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A class of projectively flat and dually flat Finsler metrics

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Abstract: Finsler geometry is just Riemannian geometry without quadratic restriction, and the projectively flat and dually flat Finsler metrics are very important in Finsler geometry. Here a class of 3-parameter of Finsler metrics were studied, and the necessary and sufficient conditions for the Finsler metrics to be projectively flat and dually flat were obtained.

Key words: Finsler metrics; spherically symmetric; projectively flat; dually flat

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一类射影平坦和对偶平坦的 Finsler 度量

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摘要:Finsler 几何是没有二次型限制的黎曼几何,Finsler 几何中两个非常重要的问题是射影平坦和对偶平坦的 Finsler 度量.主要研究了一类含有 3 个参数的 Finsler 度量,得到了其为射影平坦和对偶平坦的充要条件.

关键词:Finsler 度量;球对称;射影平坦;对偶平坦

0 Introduction

Finsler metric defined on an open domain in $\mathbb{R}^n$ is said to be projectively flat if its geodesics are straight lines[1,2]. The regular case of Hilbert's 4th problem[3] is to study and characterize the projectively flat metrics on an open domain $U\subset\mathbb{R}^n$. As an important case, projectively flat Riemannian metrics are with constant sectional curvature and vice versa by Beltrami's theorem [4]. Beltrami's theorem tells us that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. It is known that the Beltrami theorem is no longer true in Finsler geometry. So, it is important to construct non-Riemannnian projectively flat Finsler metric with constant flag curvature and some famous metrics are found in the history. For example, the well-

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known Berwald's metric $F=F(x,y)$ on a strongly convex domain in $\mathbb{R}^n$ is projectively flat with the constant flag curvature $K=0$. When the domain is the unit ball $\mathbb{B}^n(1)\subset\mathbb{R}^n$, the Berwald metric is given by

$$F=\frac{(\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}+\langle x,y\rangle)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}} \tag{1}$$

where $|\cdot|$ and $\langle,\rangle$ are the standard Euclidean norm and inner product in $\mathbb{R}^n$, respectively.

Recently, Huang and Mo discussed a class of interesting Finsler metrics [5,6] satisfying

$$F(Ax,Ay)=F(x,y) \tag{2}$$

for all $A\in O(n)$. A Finsler metric F is said to be spherically symmetric if F satisfies (2) for all $A\in O(n)$. Besides, it was pointed out in Ref.[7] that a Finsler metric F on $\mathbb{B}^n(r)$ is spherically symmetrical if and only if there is a function $\phi:[0,r)\times\mathbb{R}\to\mathbb{R}$ such that

$$F(x,y)=|y|\phi\left(|x|,\frac{\langle x,y\rangle}{|y|}\right) \tag{3}$$

where $(x,y)\in T\mathbb{R}^n(r)\backslash\{0\}$. Moreover, the spherically symmetric Finsler metric of the form (3) can be rewritten as the following form [8]:

$$F=|y|\sqrt{f(\frac{|x|^2}{2},\frac{\langle x,y\rangle}{|y|})}.$$

In this paper, we construct a new class of three-parameter Finsler metrics. Let ζ be an arbitrary constant and $\Omega=\mathbb{B}^n(r)\subset\mathbb{R}^n$ where $r=\frac{1}{\sqrt{-\zeta}}$ if $\zeta<0$ and $r=+\infty$ if $\zeta\geqslant 0$. Define $F=F(x,y):T\Omega\to[0,+\infty]$ by

$$F(x,y)=\frac{(\sqrt{\varepsilon(1+\zeta|x|^2)|y|^2+\kappa^2\langle x,y\rangle^2}+\kappa\langle x,y\rangle)^2}{(1+\zeta|x|^2)^2\sqrt{\varepsilon(1+\zeta|x|^2)|y|^2+\kappa^2\langle x,y\rangle^2}} \tag{4}$$

and define $\widetilde{F}=\widetilde{F}(x,y):T\Omega\to[0,+\infty]$ by

$$\widetilde{F}=\sqrt{\frac{\kappa(\sqrt{\varepsilon(1+\zeta|x|^2)|y|^2+\kappa^2\langle x,y\rangle^2}+\kappa\langle x,y\rangle)^3}{(1+\zeta|x|^2)^3\sqrt{\varepsilon(1+\zeta|x|^2)|y|^2+\kappa^2\langle x,y\rangle^2}}} \tag{5}$$

where ε is an arbitrary positive constant, κ is an arbitrary constant.

Remark 0.1 When $\kappa^2=1$, $\zeta=-1$, $\varepsilon=1$, (4) is due to Berwald metric.

Through a simple deformation, F in (4) can be expressed as the following form:

$$F=|y|\frac{(\sqrt{\varepsilon(1+\zeta|x|^2)+\frac{\kappa^2\langle x,y\rangle^2}{|y|^2}}+\frac{\kappa\langle x,y\rangle}{|y|})^2}{(1+\zeta|x|^2)^2\sqrt{\varepsilon(1+\zeta|x|^2)+\frac{\kappa^2\langle x,y\rangle^2}{|y|^2}}}=|y|\phi(|x|,\frac{\langle x,y\rangle}{|y|})=\alpha\phi(b,s) \tag{6}$$

where

$$\phi=\phi(b,s)=\frac{(\sqrt{\varepsilon(1+\zeta|x|^2)+\frac{\kappa^2\langle x,y\rangle^2}{|y|^2}}+\frac{\kappa\langle x,y\rangle}{|y|})^2}{(1+\zeta|x|^2)^2\sqrt{\varepsilon(1+\zeta|x|^2)+\frac{\kappa^2\langle x,y\rangle^2}{|y|^2}}},$$

$$\alpha=|y|,\ \beta=\langle x,y\rangle,\ b=\|\beta\|_\alpha=|x|,\ s=\frac{\beta}{\alpha}.$$

And $\widetilde{F}$ in Ref.(5) can be expressed as the following form:

$$\widetilde{F}=|y|\sqrt{\frac{\kappa(\sqrt{\varepsilon(1+\zeta|x|^2)+\frac{\kappa^2\langle x,y\rangle^2}{|y|^2}}+\frac{\kappa\langle x,y\rangle}{|y|})^3}{\varepsilon(1+\zeta|x|^2)^3\sqrt{\varepsilon(1+\zeta|x|^2)+\frac{\kappa^2\langle x,y\rangle^2}{|y|^2}}}}=|y|\sqrt{f(\frac{|x|^2}{2},\frac{\langle x,y\rangle}{|y|})}=|y|\sqrt{f(t,s)} \tag{7}$$

where

$$f=f(t,s)=\frac{\kappa(\sqrt{\varepsilon(1+\zeta|x|^2)+\frac{\kappa^2\langle x,y\rangle^2}{|y|^2}}+\frac{\kappa\langle x,y\rangle}{|y|})^3}{\varepsilon(1+\zeta|x|^2)^3\sqrt{\varepsilon(1+\zeta|x|^2)+\frac{\kappa^2\langle x,y\rangle^2}{|y|^2}}},$$

$$t=\frac{|x|^2}{2},\ s=\frac{\langle x,y\rangle}{y}.$$

Clearly, F and $\widetilde{F}$ are spherically symmetric metrics. By using the necessary and sufficient conditions for the spherically symmetric to be projectively flat and dually flat, we obtain the following results.

Theorem 0.1 Let $F = F(x, y): T\Omega \to [0,\infty)$ be a function given by (4). Then, it has the following properties:

① F is Finsler metric.

② F is projectively flat Finsler metric if and only if $\kappa^2+\varepsilon\zeta=0$.

③ When $\kappa^2+\varepsilon\zeta=0$, F is projectively flat Finsler metric. Its projective factor P is given by

$$P=\frac{\kappa(\sqrt{\varepsilon(1+\zeta|x|^2)|y|^2+\kappa^2\langle x,y\rangle^2}+\kappa\langle x,y\rangle)}{\varepsilon(1+\zeta|x|^2)}$$

and its flag curvature is

$$K=0.$$

Theorem 0.2 Let $\widetilde{F}=\widetilde{F}(x,y): T\Omega\to[0,\infty)$ be a function given by (5), then $\widetilde{F}$ is a locally dual Finsler metric if and only if $\kappa^2+\varepsilon\zeta=0$.

1 Preliminaries

A Minkowski norm $\Psi(y)$ on a vector space V is a C^∞ function on $V_1\setminus\{0\}$ with the following properties:

① $\Psi(y)\geqslant 0$ and $\Psi(y)=0$ if and only if $y=0$;

② $\Psi(y)$ is positively homogeneous function of degree one, i.e., $\Psi(ty)=t\Psi(y)$, $t\geqslant 0$;

③ $\Psi(y)$ is strongly convex, i.e., for any $y\neq 0$, the matrix $g_{ij}(x,y):=\frac{1}{2}[F^2]_{y^iy^j}(x,y)$ is positively definite.

A Finsler metric F on a manifold M is C^∞ function on $TM\setminus\{0\}$ such that $F_x:=F|_{T_xM}$ is a Minkowski norm on T_xM for any $x\in M$. The fundamental tensor $g_{ij}(x,y):=\frac{1}{2}[F^2]_{y^iy^j}(x,y)$ is positively definite. If $g_{ij}(x,y)=g_{ij}(x)$, F is a Riemannian metric. If $g_{ij}(x,y)=g_{ij}(y)$, F is a Minkowski metric. If all geodesics are straight lines, F is projectively flat. This is equivalent to $G^i=P(x,y)y^i$ being geodesic coefficients of F, and G^i are given by

$$G^i=\frac{g^{il}}{4}([F^2]_{x^my^l}y^m-[F^2]_{x^l}).$$

For each tangent plane $\Pi\subset T_xM$ and $y\in\Pi$, the flag curvature of (Π,y) is defined by

$$K(\Pi,y)=\frac{g_{im}R^i_ku^ku^m}{F^2g_{ij}u^iu^j-[g_{ij}y^iu^j]^2},$$

where $\Pi=\operatorname{span}\{y,u\}$, and

$$R^i_k=2\frac{\partial G^i}{\partial x^k}-y^i\frac{\partial^2G^i}{\partial y^i\partial y^k}+2G^j\frac{\partial^2G^i}{\partial y^j\partial y^k}-\frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

We need the following lemmas for later use.

Lemma 1.1[9] Let M be an n-dimensional mainfold. $F=\alpha\phi(b,\frac{\beta}{\alpha})$ is a Finsler metric on M for any Riemannian metric α and 1-form β with $\|\beta\|_\alpha<b_0$ if and only if $\phi=\phi(b,s)$ is a positive C^∞ function satisfying

$$\phi-s\phi_2>0,\ \phi-s\phi_2+(b^2-s^2)\phi_{22}>0 \tag{8}$$

when $n\geqslant 3$ or

$$\phi-s\phi_2+(b^2-s^2)\phi_{22}>0$$

when $n=2$, where s and b are arbitrary numbers with $|s|\leqslant b<b_0$. ϕ_2 means derivation of ϕ with respect to the second variable s.

Lemma 1.2[7] Let $F=|y|\phi(|x|,\frac{\langle x,y\rangle}{|y|})$ be a spherically symmetric Finlser metric on an $\mathbb{B}^n(r)$. Then $F=F(x,y)$ is projectively flat if and only if $\phi=\phi(b,s)$ satisfies

$$s\phi_{bs}+b\phi_{ss}-\phi_b=0 \tag{9}$$

where $b=\|\beta\|_\alpha$, $s=\frac{\beta}{\alpha}$, $\phi=\phi(|x|,\frac{\langle x,y\rangle}{|y|})$. ϕ_b means derivation of ϕ with respect to the first variable b.

If F is projectively flat, its flag curvature K is given by Ref.[10]:

$$K=\frac{P^2-P_{x^k}y^k}{F^2} \tag{10}$$

where the projective factor is

$$P=\frac{F_{x^m}y^m}{2F} \tag{11}$$

In Ref.[11], Shen proved that a Finsler metric

$F=F(x,y)$ on an open subset $U\subset\mathbb{R}^n$ is dually flat if and only if satisfies the following PDE

$$[F^2]_{x^k y^l}y^k=2[F^2]_{x^l} \tag{12}$$

Recently, Huang and Mo provided a necessary and sufficient condition for the spherically symmetric metric to be dually flat.

Lemma 1.3[12] Let $F=|y|\sqrt{f(\frac{|x|^2}{2}),\frac{\langle x,y\rangle}{|y|}}$ be a spherically symmetric Finlser metric on $T\mathbb{R}^n(r)$. Then F is a solution of the dually flat equaition (12) if and only if

$$sf_{bs}+f_{ss}-2f_t=0 \tag{13}$$

where $f=f(t,s)$, $t=\frac{|x|^2}{2}$, $s=\frac{\langle x,y\rangle}{|y|}$.

2 Proof of Theorem 0.1

① Firstly, we prove that F is a Finsler metric.

By Eq.(6), we have

$$F=|y|\frac{\left(\sqrt{\varepsilon(1+\zeta|x|^2)+\frac{\kappa^2\langle x,y\rangle^2}{|y|^2}}+\frac{\kappa\langle x,y\rangle}{|y|}\right)^2}{(1+\zeta|x|^2)^2\sqrt{\varepsilon(1+\zeta|x|^2)+\frac{\kappa^2\langle x,y\rangle^2}{|y|^2}}}.$$

Let $\alpha=|y|$, $\beta=\langle x,y\rangle$, $b=\|\beta\|_\alpha=|x|$, so F can be expressed as

$$F=\alpha\frac{(\sqrt{\kappa^2s^2+\varepsilon(1+\zeta b^2)}+\kappa s)^2}{(1+\zeta b^2)^2\sqrt{\kappa^2s^2+\varepsilon(1+\zeta b^2)}}=\alpha\phi(b,s) \tag{14}$$

Set

$$\triangle=\kappa^2s^2+\varepsilon(1+\zeta b^2) \tag{15}$$

After substituting (15) into (14), we have

$$\phi=\phi(b,s)=\frac{(\sqrt{\triangle}+\kappa s)^2}{(1+\zeta b^2)^2\sqrt{\triangle}}=\frac{2\sqrt{\triangle}}{(1+\zeta b^2)^2}+\frac{2\kappa s}{(1+\zeta b^2)^2}-\frac{\varepsilon\triangle^{-\frac{1}{2}}}{1+\zeta b^2} \tag{16}$$

Differentiating ϕ with respect to s, we have

$$\phi_s=\frac{2\triangle^{-\frac{1}{2}}\kappa^2s}{(1+\zeta b^2)^2}+\frac{2\kappa}{(1+\zeta b^2)^2}+\frac{\varepsilon\triangle^{-\frac{3}{2}}\kappa^2s}{1+\zeta b^2} \tag{17}$$

it follows that

$$\phi_{ss}=\frac{2\kappa^2\triangle^{-\frac{1}{2}}-2\triangle^{-\frac{3}{2}}\kappa^4s^2}{(1+\zeta b^2)^2}+\frac{\varepsilon\triangle^{-\frac{3}{2}}\kappa^2-3\varepsilon\kappa^4s^2\triangle^{-\frac{5}{2}}}{1+\zeta b^2}=$$

$$\frac{2\kappa^2\triangle^{-\frac{1}{2}}-2\triangle^{-\frac{3}{2}}\kappa^2[\triangle-\varepsilon(1+\zeta b^2)]}{(1+\zeta b^2)^2}+\frac{\varepsilon\triangle^{-\frac{3}{2}}\kappa^2-3\varepsilon\kappa^2[\triangle-\varepsilon(1+\zeta b^2)]\triangle^{-\frac{5}{2}}}{1+\zeta b^2}=$$

$$3\varepsilon^2\kappa^2\triangle^{-\frac{5}{2}} \tag{18}$$

Using Eqs.(16) and (17), we have

$$\phi-s\phi_s=\frac{2\sqrt{\triangle}}{(1+\zeta b^2)^2}+\frac{2\kappa s}{(1+\zeta b^2)^2}-\frac{\varepsilon\triangle^{-\frac{1}{2}}}{1+\zeta b^2}-s\left[\frac{2\triangle^{-\frac{1}{2}}\kappa^2s}{(1+\zeta b^2)^2}+\frac{2\kappa}{(1+\zeta b^2)^2}+\frac{\varepsilon\triangle^{-\frac{3}{2}}\kappa^2s}{1+\zeta b^2}\right]$$

$$=\frac{2\sqrt{\triangle}+2\kappa s}{(1+\zeta b^2)^2}-\frac{\varepsilon\triangle^{-\frac{1}{2}}}{1+\zeta b^2}-\left[\frac{2\sqrt{\triangle}}{(1+\zeta b^2)^2}-\frac{2\varepsilon\triangle^{-\frac{1}{2}}}{1+\zeta b^2}+\frac{2\kappa s}{(1+\zeta b^2)^2}+\frac{\varepsilon\triangle^{-\frac{1}{2}}\kappa^2s}{1+\zeta b^2}-\varepsilon^2\triangle^{-\frac{3}{2}}\right]=\varepsilon^2\triangle^{-\frac{3}{2}}>0 \tag{19}$$

Combing Eqs.(18) and (19) gives

$$\phi-s\phi_s+(b^2-s^2)\phi_{ss}=\varepsilon^2\triangle^{-\frac{3}{2}}+3\varepsilon^2\kappa^2\triangle^{-\frac{5}{2}}(b^2-s^2)>0.$$

Then, according to Lemma 1.1, we know F is a Finsler metric.

② In this part, we will prove that F is a projectively flat Finsler metric if and only if $\kappa^2+\varepsilon\zeta=0$.

From Eq.(16) we have

$$\phi_b=\frac{4\triangle^{-\frac{1}{2}}\varepsilon\zeta b}{(1+\zeta b^2)^2}-\frac{8\sqrt{\triangle}\zeta b+8\kappa\zeta sb}{(1+\zeta b^2)^3}+\frac{\triangle^{-\frac{3}{2}}\varepsilon^2\zeta b}{(1+\zeta b^2)} \tag{20}$$

$$\phi_{bs}=\frac{-4\triangle^{-\frac{3}{2}}\varepsilon\zeta b\kappa^2s}{(1+\zeta b^2)^2}-\frac{8\triangle^{-\frac{1}{2}}\zeta b\kappa^2s+8\kappa\zeta b}{(1+\zeta b^2)^3}-\frac{\triangle^{-\frac{5}{2}}\varepsilon^2\zeta b\kappa^2s}{(1+\zeta b^2)} \tag{21}$$

Using Eqs.(18),(20) and (21), we have

$$s\phi_{bs}+b\phi_{ss}-\phi_b=\frac{-4\triangle^{-\frac{3}{2}}\varepsilon\zeta b\kappa^2s^2}{(1+\zeta b^2)^2}-\frac{8\triangle^{-\frac{1}{2}}\zeta b\kappa^2s^2+8\kappa s\zeta b}{(1+\zeta b^2)^3}-\frac{\triangle^{-\frac{5}{2}}\varepsilon^2\zeta b\kappa^2s^2}{(1+\zeta b^2)}+3\varepsilon^2\kappa^2\triangle^{-\frac{5}{2}}b-\frac{4\triangle^{-\frac{1}{2}}\varepsilon\zeta b}{(1+\zeta b^2)^2}-$$

$$\frac{8\sqrt{\triangle}\zeta b+8\kappa\zeta sb}{(1+\zeta b^2)^3}+\frac{\triangle^{-\frac{3}{2}}\varepsilon^2\zeta b}{(1+\zeta b^2)}=$$

$$\frac{-4\triangle^{-\frac{1}{2}}\varepsilon\zeta b}{(1+\zeta b^2)^2}+\frac{4\triangle^{-\frac{3}{2}}\varepsilon^2\zeta b}{1+\zeta b^2}-$$

$$\frac{8\sqrt{\triangle}\zeta b}{(1+\zeta b^2)^3}+\frac{8\triangle^{-\frac{1}{2}}\varepsilon\zeta b}{(1+\zeta b^2)^3}-\frac{8\kappa s\zeta b}{(1+\zeta b^2)^3}-$$

$$\frac{3\triangle^{-\frac{3}{2}}\varepsilon^2\zeta b}{1+\zeta b^2}+3\varepsilon^2\kappa^2\triangle^{-\frac{5}{2}b}+3\varepsilon^3\zeta\triangle^{-\frac{5}{2}b}-$$

$$\frac{4\triangle^{-\frac{1}{2}}\varepsilon\zeta b}{(1+\zeta b^2)^3}+\frac{8\sqrt{\triangle}\zeta b}{(1+\zeta b^2)^3}+\frac{8\kappa s\zeta b}{(1+\zeta b^2)^3}-$$

$$\frac{\triangle^{-\frac{3}{2}}\varepsilon^2\zeta b}{1+\zeta b^2}=3\triangle^{-\frac{5}{2}}\varepsilon^2(\varepsilon\zeta+\kappa^2)b.$$

It's easy to see that $s\phi_{bs}+b\phi_{ss}-\phi_b=0$ is equivalent to $\kappa^2+\varepsilon\zeta=0$. Then from Lemma 1.3, we know that F is projectively flat Finsler metric if and only if $\kappa^2+\varepsilon\zeta=0$.

③ From Lemma 1.2, we know that F is projectively flat Finsler metric and its projective factor and flag curvature are given by

$$P=\frac{F_{x^m}y^m}{2F},\ K=\frac{P^2-P_{x^k}y^k}{F^2}.$$

From Eq.(4), we obtain

$$F_{x^k}y^k=$$

$$\frac{1}{(1+\zeta|x|^2)^3}\{[\Gamma^{-\frac{1}{2}}(\kappa^2+\varepsilon\zeta)|y|^2\langle x,y\rangle+$$
$$2\Gamma^{-\frac{1}{2}}\kappa^2\langle x,y\rangle|y|^2-\kappa^2\langle x,y\rangle^2\Gamma^{-\frac{3}{2}}\cdot$$
$$(\kappa^2+\varepsilon\zeta)\langle x,y\rangle|y|^2](1+\zeta|x|^2)+$$
$$2\kappa|y|^2-4\zeta\langle x,y\rangle(\sqrt{\Gamma}+2\kappa\langle x,y\rangle+$$
$$\kappa^2\langle x,y\rangle^2\Gamma^{-\frac{1}{2}})\}\tag{22}$$

As $\Gamma=\kappa^2\langle x,y\rangle^2+\varepsilon|y|^2(1+\zeta|x|^2)$, we have

$$(1+\zeta|x|^2)|y|^2=\frac{1}{\varepsilon}(\Gamma-\kappa^2\langle x,y\rangle^2)\tag{23}$$

and we know that F is projectively flat if and only if

$$\kappa^2+\varepsilon\zeta=0\tag{24}$$

By Eqs.(22),(23) and (24), we get

$$F_{x^k}y^k=\frac{2}{(1+\zeta|x|^2)^3}\frac{\kappa}{\varepsilon}(\sqrt{\Gamma}+\kappa\langle x,y\rangle)^3\Gamma^{-\frac{1}{2}}\tag{25}$$

By Eqs.(11) and (25), we have

$$P=\frac{F_{x^m}y^m}{2F}=$$

$$\frac{\frac{2}{(1+\zeta|x|^2)^3}\frac{\kappa}{\varepsilon}(\sqrt{\Gamma}+\kappa\langle x,y\rangle)^3\Gamma^{-\frac{1}{2}}}{2\frac{(\sqrt{\Gamma}+\kappa\langle x,y\rangle)^2}{(1+\zeta|x|^2)^2\sqrt{\Gamma}}}=$$

$$\frac{\kappa(\sqrt{\Gamma}+\kappa\langle x,y\rangle)}{\varepsilon(1+\zeta|x|^2)}\tag{26}$$

where $\Gamma=\kappa^2\langle x,y\rangle^2+\varepsilon|y|^2(1+\zeta|x|^2)$.

Using (26), we get

$$P_{x^k}y^k=$$

$$\frac{1}{(1+\zeta|x|^2)^2}\left\{\frac{\kappa}{\varepsilon}[\Gamma^{-\frac{1}{2}}(\kappa^2+\varepsilon\zeta)\langle x,y\rangle|y|^2+\right.$$
$$\left.\kappa|y|^2](1+\zeta|x|^2)-(\sqrt{\Gamma}+\kappa\langle x,y\rangle)\frac{2\kappa\zeta\langle x,y\rangle}{\varepsilon}\right\}$$

Nothing that $\kappa^2+\varepsilon\zeta=0$ and

$$\varepsilon(1+\zeta|x|^2)|y|^2=\Gamma-\kappa^2\langle x,y\rangle^2,$$

we have

$$P_{x^k}y^k=\frac{1}{(1+\zeta|x|^2)^2}\frac{\kappa^2}{\varepsilon^2}(\sqrt{\Gamma}+\kappa\langle x,y\rangle)^2\tag{27}$$

By Eqs.(26) and (27), we have

$$P^2-P_{x^k}y^k=\left[\frac{(\sqrt{\Gamma}+\kappa\langle x,y\rangle)\frac{\kappa}{\varepsilon}}{1+\zeta|x|^2}\right]^2-$$
$$\frac{1}{(1+\zeta|x|^2)^2}\frac{\kappa^2}{\varepsilon^2}(\sqrt{\Gamma}+\kappa\langle x,y\rangle)^2=0,$$

which implies $K=0$.

3 Proof of Theorem 0.2

By Eq.(7), we have

$$\widetilde{F}=\sqrt{\frac{\kappa(\sqrt{\varepsilon(1+\zeta|x|^2)|y|^2+\kappa^2\langle x,y\rangle^2}+\kappa\langle x,y\rangle)^3}{\varepsilon(1+\zeta|x|^2)^3\sqrt{\varepsilon(1+\zeta|x|^2)|y|^2+\kappa^2\langle x,y\rangle^2}}}=|y|\sqrt{\frac{\kappa(\sqrt{\Upsilon}+\kappa s)^3}{\varepsilon(1+2\zeta t)^3\sqrt{\Upsilon}}}\tag{28}$$

where $t=\frac{|x|^2}{2}$, $s=\frac{\langle x,y\rangle}{|y|}$, $\Upsilon=\kappa^2s^2+\varepsilon(1+2\zeta t)$.

If we set

$$f=\frac{\kappa\left(\sqrt{\varepsilon(1+\zeta\mid x\mid^{2})+\frac{\kappa^{2}\langle x,y\rangle^{2}}{\mid y\mid^{2}}}+\frac{\kappa\langle x,y\rangle}{\mid y\mid}\right)^{3}}{\varepsilon(1+\zeta\mid x\mid^{2})^{3}\sqrt{\varepsilon(1+\zeta\mid x\mid^{2})+\frac{\kappa^{2}\langle x,y\rangle^{2}}{\mid y\mid^{2}}}}=\frac{\kappa(\sqrt{\Upsilon}+\kappa s)^{3}}{\varepsilon(1+2\zeta t)^{3}\sqrt{\Upsilon}},$$

$\widetilde{F}$ also can be expressed as the following form:

$$\widetilde{F}=\mid y\mid\sqrt{f\left(\frac{\mid x\mid^{2}}{2},\frac{\langle x,y\rangle}{\mid y\mid}\right)}.$$

Noting that

$$f=\frac{\kappa(\sqrt{\Upsilon}+\kappa s)^{3}}{\varepsilon(1+2\zeta t)^{3}\sqrt{\Upsilon}}=\frac{(\kappa\Upsilon+3\sqrt{\Upsilon}\kappa s+3\kappa^{2}s^{2}+\kappa^{3}s^{3}\Upsilon^{-\frac{1}{2}})}{\varepsilon(1+2\zeta t)^{3}} \tag{29}$$

we have

$$f_{t}=\frac{\kappa}{\varepsilon(1+2\zeta t)^{4}}[(2\varepsilon\zeta+3\Upsilon^{-\frac{1}{2}}\varepsilon\zeta\kappa s-\varepsilon\zeta\kappa^{3}s^{3}\Upsilon^{-\frac{3}{2}})(1+2\zeta t)-6\zeta(\Upsilon+3\sqrt{\Upsilon}\kappa s+3\kappa^{2}s^{2}+\kappa^{3}s^{3}\Upsilon^{-\frac{1}{2}})] \tag{30}$$

$$f_{ts}=\frac{\kappa}{\varepsilon(1+2\zeta t)^{4}}[(-6\Upsilon^{-\frac{3}{2}}\varepsilon\zeta\kappa^{3}s^{2}+3\Upsilon^{-\frac{1}{2}}\varepsilon\zeta\kappa+3\Upsilon^{-\frac{5}{2}}\varepsilon\zeta\kappa^{5}s^{4})(1+2\zeta t)-6\zeta(8\kappa^{2}s+6\Upsilon^{-\frac{1}{2}}\kappa^{3}s^{2}+3\sqrt{\Upsilon}\kappa-\Upsilon^{-\frac{3}{2}}\kappa^{5}s^{4})] \tag{31}$$

$$f_{ss}=\frac{\kappa}{\varepsilon(1+2\zeta t)^{3}}[8\kappa^{2}+15\Upsilon^{-\frac{1}{2}}\kappa^{3}s-10\Upsilon^{-\frac{3}{2}}\kappa^{5}s^{3}+3\Upsilon^{-\frac{5}{2}}\kappa^{7}s^{5}] \tag{32}$$

Using Eqs.(30)～(32) and $\Upsilon=\kappa^{2}s^{2}+\varepsilon(1+2\zeta t)$, we have

$$\begin{aligned}sf_{ts}+f_{ss}-2f_{t}=&\frac{\kappa}{\varepsilon(1+2\zeta t)^{4}}[(1+2\zeta t)(-6\Upsilon^{-\frac{3}{2}}\varepsilon\zeta\kappa^{3}s^{3}+3\Upsilon^{-\frac{1}{2}}\varepsilon\zeta\kappa s+3\Upsilon^{-\frac{5}{2}}\varepsilon\zeta\kappa^{5}s^{5}+8\kappa^{2}+\\&15\Upsilon^{-\frac{1}{2}}\kappa^{3}s-10\Upsilon^{-\frac{3}{2}}\kappa^{5}s^{3}+3\Upsilon^{-\frac{5}{2}}\kappa^{7}s^{5}-4\varepsilon\zeta-6\Upsilon^{-\frac{1}{2}}\varepsilon\zeta\kappa s+2\varepsilon\zeta\kappa^{3}s^{3}\Upsilon^{-\frac{3}{2}})-\\&6\zeta(8\kappa^{2}s+6\Upsilon^{-\frac{1}{2}}\kappa^{3}s^{2}+3\sqrt{\Upsilon}\kappa-\Upsilon^{-\frac{3}{2}}\kappa^{5}s^{4})+12\zeta(\Upsilon+3\sqrt{\Upsilon}\kappa s+3\kappa^{2}s^{2}+\kappa^{3}s^{2}\Upsilon^{-\frac{1}{2}})]=\\&\frac{\kappa}{\varepsilon(1+2\zeta t)^{3}}(\kappa^{2}+\varepsilon\zeta)[8\Upsilon^{-\frac{1}{2}}\kappa s+4\Upsilon^{-\frac{3}{2}}\kappa s\varepsilon(1+2\zeta t)+3\Upsilon^{-\frac{5}{2}}\kappa s\varepsilon^{2}(1+2\zeta t)^{2}+8]\end{aligned} \tag{33}$$

From Eq.(33), we can see that $sf_{ts}+f_{ss}-2f_{t}=0$ if and only if $\kappa^{2}+\varepsilon\zeta=0$. So when $\kappa^{2}+\varepsilon\zeta=0$, $\widetilde{F}$ is a locally dually flat Finsler metric.

Remark 3.1 $\widetilde{F}$ can also be expressed as the form: $\widetilde{F}=\sqrt{PF}$. When $\kappa^{2}+\varepsilon\zeta=0$, F is a projectively flat Finsler metric. At this time, P is the projective factor of F and it is a Finsler metric. Thus, $\widetilde{F}$ is a dually flat Finsler metric [13].

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