第48卷第2期 中国科学技术大学学报 Vol.48,No.2
2018年2月 JOURNAL OF UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA Feb. 2018

文章编号:0253-2778(2018)2-0089-08

Marcinkiewicz type complete convergence for weighted sums under sub-linear expectations

YU Donglin, WU Qunying

(*College of Science, Guilin University of Technology, Guilin* 541004, *China*)

Abstract: The complete convergence theorems under sub-linear expectations was studied. As applications, Marcinkiewicz type complete convergence for weighted sums of END random variables under sub-linear expectation with the moment condition of $\hat{\mathbb{E}}(|X|^{\beta})<\infty$, $\beta=\max(\alpha,\gamma)$ for some $0<\alpha\leqslant 2$, $\gamma>0$ and $\alpha\neq\gamma$ has been obtained. The corresponding result of predecessors to END random variables under sub-linear expectations has been generalized and improved.

Key words: sub-linear expectation; complete convergence; limit theorem; END random variables

CLC number: O211.3 **Document code**: A doi:10.3969/j.issn.0253-2778.2018.02.001

2010 Mathematics Subject Classification: 60F15

Citation: YU Donglin, WU Qunying. Marcinkiewicz type complete convergence for weighted sums under sub-linear expectations[J]. Journal of University of Science and Technology of China, 2018,48(2):89-96.
余东林,吴群英. 次线性期望空间下 Marcinkiewicz 型加权和的完全收敛性[J]. 中国科学技术大学学报,2018,48(2):89-96.

次线性期望空间下 Marcinkiewicz 型加权和的完全收敛性

余东林,吴群英

(桂林理工大学理学院,广西桂林 541004)

摘要:研究了满足矩条件为$\hat{\mathbb{E}}(|X|^{\beta})<\infty$, $\beta=\max(\alpha,\gamma)$,其中$0<\alpha\leqslant 2$, $\gamma>0$且$\alpha\neq\gamma$情形下的次线性期望空间中END序列加权和的完全收敛性.对前人工作的相应结果进行了改进,并将其推广到了次线性期望空间下END序列加权和的情形.

关键词:次线性期望空间;完全收敛性;极限定理;END随机变量

0 Introduction

In probability and statistics, classical limit theorems are established under the precondition of additive probabilities and linear expectations. However, in practice, many uncertainty phenomena cannot be modeled using model certainty without the precondition. So Peng[1-3] introduced the concept of sub-linear expectation as an extension of the original probability space and constructed the general theoretical framework of the sub-linear expectation space. Sub-linear expectation has been widely used in many fields such as finance, statistics and measures for

Received: 2017-12-15; **Revised**: 2018-01-19

Foundation item: Supported by National Science Foundation of China (11661029), Program of the Guangxi China Science Foundation (2015GXNSFAA139008)

Biography: YU Donglin, male, born in 1993, master. Research field: Probability limit theory. E-mail: daliyu1st@163.com

Corresponding author: WU Qunying, PhD/Prof. E-mail: wqy666@glut.edu.cn

handling risk measurement and it has many different properties of linear expectation and some interesting challenging questions that have attracted more and more attention from scholars. For instance, Zhang[4-7] demonstrated and obtained the exponential inequalities, Rosenthal's inequalities, Kolmogorov's strong law of larger numbers and Hartman-Wintner's law of iterated logarithm and Wu and Jiang[8] established strong law of large numbers and Chover's law of the iterated logarithm.

For complete convergence, there have been few reports under sub-linear expectations and the aim of this article is to obtain Marcinkiewicz type complete convergence for weighted sums of extended negatively dependent random variables. We first introduce the concept of extended negatively dependent in sub-linear expectation space[4].

Definition 0.1 A sequence of random variables $\{X_n;\ n\geqslant 1\}$ is said to be upper (resp. lower) extended negatively dependent if there is some dominating constant $K\geqslant 1$ such that

$$\hat{\mathbb{E}}\left(\prod_{i=1}^{n}\varphi_i(X_i)\right)\leqslant K\prod_{i=1}^{n}\hat{\mathbb{E}}(\varphi_i(X_i)),\ \forall n\geqslant 2,$$

whenever the non-negative functions $\varphi_i(x)\in C_{b,Lip}(\mathbb{R})$, $i=1,2,\cdots$, are all non-decreasing (resp. all non-increasing). They are called extended negatively dependent if they are both upper extended negatively dependent and lower extended negatively dependent.

What's more, let's recall that the concept of complete convergence was introduced by Hsu and Robbin[9]. A sequence of random variables $\{U_n,\ n\geqslant 1\}$ is said to converge completely to a constant C, if

$$\sum_{n=1}^{\infty}P(|U_n-C|>\epsilon)<\infty \text{ for all } \epsilon>0.$$

Since then, many valuable results have been established, such as Refs.[10-14] and so on.

Cai[11] studied complete convergence for weighted sums of NA random variables and got the main result as follows.

Theorem 0.1 Let $\{X_n;\ n\geqslant 1\}$ be a sequence of NA random variables with identical distribution $\sum_{i=1}^{n}|a_{ni}|^{\alpha}=O(n)$ for $0<\alpha\leqslant 2$. Let $T_n=\sum_{i=1}^{n}a_{ni}X_i$, $n\geqslant 1$, $b_n=n^{1/\alpha}(\ln n)^{1/\gamma}$. $EX_n=0$ when $1<\alpha\leqslant 2$. We assume that for some h, $\gamma>0$, $E\exp(|X|^{\gamma})<\infty$. Then

$$\forall\varepsilon>0,\ \sum_{n=1}^{\infty}n^{-1}P(\max_{1\leqslant j\leqslant n}|T_n|>\varepsilon b_n)<\infty.$$

Inspired by Ref.[11], we extend the results to Marcinkiewicz type complete convergence for weighted sums of extended negatively dependent random variables under sub-linear expectations. It's worth noting that the concept of complete convergence is no longer defined by probability but by capacity under sub-linear expectations. A sequence of random variables $\{X_n;\ n\geqslant 1\}$ is said to completely converge to X, if

$$\sum_{n=1}^{\infty}\mathbb{V}(|X_n-X|\geqslant\varepsilon)<\infty,\ \text{for any } \varepsilon>0.$$

which is denoted by $X_n\xrightarrow{c}X$ as $n\to\infty$. Because of the uncertainty and no additive of expectation and capacity, the commonly used powerful tools and methods of classical linear expectation no longer apply in sub-linear expectation space. It leads to the complete convergence that essentially different from the classical probability space and the study of complete convergence under sub-linear expectations is more difficult.

Throughout this paper, let $\{X_n;\ n\geqslant 1\}$ be a random variable sequence in $(\Omega,\mathscr{H},\hat{\mathbb{E}})$. C will signify a positive constant that may have different values in different places. $a_n\ll b_n$ denote that for a sufficiently large n, there exists $C>0$ such that $a_n\leqslant C\,b_n$ and $I(\cdot)$ denotes an indicator function. It proves convenient to define $\log x=\ln(e\vee x)$, where $\ln x$ denotes the natural logarithm.

Now, we state the main results of this article.

Theorem 0.2 Suppose that $\{X_n,\ n\geqslant 1\}$ is a sequence of upper END random variables, there exist a random variable X and a constant C satisfying

$$\left.\begin{array}{l}\hat{\mathbb{E}}(h(|X_n|)) \leqslant C\,\hat{\mathbb{E}}(h(|X|)),\\ \text{for all } n \geqslant 1,\ 0 \leqslant h \in C_{l,Lip}(\mathbb{R})\end{array}\right\} \quad (1)$$

Let $\beta = \max(\alpha, \gamma)$ for some $0 < \alpha \leqslant 2, \gamma > 0$ and $\alpha \neq \gamma$. Assume that $\{a_{ni};\ 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ is an array of real positive constants satisfying

$$\limsup_{n\to\infty} n^{-1} \sum_{i=1}^{n} a_{ni}^{\beta} < \infty \quad (2)$$

and further assume that

$$\hat{\mathbb{E}}[|X|^{\beta}] < \infty \quad (3)$$

Then for $b_n = n^{1/\alpha} \log^{1/\gamma} n$, we have

$$\sum_{n=1}^{\infty} n^{-1} \mathbb{V}\Big(\sum_{i=1}^{n} a_{ni}(X_i - c_i) > \epsilon b_n\Big) < \infty \quad (4)$$

where $c_i = 0$ if $\alpha \leqslant 1$, and $c_i = \hat{\mathbb{E}} X_i$ if $\alpha > 1$.

Further, if $\{X_n, n \geqslant 1\}$ is extended negatively dependent, then

$$\sum_{n=1}^{\infty} n^{-1} \mathbb{V}\Big(\sum_{i=1}^{n} a_{ni}(X_i - \tilde{c}_i) < -\epsilon b_n\Big) < \infty \quad (5)$$

where $\tilde{c}_i = 0$ if $\alpha \leqslant 1$, and $\tilde{c}_i = \varepsilon X_i$ if $\alpha > 1$.

In particular, if $\{X_n, n \geqslant 1\}$ is extended negatively dependent and $\hat{\mathbb{E}} X_i = \hat{\varepsilon} X_i$ for $\alpha > 1$, then

$$\sum_{n=1}^{\infty} n^{-1} \mathbb{V}\Big(\Big|\sum_{i=1}^{n} a_{ni}(X_i - c_i)\Big| > \epsilon b_n\Big) < \infty \quad (6)$$

Remark 0.1 END random variables include ND random variables in sub-linear expectation space and extended independent random variables are END random variables with $K = 1$ by Ref.[4, Definition 2.3], so for ND random variables and extended independent random variables under sub-linear expectations, Theorem 0.2 also holds.

Remark 0.2 Theorem 0.2 not only generalizes Theorem 0.1 to the case of END and arrays of rowwise END random variables under sub-linear expectations, but also partly improves the corresponding results of Refs.[11,15].

1 Preliminaries

We use the framework and notions of Peng[3]. Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, X_2, \cdots, X_n \in \mathcal{H}$ then $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}_n)$, where $C_{l,Lip}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leqslant c(1 + |x|^m + |y|^m)|x - y|,\ \forall\ x, y \in \mathbb{R}_n,$$

for some $c > 0$, $m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of random variables. In this case we denote $X \in \mathcal{H}$.

Definition 1.1 A sub-linear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\hat{\mathbb{E}}: \mathcal{H} \to \overline{\mathbb{R}}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) monotonicity: If $X \geqslant Y$ then $\hat{\mathbb{E}}X \geqslant \hat{\mathbb{E}}Y$;

(b) constant preserving: $\hat{\mathbb{E}}c = c$;

(c) sub-additivity: $\hat{\mathbb{E}}(X + Y) \leqslant \hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$; whenever $\hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;

(d) positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda \hat{\mathbb{E}}X$, $\lambda \geqslant 0$.

Here $\overline{\mathbb{R}} = [-\infty, \infty]$. The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sub-linear expectation space.

Given a sub-linear expectation $\hat{\mathbb{E}}$, let us denote the conjugate expectation $\hat{\varepsilon}$ of $\hat{\mathbb{E}}$ by

$$\hat{\varepsilon}X := -\hat{\mathbb{E}}(-X),\ \forall X \in \mathcal{H}.$$

From the definition, it is easily shown that for all $X, Y \in \mathcal{H}$

$$\hat{\varepsilon}X \leqslant \hat{\mathbb{E}}X,\ \hat{\mathbb{E}}(X + c) = \hat{\mathbb{E}}X + c,$$

$$|\hat{\mathbb{E}}(X - Y)| \leqslant \hat{\mathbb{E}}|X - Y|,$$

$$\hat{\mathbb{E}}(X - Y) \geqslant \hat{\mathbb{E}}X - \hat{\mathbb{E}}Y.$$

If $\hat{\mathbb{E}}X = \hat{\varepsilon}X$, then $\hat{\mathbb{E}}(X + aY) = \hat{\mathbb{E}}X + a\hat{\mathbb{E}}Y$ for any $a \in \mathbb{R}$.

Next, we consider the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V: \mathcal{G} \to [0,1]$ is called a capacity if

$$V(\varnothing) = 0,\ V(\Omega) = 1,$$

$$V(A) \leqslant V(B) \text{ for } \forall A \subseteq B,\ A, B \in \mathcal{G}.$$

It is called sub-additive if $V(A \cup B) \leqslant V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$. In the sub-linear space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, we denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A):=\inf\{\hat{\mathbb{E}}\xi;\ I(A)\leqslant\xi,\ \xi\in\mathscr{H}\},$$
$$\mathcal{V}(A):=1-\mathbb{V}(A^c),\ \forall A\in\mathscr{F},$$

where $\mathbb{V}(A^c)$ is the complement set of A. By definition of $\mathbb{V}$ and $\mathcal{V}$, it is obvious that $\mathbb{V}$ is sub-additive, and

$$\mathcal{V}\leqslant\mathbb{V},\ \forall A\in\mathscr{F};\ \mathbb{V}(A)=\hat{\mathbb{E}}(I(A)),$$
$$\mathcal{V}(A)=\hat{\varepsilon}(I(A)),\ \text{if } I(A)\in\mathscr{H},$$
$$\hat{\mathbb{E}}f\leqslant\mathbb{V}(A)\leqslant\hat{\mathbb{E}}g,\ \hat{\varepsilon}f\leqslant\mathcal{V}(A)\leqslant\hat{\varepsilon}g,$$
$$\text{if } f\leqslant I(A)\leqslant g,\ f,g\in\mathscr{H} \tag{7}$$

This implies Markov inequality: $\forall X\in\mathscr{H}$,

$$\mathbb{V}(|X|\geqslant x)\leqslant\hat{\mathbb{E}}(|X|^p)/x^p,$$
$$\forall x>0,p>0,$$

from $I(|X|\geqslant x)\leqslant|X|^p/x^p\in\mathscr{H}$. By Ref.[16, Proposition 16], we have Hölder inequality : $\forall X, Y\in\mathscr{H}, p,q>1$ satisfying $p^{-1}+q^{-1}=1$,

$$\hat{\mathbb{E}}(|XY|)\leqslant(\hat{\mathbb{E}}(|X|^p))^{1/p}(\hat{\mathbb{E}}(|Y|^q))^{1/q},$$

particularly, Jensen inequality : $\forall X\in\mathscr{H}$,

$$(\hat{\mathbb{E}}(|X|^r))^{1/r}\leqslant(\hat{\mathbb{E}}(|X|^s))^{1/s}\text{for } 0<r\leqslant s.$$

Lemma 1.1[4, Definiton 2.4] If $\{X_n;\ n\geqslant1\}$ is a sequence of upper (resp. lower) extended negatively dependent random variables and functions $f_1(x), f_2(x),\cdots\in C_{l,Lip}(\mathbb{R})$ are all non-decreasing (resp. all non-increasing), then $\{f_n(X_n);\ n\geqslant1\}$ is also a sequence of upper (resp. lower) extended negatively dependent random variables.

Lemma 1.2 [4, Theorem 3.1] Let $\{X_k;\ k\geqslant1\}$ be a sequence of upper extended negatively dependent random variables in $(\Omega, \mathscr{H}, \hat{\mathbb{E}})$ with $\hat{\mathbb{E}}X_k\leqslant0$. Then for any $p\geqslant2$, there exists a constant $C_p\geqslant1$ such that for all $x>0$ and $0<\delta\leqslant1$,

$$\mathbb{V}(S_n\geqslant x)\leqslant C_p\delta^{-2p}K\frac{\sum_{k=1}^n\hat{\mathbb{E}}|X_k|^p}{x^p}+K\exp\left(-\frac{x^2}{2B_n(1+\delta)}\right),$$

where $B_n=\sum_{k=1}^n\hat{\mathbb{E}}X_k^2$.

2 Proof of main result

Proof of Theorem 0.2 Without loss of generality, we can assume that $\hat{\mathbb{E}}X_n=0$ when $\alpha>1$. We just need to prove (4). Because of considering $\{-X_n, n\geqslant1\}$ instead of $\{X_n, n\geqslant1\}$ in (4), we can obtain (5). Noting that $a_{ni}\geqslant0$, for all $1\leqslant i\leqslant n$, $n\geqslant1$. It follows by (2) that

$$\sum_{i=1}^n a_{ni}^\beta\leqslant Cn.$$

For upper extended negatively dependent random variables $\{X_n;\ n\geqslant1\}$, in order to ensure that the truncated random variables are also upper extended negatively dependent, we need that truncated functions belong to $C_{l,Lip}$ and are non-decreasing. Let

$$f_c(x)=-cI(x<-c)+xI(|x|\leqslant c)+cI(x>c),$$

for any $1\leqslant i\leqslant n$, $n\geqslant1$,

$$Y_i:=f_{b_na_{ni}^{-1}}(X_i)=-\frac{b_n}{a_{ni}}I(a_{ni}X_i<-b_n)+X_iI(|a_{ni}X_i|\leqslant b_n)+\frac{b_n}{a_{ni}}I(a_{ni}X_i>b_n)$$

where $b_n=n^{1/\alpha}\log^{1/\gamma}n$, and

$$T_n:=\sum_{i=1}^n a_{ni}(Y_i-\hat{\mathbb{E}}Y_i).$$

Then $\{Y_i;\ 1\leqslant i\leqslant n,\ n\geqslant1\}$ is also a sequence of upper extended negatively dependent random variables.

It is easily checked that for any $\epsilon>0$,

$$\left(\sum_{i=1}^n a_{ni}X_i>\epsilon b_n\right)\subset\bigcup_{i=1}^n\left(|X_i|>\frac{b_n}{a_{ni}}\right)\cup\left(\sum_{i=1}^n a_{ni}Y_i>\epsilon b_n\right),$$

which implies that

$$\sum_{n=1}^\infty n^{-1}\mathbb{V}\left(\sum_{i=1}^n a_{ni}X_i>\epsilon b_n\right)\leqslant\sum_{n=1}^\infty n^{-1}\sum_{i=1}^n\mathbb{V}\left(|X_i|>\frac{b_n}{a_{ni}}\right)+\sum_{n=1}^\infty n^{-1}\mathbb{V}\left(T_n>\epsilon b_n-\left|\sum_{i=1}^n a_{ni}\hat{\mathbb{E}}Y_i\right|\right):=I_1+I_2 \tag{8}$$

For $2^{-1/\beta}<\mu<1$, let $g(x)$ be an even function and $g(x)\in C_{l,Lip}(\mathbb{R})$ such that $0\leqslant g(x)\leqslant1$ for all x and $g(x)=1$ if $|x|\leqslant\mu$, $g(x)=0$ if $|x|>1$. Then

$$I(|x|\leqslant\mu)\leqslant g(x)\leqslant I(|x|\leqslant1),$$

$$I(|x|>1)\leqslant 1-g(x)\leqslant I(|x|>\mu) \quad (9)$$

To prove (4), it suffices to demonstrate $I_1<\infty$ and $I_2<\infty$. By (2), (3), (9) and Markov's inequality, we can show that

$$I_1=\sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}\mathbb{V}(|a_{ni}X_i|>b_n)\leqslant \sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[1-g\left(\frac{|a_{ni}X|}{b_n}\right)\right]\ll \sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}\frac{a_{ni}^{\beta}}{b_n^{\beta}}\hat{\mathbb{E}}[|X|^{\beta}]\ll \sum_{n=1}^{\infty}\frac{1}{n^{\beta/\alpha}\log^{\beta/\gamma}n}<\infty \quad (10)$$

where $\beta=\max(\alpha,\gamma)$ for some $0<\alpha\leqslant 2$ and $\gamma>0$.

Next, we will illustrate $I_2<\infty$. We firstly show that

$$|b_n^{-1}\sum_{i=1}^{n}a_{ni}\hat{\mathbb{E}}Y_i|\to 0 \text{ , as } n\to\infty \quad (11)$$

For any $r>0$, by the c_r inequality, (1) and (9),

$$|Y_i|^r\ll |X_i|^r I[|a_{ni}X_i|\leqslant b_n]+\left(\frac{b_n}{a_{ni}}\right)^r I[|a_{ni}X_i|>b_n]\leqslant |X_i|^r g\left(\frac{\mu a_{ni}X_i}{b_n}\right)+\left(\frac{b_n}{a_{ni}}\right)^r\left[1-g\left(\frac{a_{ni}X_i}{b_n}\right)\right],$$

$$\hat{\mathbb{E}}|Y_i|^r\ll \hat{\mathbb{E}}\left[|X|^r g\left(\frac{\mu a_{ni}X}{b_n}\right)\right]+\left(\frac{b_n}{a_{ni}}\right)^r\hat{\mathbb{E}}\left[1-g\left(\frac{a_{ni}X}{b_n}\right)\right]\leqslant \hat{\mathbb{E}}\left[|X|^r g\left(\frac{\mu a_{ni}X}{b_n}\right)\right]+\left(\frac{b_n}{a_{ni}}\right)^r\mathbb{V}(|a_{ni}X|>\mu b_n) \quad (12)$$

Case 1: $0<\alpha\leqslant 1$

Since $g\left(\frac{\mu a_{ni}X}{b_n}\right)\leqslant I\left(\frac{|\mu a_{ni}X|}{b_n}\leqslant 1\right)$, we have $\frac{|\mu a_{ni}X|}{b_n}\leqslant\frac{|\mu a_{ni}X|^{\alpha}}{b_n^{\alpha}}\leqslant 1$ and $g\left(\frac{\mu a_{ni}X}{b_n}\right)\leqslant 1$. So, by (2), (3), (9), (12) and Markov's inequality, we have

$$|b_n^{-1}\sum_{i=1}^{n}a_{ni}\hat{\mathbb{E}}Y_i|\leqslant b_n^{-1}\sum_{i=1}^{n}a_{ni}\hat{\mathbb{E}}|Y_i|\ll b_n^{-1}\sum_{i=1}^{n}a_{ni}\hat{\mathbb{E}}|Y|\ll b_n^{-1}\sum_{i=1}^{n}a_{ni}\hat{\mathbb{E}}\left[|X|g\left(\frac{\mu a_{ni}X}{b_n}\right)\right]+\sum_{i=1}^{n}\mathbb{V}(|a_{ni}X|>\mu b_n)\leqslant$$

$$C\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\frac{|\mu a_{ni}X|^{\alpha}}{b_n^{\alpha}}g\left(\frac{\mu a_{ni}X}{b_n}\right)\right]+C\sum_{i=1}^{n}\frac{a_{ni}^{\alpha}}{b_n^{\alpha}}\hat{\mathbb{E}}[|X|^{\alpha}]\leqslant C\sum_{i=1}^{n}\frac{a_{ni}^{\alpha}}{b_n^{\alpha}}\hat{\mathbb{E}}[|X|^{\alpha}]\leqslant C\frac{n}{b_n^{\alpha}}\hat{\mathbb{E}}[|X|^{\alpha}]\leqslant \frac{C}{\log^{\alpha/\gamma}n}\to 0,\text{ as } n\to\infty \quad (13)$$

Case 2: $1<\alpha\leqslant 2$

Noting that $\frac{|a_{ni}X|^{\alpha-1}}{(\mu b_n)^{\alpha-1}}>1$, since

$$1-g\left(\frac{a_{ni}X}{b_n}\right)\leqslant I\left(\frac{|a_{ni}X|}{b_n}>\mu\right).$$

Then by $\hat{\mathbb{E}}X_n=0$, (1)~(3), (9) and Markov's inequality, we have

$$|b_n^{-1}\sum_{i=1}^{n}a_{ni}\hat{\mathbb{E}}Y_i|=b_n^{-1}\sum_{i=1}^{n}a_{ni}|\hat{\mathbb{E}}X_i-\hat{\mathbb{E}}Y_i|\leqslant b_n^{-1}\sum_{i=1}^{n}a_{ni}\hat{\mathbb{E}}[|X_i-Y_i|]=$$

$$b_n^{-1}\sum_{i=1}^{n}a_{ni}\hat{\mathbb{E}}\left[\left|\left(X_i+\frac{b_n}{a_{ni}}\right)I\left[X_i<-\frac{b_n}{a_{ni}}\right]+\left(X_i-\frac{b_n}{a_{ni}}\right)I\left[X_i>\frac{b_n}{a_{ni}}\right]\right|\right]\ll b_n^{-1}\sum_{i=1}^{n}a_{ni}\hat{\mathbb{E}}\left[|X|\left(1-g\left(\frac{a_{ni}X}{b_n}\right)\right)\right]\leqslant$$

$$C\,b_n^{-1}\sum_{i=1}^{n}a_{ni}\frac{\hat{\mathbb{E}}[a_{ni}^{\alpha-1}|X||X|^{\alpha-1}]}{b_n^{\alpha-1}}\leqslant Cb_n^{-1}\sum_{i=1}^{n}\frac{a_{ni}^{\alpha}}{b_n^{\alpha-1}}\hat{\mathbb{E}}[|X|^{\alpha}]\leqslant \frac{C}{\log^{\alpha/\gamma}n}\to 0,\ n\to\infty \quad (14)$$

Combining (13) with (14), (11) holds. Then, for every $\epsilon>0$ and for all n large enough, we can get that

$$|b_n^{-1}\sum_{i=1}^{n}a_{ni}\hat{\mathbb{E}}Y_i|<\epsilon/2$$

which implies that

$$I_2\leqslant\sum_{n=1}^{\infty}n^{-1}\mathbb{V}(T_n>\epsilon b_n/2) \quad (15)$$

Since $\{Y_i-\hat{\mathbb{E}}Y_i,\ i\leqslant n,\ n\geqslant 1\}$ is also upper extended negatively dependent with $\hat{\mathbb{E}}[Y_i-\hat{\mathbb{E}}Y_i]=0$. Choosing p such that $p>\max(\alpha,\gamma,2)$ and $\delta=1$ in Lemma 1.2 for $\{Y_i;1\leqslant i\leqslant n,n\geqslant 1\}$, we obtain for every $\epsilon>0$,

$$I_2\leqslant\sum_{n=1}^{\infty}n^{-1}\mathbb{V}(T_n>\epsilon b_n/2)\ll \sum_{n=1}^{\infty}n^{-1}b_n^{-p}\sum_{i=1}^{n}a_{ni}^p\hat{\mathbb{E}}[|Y_i-\hat{\mathbb{E}}Y_i|^p]+ \sum_{n=1}^{\infty}n^{-1}\exp\left(-\frac{\epsilon^2b_n^2}{4\sum_{i=1}^{n}a_{ni}^2\hat{\mathbb{E}}[(Y_i-\hat{\mathbb{E}}Y_i)^2]}\right):= I_{21}+I_{22} \tag{16}$$

Firstly, we show $I_{21}<\infty$. If $p\geqslant\max(2,\alpha,\gamma)$, note that $\beta=\max(\alpha,\gamma)$. Then by (1)～(3),(9),(12) and Markov's inequality, we have

$$\begin{aligned}I_{21}&=\sum_{n=1}^{\infty}n^{-1}b_n^{-p}\sum_{i=1}^{n}a_{ni}^p\hat{\mathbb{E}}[|Y_i-\hat{\mathbb{E}}Y_i|^p]\ll \sum_{n=1}^{\infty}n^{-1}b_n^{-p}\sum_{i=1}^{n}a_{ni}^p\hat{\mathbb{E}}[|Y_i|^p]\ll\\ &\sum_{n=1}^{\infty}n^{-1}b_n^{-p}\sum_{i=1}^{n}a_{ni}^p\hat{\mathbb{E}}\left[|X|^pg\left(\frac{\mu a_{ni}X}{b_n}\right)\right]+ \sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}\mathbb{V}(|a_{ni}X|>\mu b_n)\leqslant\\ &\sum_{n=1}^{\infty}n^{-1}\mu^{-p}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\frac{|\mu a_{ni}X|^p}{b_n{}^p}g\left(\frac{\mu a_{ni}X}{b_n}\right)\right]+ \sum_{n=1}^{\infty}C\,n^{-1}\sum_{i=1}^{n}\frac{a_{ni}^{\beta}}{b_n^{\beta}}\hat{\mathbb{E}}[|X|^{\beta}]\leqslant\\ &\sum_{n=1}^{\infty}n^{-1}\mu^{-p}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\frac{\mu^{\beta}a_{ni}^{\beta}|X|^{\beta}}{b_n^{\beta}}g\left(\frac{\mu a_{ni}X}{b_n}\right)\right]+ \sum_{n=1}^{\infty}C\,b_n^{-\beta}\hat{\mathbb{E}}[|X|^{\beta}]\leqslant\\ &C\sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}\frac{a_{ni}^{\beta}}{b_n^{\beta}}\hat{\mathbb{E}}[|X|^{\beta}]+ \sum_{n=1}^{\infty}\frac{C}{n^{\beta/\alpha}\log^{\beta/\gamma}n}\hat{\mathbb{E}}[|X|^{\beta}]\leqslant \sum_{n=1}^{\infty}\frac{C}{n^{\beta/\alpha}\log^{\beta/\gamma}n}\end{aligned} \tag{17}$$

If $\alpha>\gamma$, we have $\beta=\alpha$ and

$$I_{21}\ll\sum_{n=1}^{\infty}\frac{1}{n\log^{\alpha/\gamma}n}<\infty.$$

And if $\alpha<\gamma$, then $\beta=\gamma$, $I_{21}\ll\sum_{n=1}^{\infty}\frac{1}{n^{\gamma/\alpha}\log n}<\infty$.

Secondly, we will prove $I_{22}<\infty$. Note that $\lim_{y\to\infty}\frac{2\ln y}{y^{\theta}}=0$ for all $\theta>0$, we can have for any $\theta>0$, there exist a constant $y_0\geqslant 1$ such that for all $y\geqslant y_0$,

$$\frac{2\ln y}{y^{\theta}}\leqslant 1$$

which implies that

$$\exp(y^{\theta})\geqslant\exp(2\ln y)=y^2,$$

$$y\geqslant y_0\Rightarrow\frac{1}{\exp(y^{\theta})}\leqslant\frac{1}{y^2},\ y\geqslant y_0.$$

Then, let $y=\log x$, we have

$$\int_1^{\infty}\frac{1}{x\exp(\log^{\theta}x)}\mathrm{d}x=C\int_1^{\infty}\frac{1}{\exp(\log^{\theta}x)}\mathrm{d}(\log x)= C\int_0^{\infty}\frac{1}{\exp(y^{\theta})}\mathrm{d}y\leqslant\int_0^{y_0}\frac{1}{\exp(y^{\theta})}\mathrm{d}y+\int_{y_0}^{\infty}\frac{1}{y^2}\mathrm{d}y<\infty.$$

So, for any $\theta>0$,

$$\int_1^{\infty}\frac{1}{x\exp(\log^{\theta}x)}\mathrm{d}x<\infty\Leftrightarrow\sum_{n=1}^{\infty}\frac{1}{n\exp(\log^{\theta}n)}<\infty \tag{18}$$

If $\alpha<\gamma\leqslant 2$ or $\gamma<\alpha\leqslant 2$, by (1)～(3),(9),(12) and Markov's inequality, we have

$$\begin{aligned}&b_n^{-2}\sum_{i=1}^{n}a_{ni}^2\hat{\mathbb{E}}[(Y_i-\hat{\mathbb{E}}Y_i)^2]\ll b_n^{-2}\sum_{i=1}^{n}a_{ni}^2\hat{\mathbb{E}}Y_i{}^2\ll b_n^{-2}\sum_{i=1}^{n}a_{ni}^2\hat{\mathbb{E}}\left[X^2g\left(\frac{\mu a_{ni}X}{b_n}\right)\right]+ \sum_{i=1}^{n}\mathbb{V}(|a_{ni}X|>\mu b_n)\leqslant\\ &C\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\frac{\mu^{\beta}a_{ni}^{\beta}|X|^{\beta}}{b_n^{\beta}}g\left(\frac{\mu a_{ni}X}{b_n}\right)\right]+ C\sum_{i=1}^{n}\frac{a_{ni}^{\beta}}{b_n^{\beta}}\hat{\mathbb{E}}[|X|^{\beta}]\leqslant\frac{C}{n^{\beta/\alpha-1}\log^{\beta/\gamma}n}\end{aligned} \tag{19}$$

So, by (18) and (19), we have

$$I_{22}=\sum_{n=1}^{\infty}n^{-1}\exp\left(-\frac{\epsilon^2b_n{}^2}{4\sum_{i=1}^{n}a_{ni}^2\hat{\mathbb{E}}[(Y_i-\hat{\mathbb{E}}Y_i)^2]}\right)\ll \sum_{n=1}^{\infty}\frac{1}{n\exp(n^{\beta/\alpha-1}\log^{\beta/\gamma}n)}\ll \sum_{n=1}^{\infty}\frac{1}{n\exp(\log^{\beta/\gamma}n)}<\infty \tag{20}$$

If $\alpha\leqslant 2<\gamma$ or $\alpha<2\leqslant\gamma$, then $\beta=\gamma$. By (2), the Hölder inequality and the Jessen inequality, we have

$$\sum_{i=1}^{n}a_{ni}^{\alpha}\leqslant\left(\sum_{i=1}^{n}a_{ni}^{\gamma}\right)^{\alpha/\gamma}\left(\sum_{i=1}^{n}1\right)^{1-\alpha/\gamma}\ll n$$

and

$$(\hat{\mathbb{E}}[|X|^{\alpha}])^{1/\alpha}\leqslant(\hat{\mathbb{E}}[|X|^{\gamma}])^{1/\gamma}\ \text{for}\ 0<\alpha\leqslant\gamma.$$

So, by (1), (3), (9), (12) and Markov's inequality,

$$b_n^{-2}\sum_{i=1}^{n}a_{ni}^2\,\hat{\mathbb{E}}[(Y_i-\hat{\mathbb{E}}Y_i)^2]\ll b_n^{-2}\sum_{i=1}^{n}a_{ni}^2\,\hat{\mathbb{E}}Y_i^2\ll \sum_{i=1}^{n}\frac{a_{ni}^2}{b_n^2}\hat{\mathbb{E}}\left[X^2g\left(\frac{\mu a_{ni}X}{b_n}\right)\right]+\sum_{i=1}^{n}\mathbb{V}(|a_{ni}X|>\mu b_n)\leqslant C\sum_{i=1}^{n}\frac{a_{ni}^{\alpha}}{b_n^{\alpha}}\hat{\mathbb{E}}\left[|X|^{\alpha}g\left(\frac{\mu a_{ni}X}{b_n}\right)\right]+\sum_{i=1}^{n}\frac{a_{ni}^{\gamma}}{b_n^{\gamma}}\hat{\mathbb{E}}[|X|^{\gamma}]\leqslant \frac{C}{\log^{\alpha/\gamma}n}\hat{\mathbb{E}}[|X|^{\alpha}]+\frac{Cn}{n^{\gamma/\alpha}\log n}\hat{\mathbb{E}}[|X|^{\gamma}]\leqslant \frac{C}{\log^{\alpha/\gamma}n}+\frac{C}{n^{\gamma/\alpha-1}\log n} \tag{21}$$

By (18) and (21), we have

$$I_{22}=\sum_{n=1}^{\infty}n^{-1}\exp\left(-\frac{\epsilon^2b_n^2}{4\sum_{i=1}^{n}a_{ni}^2\,\hat{\mathbb{E}}[(Y_i-\hat{\mathbb{E}}Y_i)^2]}\right)\ll \sum_{n=1}^{\infty}n^{-1}\exp\left(-\frac{n^{\gamma/\alpha-1}\log^{1+\alpha/\gamma}n}{n^{\gamma/\alpha-1}\log n+\log^{\alpha/\gamma}n}\right)\ll \sum_{n=1}^{\infty}\frac{1}{n\exp(\log^{\alpha/\gamma}n)}<\infty \tag{22}$$

Together with (10), (11) and (16), (18) holds. This completes the proof of Theorem 0.2.

Corollary 2.1 Suppose that $0<\beta\leqslant 2$ and $\{X_n, n\geqslant 1\}$ is a sequence of upper END random variables, there exist a random variable X and a constant C satisfying (1). Assume $\{a_{ni}; 1\leqslant i\leqslant n, n\geqslant 1\}$ is an array of real positive constants such that (2) holds, and further assume that

$$\hat{\mathbb{E}}[|X|^{\beta}]<\infty \tag{23}$$

Then for $b_n=(n\log^{1+\delta}n)^{1/\beta}, \delta>0$, we have

$$\sum_{n=1}^{\infty}n^{-1}\mathbb{V}\left(\sum_{i=1}^{n}a_{ni}(X_i-c_i)>\epsilon b_n\right)<\infty \tag{24}$$

where $c_i=0$ if $\beta\leqslant 1$, and $c_i=\hat{\mathbb{E}}X_i$ if $\beta>1$.

Further, if $\{X_n, n\geqslant 1\}$ is extended negatively dependent, then

$$\sum_{n=1}^{\infty}n^{-1}\mathbb{V}\left(\sum_{i=1}^{n}a_{ni}(X_i-\tilde{c}_i)<-\epsilon b_n\right)<\infty \tag{25}$$

where $\tilde{c}_i=0$ if $\beta\leqslant 1$, and $\tilde{c}_i=\hat{\varepsilon}X_i$ if $\beta>1$.

In particular, if $\{X_n, n\geqslant 1\}$ is extended negatively dependent and $\hat{\mathbb{E}}X_i=\hat{\varepsilon}X_i$ for $\beta>1$, then

$$\sum_{n=1}^{\infty}n^{-1}\mathbb{V}\left(\left|\sum_{i=1}^{n}a_{ni}(X_i-c_i)\right|>\epsilon b_n\right)<\infty \tag{26}$$

Proof We use the same notations as those in Theorem 0.2 and the proof is similar to that of Theorem 0.2. Without loss of generality, we can assume that $\hat{\mathbb{E}}X_n=0$ when $\beta>1$. To prove (24), it suffices to demonstrate $I_1<\infty$ and $I_2<\infty$. Note that $b_n=n^{1/\alpha}\log^{1/\gamma}n$ in Theorem 0.2, if $\beta=\alpha=\gamma$, we can not get $I_1<\infty$ and $I_{21}<\infty$. Therefore, we can modify and improve the condition of $b_n=n^{1/\alpha}\log^{1/\gamma}n$ to $b_n=(n\log^{1+\delta}n)^{1/\beta}, \delta>0$ and we just need to prove $I_1<\infty$ and $I_{21}<\infty$.

By (2), (23), (9) and Markov's inequality, we can show that

$$I_1=\sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}\mathbb{V}(|a_{ni}X_i|>b_n)\leqslant \sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[1-g\left(\frac{|a_{ni}X|}{b_n}\right)\right]=\sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}\mathbb{V}(|a_{ni}X|>\mu b_n)\ll \sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}\frac{a_{ni}^{\beta}}{b_n^{\beta}}\hat{\mathbb{E}}[|X|^{\beta}]\ll \sum_{n=1}^{\infty}\frac{1}{n\log^{1+\delta}n}<\infty \tag{27}$$

where $0<\beta\leqslant 2$. And by (1), (2), (9), (12), (23) and Markov's inequality, we have

$$I_{21}=\sum_{n=1}^{\infty}n^{-1}b_n^{-p}\sum_{i=1}^{n}a_{ni}^{p}\,\hat{\mathbb{E}}[|Y_i-\hat{\mathbb{E}}Y_i|^{p}]\ll \sum_{n=1}^{\infty}n^{-1}b_n^{-p}\sum_{i=1}^{n}a_{ni}^{p}\,\hat{\mathbb{E}}[|Y_i|^{p}]\ll \sum_{n=1}^{\infty}n^{-1}b_n^{-p}\sum_{i=1}^{n}a_{ni}^{p}\hat{\mathbb{E}}\left[|X|^{p}g\left(\frac{\mu a_{ni}X}{b_n}\right)\right]+\sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}\mathbb{V}(|a_{ni}X|>\mu b_n)\leqslant \sum_{n=1}^{\infty}n^{-1}\mu^{-p}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\frac{\mu^{\beta}a_{ni}^{\beta}|X|^{\beta}}{b_n^{\beta}}g\left(\frac{\mu a_{ni}X}{b_n}\right)\right]+\sum_{n=1}^{\infty}C\,b_n^{-\beta}\hat{\mathbb{E}}[|X|^{\beta}]\leqslant$$

$$C\sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}\frac{a_{ni}^{\beta}}{b_{n}^{\beta}}\hat{\mathbb{E}}[|X|^{\beta}]\leqslant$$
$$\sum_{n=1}^{\infty}\frac{C}{n\log^{1+\delta}n}<\infty \tag{28}$$

This completes the proof of Corollary (2.1).

Acknowledgements The authors are very grateful to the referees and the editors for their valuable comments and helpful suggestions that improved the clarity and readability of the paper.

References

[1] PENG S. G-expectation, g-brownian motion and related stochastic calculus of ito type[J]. Stochastic Analysis & Applications, 2006, 2(4),541-567.

[2] PENG S. Multi-dimensional g-brownian motion and related stochastic calculus under g-expectation [J]. Stochastic Processes & Their Applications, 2008, 118 (12): 2223-2253.

[3] PENG S. A new central limit theorem under sublinear expectations [J]. Mathematics, 2008, 53 (8): 1989-1994.

[4] ZHANG L X. Strong limit theorems for extended independent and extended negatively dependent random variables under non-linear expectations [DB/OL]. arXiv.ORG: 1608.00710, 2016.

[5] ZHANG L X. Rosenthals inequalities for independent and negatively dependent random variables under sub-linear expectations with applications[J]. Science China Mathematics, 2016, 59(4): 751-768.

[6] ZHANG L X. Self-normalized moderate deviation and laws of the iterated logarithm under G-Expectation[J]. Communications in Mathematics and Statistics, 2016, 4(2): 229-263.

[7] ZHANG L X. Exponential inequalities under the sub-linear expectations with applications to laws of the iterated[J]. Science China Mathematics, 2016, 59 (12): 2503-2526.

[8] WU Q, JIANG Y. Strong law of large numbers and Chover's law of the iterated logarithm under sub-linear expectations[J]. Journal of Mathematical Analysis and Applications, 2018,460(1):252-270.

[9] HSU P L, ROBBINS H. Complete convergence and the law of large numbers [J]. Proceedings of the National Academy of Sciences of the United States of America, 1947, 33(2): 25-31.

[10] SUNG S H. On the strong convergence for weighted sums of random variables[J]. Statistical Papers, 2011, 52(2): 447-454.

[11] CAI G H. Strong laws for weighted sums of NA random variables[J]. Metrika, 2008, 68(3): 323-331.

[12] WU Q, JIANG Y. Complete convergence and complete moment convergence for negatively associated sequences of random variables [J]. Journal of Inequalities & Applications, 2016, 2016:157.

[13] HUANG H, PENG J, WU X, et al. Complete convergence and complete moment convergence for arrays of rowwise ANA random variables[J]. Journal of Inequalities & Applications,2016, 2016:72.

[14] SHEN A, XUE M, WANG W. Complete convergence for weighted sums of extended negatively dependent random variables [J]. Communications in Statistics: Theory and Methods,2017, 46(3): 1433-1444.

[15] HUANG H, WANG D. A note on the strong limit theorem for weighted sums of sequences of negatively dependent random variables[J]. Journal of Inequalities & Applications, 2012, 2012:233.

[16] DENIS L, HU M, PENG S. Function spaces and capacity related to a sublinear expectation: Application to g-brownian motion paths [J]. Potential Analysis, 2011, 34(2): 139-161.