the above lemma, i.e., if $P(x)=p_0+p_1x+\cdots+p_{n-1}x^{n-1}\in R[x]/(x^n)$. Then

$$\sum_{a\in R}\chi(\langle P(x),ax^i\rangle)y^{W_L(a)}=$$
$$[1+(l-1)y]^{k-W_L(p_{n-i-1})}(1-y)^{W_L(p_{n-i-1})}.$$

In connection with the preceding Lemma 2.1, we mention the following equation which will be useful for the next theorem.

Lemma 2.4 Let C be a linear code over $M_{n\times s}(R)$, f: $M_{n\times 1}(R[x]/(x^s))\rightarrow\mathbb{C}[Y_{ns}]$. Then

$$\sum_{Q(x)\in C^{\perp}}f(Q(x))=\frac{1}{|C|}\sum_{P(x)\in C}\hat{f}(P(x)),$$

where

$$\hat{f}(P(x))=\sum_{Q(x)\in M_{n\times 1}(R[x]/(x^s))}\chi(\langle P(x),Q(x)\rangle)f(Q(x)).$$

Proof　We have

$$\sum_{P(x)\in C}\hat{f}(P(x))=$$
$$\sum_{P(x)\in C}\sum_{Q(x)\in M_{n\times 1}(R[x]/(x^s))}\chi(\langle P(x),Q(x)\rangle)\times f(Q(x))=$$
$$\sum_{P(x)\in C}\sum_{Q(x)\in C^{\perp}}\chi(\langle P(x),Q(x)\rangle)f(Q(x))+$$
$$\sum_{P(x)\in C}\sum_{Q(x)\notin C^{\perp}}\chi(\langle P(x),Q(x)\rangle)f(Q(x))=$$
$$|C|\sum_{Q(x)\in C^{\perp}}f(Q(x)).$$

Thus the lemma is proved.

We are now ready to give one of the most valuable results of this paper.

Theorem 2.1　Let C be a linear code over $M_{n\times s}(R)$, then

$$\sum_{Q(x)\in C^{\perp}}y_{1,0}^{W_L(q_{1,0})}\cdots y_{1,s-1}^{W_L(q_{1,s-1})}\cdots y_{n,0}^{W_L(q_{n,0})}\cdots y_{n,s-1}^{W_L(q_{n,s-1})}=$$
$$\frac{1}{|C|}\sum_{P(x)\in C}\prod_{i=1}^{n}\prod_{j=0}^{s-1}[1+(l-1)y_{i,j}]^k\prod_{r=1}^{n}\prod_{t=0}^{s-1}\left(\frac{1-y_{r,t}}{1+(l-1)y_{r,t}}\right)^{W_L(p_{r,s-1-t})}.$$

Proof　Suppose $f(Q(x))=(f(Q_1(x),\cdots,Q_n(x))^{\mathrm{T}})=\prod_{i=1}^{n}\prod_{j=0}^{s-1}y_{i,j}^{W_L(q_{i,j})}$ in Lemma 2.4. According to Remark 2.1, we have

$$\hat{f}(P(x))=\sum_{Q(x)\in M_{n\times 1}(R[x]/(x^s))}\chi(\langle P(x),Q(x)\rangle)\prod_{i=1}^{n}\prod_{j=0}^{s-1}y_{i,j}^{W_L(q_{i,j})}=$$
$$\prod_{j=0}^{s-1}\sum_{q_{1,j}\in R}\chi(\langle P_1(x),q_{1,j}x^j\rangle)y_{1,j}^{W_L(q_{1,j})}\cdots\prod_{j=0}^{s-1}\sum_{q_{n,j}\in R}\chi(\langle P_n(x),q_{n,j}x^j\rangle)y_{n,j}^{W_L(q_{n,j})}=$$
$$[1+(l-1)y_{1,0}]^{k-W_L(p_{1,s-1})}(1-y_{1,0})^{W_L(p_{1,s-1})}\cdots[1+(l-1)y_{n,s-1}]^{k-W_L(p_{n,0})}(1-y_{n,s-1})^{W_L(p_{n,0})}=$$
$$\prod_{i=1}^{n}\prod_{j=0}^{s-1}[1+(l-1)y_{i,j}]^k\prod_{r=1}^{n}\prod_{t=0}^{s-1}\left(\frac{1-y_{r,t}}{1+(l-1)y_{r,t}}\right)^{W_L(p_{r,s-1-t})}.$$

By substituting it into Lemma 2.4, the results follows.

Corollary 2.1　Let C be a linear code and $p(x)=p_0+p_1x+\cdots+p_{s-1}x^{s-1}$, $q(x)=q_0+q_1x+\cdots+q_{s-1}x^{s-1}\in R[x]/(x^s)$. Then in Theorem 2.1, when $n=1$, by properly interchanging the subscripts, we get

$$\sum_{Q(x)\in C^{\perp}}y_1^{W_L(q_0)}y_2^{W_L(q_1)}\cdots y_s^{W_L(q_{s-1})}=\frac{1}{|C|}\sum_{P(x)\in C}\prod_{i=1}^{s}[1+(l-1)y_i]^k\prod_{r=1}^{s}\left[\frac{1-y_r}{1+(l-1)y_r}\right]^{W_L(p_{k-r})}.$$

Similarly, when $s=1$, by properly interchanging the subscripts, we get

$$\sum_{Q(x)\in C^{-}}y_1^{W_L(q_0)}y_2^{W_L(q_1)}\cdots y_n^{W_L(q_{n-1})}=\frac{1}{|C|}\sum_{P(x)\in C}\prod_{i=1}^{n}[1+(l-1)y_i]^k\prod_{r=1}^{n}\left(\frac{1-y_r}{1+(l-1)y_r}\right)^{W_L(p_{r-1})}.$$

The above equalities are called MacWilliams identities for Lee complete ρ weight enumerator and Lee weight enumerator of a linear code C over R, respectively. We can see that the inner product of the dual code of a linear code in the second equation over R is just the ordinary Euclidean inner product.

Example 2.1　Consider a linear code C over $M_{1\times 2}(R_1)$, where

$$R_1=\mathbb{F}_2+v\mathbb{F}_2+v^2\mathbb{F}_2+v^3\mathbb{F}_2\ (v^4=v),$$

and the code C is generated by the set S as follows:

$$S=\{(1+v+v^2\quad 0),\ (0\quad v+v^2)\}.$$

Then we can easily list all the codewords of C, and according to Definition 2.3, we get

$$\mathrm{Lee}_C(Y)=1+y_1+3y_2^2+3y_1y_2^2+y_1^3+y_1^4+3y_1^3y_2^2+3y_1^4y_2^2.$$

In light of Theorem 2.1, we have

$$\mathrm{Lee}_{C^{-}}(Y)=\frac{1}{16}\sum_{P\in C}\prod_{i=1}^{2}[1+y_i]^4\prod_{r=1}^{2}\left(\frac{1-y_r}{1+y_r}\right)^{W_L(p_{2-r})}=$$
$$1+y_1+3y_2^2+3y_1y_2^2+y_1^3+y_1^4+3y_1^3y_2^2+3y_1^4y_2^2.$$

We can easily check the code C is self-dual, and obviously, the conclusion is correct. Next we give another example in the case of $n=s=2$, and for which the code C is not self-dual.

Example 2.2 Let C be a linear code over $M_{2\times 2}(R_1)$, where R_1 is defined as above, and the code C is generated by the set S as follows:

$$S=\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right\}.$$

By Definition 2.3, we have $\mathrm{Lee}_C(Y_{22})=1+4y_{1,0}+6y_{1,0}^2+4y_{1,0}^3+y_{1,0}^4$. In light of Theorem 2.1, we have

$$\mathrm{Lee}_{C^\perp}(Y_{22})=\frac{1}{16}\sum_{P\in C}\prod_{i=1}^{2}\prod_{j=0}^{1}[1+y_{i,j}]^4\prod_{r=1}^{2}\prod_{t=0}^{1}\left(\frac{1-y_{r,t}}{1+y_{r,t}}\right)^{W_L(p_{r,1-t})}=$$
$$1+4y_{2,1}+6y_{2,1}^2+4y_{2,1}^3+y_{2,1}^4+4y_{2,0}+\cdots+y_{1,0}^4y_{2,0}^4y_{2,1}^4.$$

On the other hand, we know the dual of C is generated by the set as follows:

$$S=\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\},$$

so we can get the Lee complete ρ weight enumerator of the dual of C according to Definition 2.3, and it is obvious that the conclusion is correct.

3 Exact complete ρ weight enumerator

In this section, we first define exact weight and exact complete ρ weight enumerators over $M_{n\times s}(R)$, and then we give a MacWilliams identity with respect to the exact complete ρ weight enumerators over $M_{n\times s}(R)$.

Definition 3.1 We define the exact weight of $\forall a_0+a_1v+\cdots+a_{k-1}v^{k-1}\in R$ as $w_e(a_0+a_1v+\cdots+a_{k-1}v^{k-1})=a_0+a_1l+\cdots+a_{k-1}l^{k-1}$, where a_0, $a_1,\cdots,a_{k-1}\in\mathbb{F}_l$.

Definition 3.2 Let $Y_{ns}=(y_{1,0},\cdots,y_{1,s-1},\cdots,y_{n,0},\cdots,y_{n,s-1})$, $P=(p_{i,j})_{n\times s}\in M_{n\times s}(R)$, where $1\leqslant i\leqslant n$ and $0\leqslant j\leqslant s-1$. Define the exact complete ρ weight enumerator of a code C over $M_{n\times s}(R)$ as

$$E_C(Y_{ns})=\sum_{P\in C}y_{1,0}^{W_e(p_{1,0})}\cdots y_{1,s-1}^{W_e(p_{1,s-1})}\cdots y_{n,0}^{W_e(p_{n,0})}\cdots y_{n,s-1}^{W_e(p_{n,s-1})}.$$

In particular, when $n=1$ in this definition, by properly interchanging the subscripts, we get the exact complete ρ weight enumerator of the code C over R, that is,

$$E_C(Y)=\sum_{P\in C}y_1^{W_e(p_0)}y_2^{W_e(p_1)}\cdots y_s^{W_e(p_{s-1})}.$$

To obtain another important theorem in this paper, we need prove the following lemma.

Lemma 3.1 Let β be a fixed element of R. Then we have

$$\sum_{a\in R}\chi(\beta a)y^{W_e(a)}=\prod_{j=0}^{k-1}\sum_{i=0}^{l-1}[\chi(\beta v^j)y^{lj}]^i.$$

Proof Let $a=a_0+a_1v+\cdots+a_{k-1}v^{k-1}\in R$, according to Definition 3.1, we have

$$\sum_{a\in R}\chi(\beta a)y^{W_e(a)}=\sum_{a_j\in\mathbf{F}_l}\prod_{j=0}^{k-1}\chi(\beta a_jv^j)y^{a_jl^j}=$$
$$\prod_{j=0}^{k-1}\Big(\sum_{a_j\in\mathbf{F}_l}(\chi(\beta v^j)y^{l^j})^{a_j}\Big)=\prod_{j=0}^{k-1}\sum_{i=0}^{l-1}[\chi(\beta v^j)y^{l^j}]^i.$$

Thus the lemma is proved.

Remark 3.1 Similar to Remark 2.1, if $P(x)=p_0+p_1x+\cdots+p_{n-1}x^{n-1}\in R[x]/(x^n)$. Then

$$\sum_{a\in R}\chi(\langle P(x),ax^r\rangle)y^{W_e(a)}=$$
$$\prod_{j=0}^{k-1}\sum_{i=0}^{l-1}[\chi(p_{n-r-1}v^j)y^{l^j}]^i.$$

We are now able to obtain the main characterization theorem.

Theorem 3.1 Let C be a linear code over $M_{n\times s}(R)$. Then

$$\sum_{Q(x)\in C^\perp}y_{1,0}^{W_e(q_{1,0})}\cdots y_{n,s-1}^{W_e(q_{n,s-1})}=\frac{1}{|C|}\sum_{P(x)\in C}\prod_{r=1}^{n}\prod_{t=0}^{s-1}\prod_{j=0}^{k-1}\times$$
$$\sum_{i=0}^{l-1}[\chi(p_{r,s-1-t}v^j)y_{r,t}^{l^j}]^i.$$

Proof Suppose $f(Q(x))=((f(Q_1(x),\cdots,Q_n(x))^{\mathrm{T}})=\prod_{i=1}^{n}\prod_{j=0}^{s-1}y_{i,j}{}^{W_e(q_{i,j})}$ in Lemma 2.4. According to Remark 3.1,

$$\hat{f}(P(x))=\sum_{Q(x)\in M_{n\times 1}(R[x]/(x^s))}\chi(\langle P(x),Q(x)\rangle)\prod_{i=1}^{n}\prod_{j=0}^{s-1}y_{i,j}^{W_e(q_{i,j})}=$$
$$\prod_{j=0}^{s-1}\sum_{q_{1,j}\in R}\chi(\langle P_1(x),q_{1,j}x^j\rangle)y_{1,j}^{W_e(q_{1,j})}\cdots\prod_{j=0}^{s-1}\sum_{q_{n,j}\in R}\chi(\langle P_n(x),q_{n,j}x^j\rangle)y_{n,j}^{W_e(q_{n,j})}=$$
$$\prod_{j=0}^{k-1}\sum_{i=0}^{l-1}[\chi(p_{1,s-1}v^j)y_{1,0}^{l^j}]^i\cdots\prod_{j=0}^{k-1}\sum_{i=0}^{l-1}[\chi(p_{n,0}v^j)y_{n,s-1}^{l^j}]^i=$$
$$\prod_{r=1}^{n}\prod_{t=0}^{s-1}\prod_{j=0}^{k-1}\sum_{i=0}^{l-1}[\chi(p_{r,s-1-t}v^j)y_{r,t}^{l^j}]^i.$$

By substituting it into Lemma 2.4, the result follows.

Corollary 3.1 Let C be a linear code and $p(x)=p_0+p_1x+\cdots+p_{s-1}x^{s-1}$, $q(x)=q_0+q_1x+\cdots+q_{s-1}x^{s-1}\in R[x]/(x^s)$. Then in Theorem 3.1, when $n=1$, by properly interchanging the subscripts, we get

$$\sum_{Q(x)\in C^-}y_1^{W_e(q_0)}y_2^{W_e(q_1)}\cdots y_s^{W_e(q_{s-1})}=$$
$$\frac{1}{|C|}\sum_{P(x)\in C}\prod_{r=1}^{s}\prod_{j=0}^{k-1}\sum_{i=0}^{l-1}[\chi(p_{s-r}v^j)y_r^{l^j}]^i.$$

Similarly, when $s=1$, by properly interchanging the subscripts, we have

$$\sum_{Q(x)\in C^\perp}y_1^{W_e(q_0)}y_2^{W_e(q_1)}\cdots y_n^{W_e(q_{n-1})}=$$
$$\frac{1}{|C|}\sum_{P(x)\in C}\prod_{r=1}^{n}\prod_{j=0}^{k-1}\sum_{i=0}^{l-1}[\chi(p_{r-1}v^j)y_r^{l^j}]^i.$$

The above equalities are called MacWilliams identities with respect to the exact complete ρ weight enumerator and exact weight enumerator of a linear code C over R, respectively. We can see that the inner product of dual code of a linear code in the second equation over R is just the ordinary Euclidean inner product.

Example 3.1 Let C be the linear code introduced in Example 2.1. Then according to Definition 3.2, we have

$$E_C(Y)=1+y_2^6+y_1^7+y_1^9+y_2^{10}+y_2^{12}+$$
$$y_1^7y_2^6+y_1^{14}+y_1^9y_2^6+y_1^7y_2^{10}+y_1^9y_2^{10}+$$
$$y_1^7y_2^{12}+y_1^{14}y_2^6+y_1^9y_2^{12}+y_1^{14}y_2^{10}+y_1^{14}y_2^{12}.$$

In light of Theorem 3.1, we can get

$$E_{C^\perp}(Y)=\frac{1}{16}\sum_{P\in C}\prod_{r=1}^{2}\prod_{j=0}^{3}\sum_{i=0}^{1}[\chi(p_{2-r}v^j)y_r^{lj}]^i=$$
$$1+y_2^6+y_1^7+y_1^9+y_2^{10}+y_2^{12}+y_1^7y_2^6+$$
$$y_1^{14}+y_1^9y_2^6+y_1^7y_2^{10}+y_1^9y_2^{10}+y_1^7y_2^{12}+$$
$$y_1^{14}y_2^6+y_1^9y_2^{12}+y_1^{14}y_2^{10}+y_1^{14}y_2^{12}.$$

Example 3.2 Let C be the linear code introduced in Example 2.2. Then according to Definition 3.2, we have $E_C(Y_{22})=\sum_{i=0}^{15}y_{1,0}^i$. In light of Theorem 3.1, we can get

$$E_{C^\perp}(Y_{22})=\frac{1}{16}\sum_{P\in C}\prod_{r=1}^{2}\prod_{t=0}^{1}\prod_{j=0}^{3}\sum_{i=0}^{1}[\chi(p_{r,1-t}v^j)y_{r,t}^{l^j}]^i=$$
$$\sum_{i=0}^{15}\sum_{j=0}^{15}\sum_{t=0}^{15}y_{1,0}^iy_{2,0}^jy_{2,1}^t.$$

Moreover, the number of codewords of $C^\perp$ in Examples 2.1 and 3.1 is not large, so we can calculate Lee complete ρ weight enumerators and exact complete ρ weight enumerators according to Definitions 2.3 and 3.2. While in Examples 2.2 and 3.2, the code is not self-dual and the number of codewords of $C^\perp$ is sufficiently large, it is efficient to use results above to obtain the Lee complete ρ weight enumerators and the exact complete ρ weight enumerators of the dual code of the linear code C over R.

4 Conclusion

MacWilliams type identities have been the most significant tool available for investigating and calculating weight distributions of linear codes. This paper is devoted to generalizing the results in Ref.[8] to the most general case. Note that we have proved the general cases for Lemmas 2.3 and 3.1, while Ref.[8] only proved the corresponding lemmas (Lemmas 2.1 and 3.1) for the special case. Moreover, Ref.[8] illustrated the main results by considering only the case of $n=1$ for $M_{n\times s}(R)$, the case of $n=1$ corresponding to the usual vectors rather than the matrices, while in this paper, we considered the nontrivial cases for $n=s=2$, which demonstrates the general situation.

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