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Regularization and robust stabilization of uncertain descriptor fractional-order systems

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Abstract: The robust stability and stabilization of uncertain descriptor fractional-order systems are investigated. In such systems parameter uncertainty is assumed to be time-invariant and norm-bounded in the state matrix. First, a predictive controller was used to regularize the uncertain descriptor fractional-order systems, and then an output feedback controller was designed to stabilize the regularized system. The sufficient conditions for robustly stabilizing such systems with the fractional order alpha in [1,2) are derived. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed approach.

Key words: descriptor fractional-order systems; parameter uncertainty; output feedback control; regularization and robust stabilization

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不确定奇异分数阶系统的正则和鲁棒稳定

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摘要: 研究了不确定奇异分数阶系统的鲁棒稳定和镇定问题. 在该系统里, 状态矩阵的不确定参数通常假定为时不变和范数有界的. 首先设计一个可预测的控制器使得不确定奇异分数阶正则; 然后设计一个输出反馈控制器使得正则后的系统稳定, 这样可以得到当分数阶阶次在[1,2)时系统鲁棒渐近稳定的充分条件; 最后通过一个数值仿真证明了该方法的有效性.

关键词: 奇异分数阶系统; 参数不确定; 输出反馈控制; 正则和鲁棒稳定

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0 Introduction

Fractional-order control systems have attracted increasing interest in recent years[1-9], and have been appropriately described such systems as electromagnetic systems[10-11], dielectric polarization[12], electrochemical process[13], thermal diffusion[14], viscoelastic systems[15-16], and so on. A typical example is the voltage-current relation of a semi-infinite lossy transmission line[17] or diffusion of the heat through a semi-infinite solid, where heat flow is equal to the half-derivative of the temperature[18]. On the other hand, descriptor systems are systems whose dynamics are governed by a mixture of differential equations and algebraic equations. A descriptor form includes information of both static and dynamic constraints[19-21]. Both continuous and discrete systems have been of interest in the literature for a long time.

The stability of fractional-order systems has been investigated from both the algebraic and the analytic point of view, and the fractional order controllers have been implemented to enhance the robustness and the performance of the closed loop control systems[22-23]. Fractional order controllers, such as CRONE controller[24], TID controller[25], fractional PID controller[26] and lead-lag compensator[27], have been implemented to improve performance and robustness. However, few works have been found that address the stabilization issue of singular fractional-order systems.

In this paper, we first regularize the singular fractional-order systems by applying a predictive controller, the necessary and sufficient conditions for the normalization of the descriptor fractional-order systems are then given. A static output feedback controller is also designed to stabilize the regularized system. The state feedback control to robustly stabilize such uncertain descriptor fractional-order systems with the fractional order $0\leqslant\alpha<2$ has been presented[28], however, in most practical applications, the systems states are not completely accessible. In this case, the method adopted by this paper is necessary. For simplicity we only discuss the case $1\leqslant\alpha<2$ in this work.

Notations: R^n denotes the n-dimensional Euclidean space, $\mathrm{Sym}\{\mathbf{X}\}$ denotes $\mathbf{X}^{\mathrm{T}}+\mathbf{X}$ and $\begin{pmatrix}\mathbf{X} & \mathbf{Y}\\ * & \mathbf{Z}\end{pmatrix}$ denotes $\begin{pmatrix}\mathbf{X} & \mathbf{Y}\\ \mathbf{Y}^{\mathrm{T}} & \mathbf{Z}\end{pmatrix}$.

1 Preliminaries and problem formulation

1.1 Preliminary results

There are different definitions of fractional derivatives [16]. In this work we adopt the following Caputo definition.

$$D^{\alpha}f(t)=\frac{\mathrm{d}^{\alpha}f(t)}{\mathrm{d}t^{\alpha}}=\frac{1}{\Gamma(\alpha-n)}\int_{a}^{t}\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\mathrm{d}\,\tau \tag{1}$$

where n is the integer satisfying $n-1\leqslant\alpha<n$, $\Gamma(\cdot)$ is the Gamma function defined by the integral

$$\Gamma(z)=\int_{a}^{\infty}e^{-t}t^{z-1}\mathrm{d}t \tag{2}$$

Consider the following descriptor fractional-order linear systems:

$$\left.\begin{aligned}\mathbf{E}D^{\alpha}x(t)&=\mathbf{A}x(t)\\ x(0)&=x_0\end{aligned}\right\} \tag{3}$$

where $0<\alpha<2$ is the fractional derivative order. $x(t)\in\mathrm{R}^n$ is the state, the matrix $\mathbf{E}\in\mathrm{R}^{n\times n}$ is a singular square matrix and $\mathbf{A}\in\mathrm{R}^{n\times n}$, as shown in Figs. 1 and 2.

Definiion 1.1 The descriptor model (3) is said to be regular if there exists a unique solution $x(t)$ for a given initial conditions.

Lemm 1.1[28] The pair $(\mathbf{E},\mathbf{A})$ is regular if and only if $\Delta(\lambda^{\alpha}\mathrm{E}-\mathbf{A})$ is not identically zero.

Proof Taking the Laplace transform, relation (3) is given by

$$(s^{\alpha}\mathbf{E}-\mathbf{A})X(s)=s^{\alpha-1}x_0 \tag{4}$$

Taking Laplace inverse transform above, then

$$x(t)=\mathbf{E}_{\alpha,1}(\mathbf{A}t^{\alpha}x_0)=\sum_{k=0}^{\infty}\frac{\mathbf{A}^k t^{k\alpha}}{\Gamma(1+k\alpha)})x_0 \tag{5}$$

Then, solution $x(t)$ exists and is unique if and

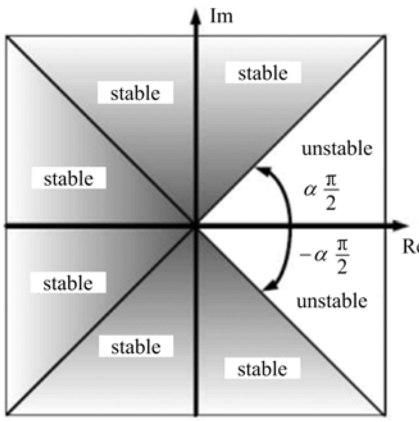

Fig. 1 Stability region of linear fractional-order systems with order $0<\alpha<1$

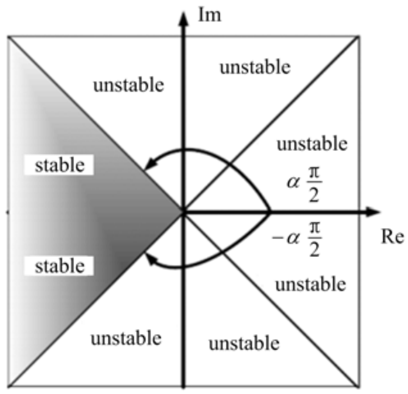

Fig. 2 Stability region of linear fractional-order systems with order $1\leqslant\alpha<2$

only if $\Delta(\lambda^{\alpha}E-\mathbf{A})$ is not identically zero with $\lambda=s$.

Lemm 1.2[22] Let $\mathbf{A}\in\mathrm{R}^{n\times n}$ be a real matrix. Then, a necessary and sufficient condition for the asymptotical stability of system (3) is satisfied for $0<\alpha<2$, and

$$|\arg(\mathrm{spec}(\mathbf{A}))|>\frac{\alpha\pi}{2} \tag{6}$$

where spec($\mathbf{A}$) is the spectrum of all eigenvalues of $\mathbf{A}$.

Lemma 1.3[30] Let $A\in\mathrm{R}^{n\times n}$ be a real matrix. Then, $|\arg(\mathrm{spec}(\mathbf{A}))|>\frac{\alpha\pi}{2}$, where $1\leqslant\alpha<2$, if and only if there exists symmetrical matrix $\mathbf{P}>0$ such that

$$\begin{bmatrix}(\mathbf{AP}+\mathbf{PA}^{\mathrm{T}})\sin\theta & (\mathbf{AP}-\mathbf{PA}^{\mathrm{T}})\cos\theta\\ * & (\mathbf{AP}+\mathbf{PA}^{\mathrm{T}})\sin\theta\end{bmatrix}<0 \tag{7}$$

where $\theta=\pi-\frac{\alpha\pi}{2}$.

Lemma 1.4[31] For any matrices X and Y with appropriate dimensions, we have

$$\mathbf{X}^{\mathrm{T}}\mathbf{Y}+\mathbf{Y}^{\mathrm{T}}\mathbf{X}\leqslant\mathbf{X}^{\mathrm{T}}\mathbf{Z}\mathbf{X}+\mathbf{Y}^{\mathrm{T}}\mathbf{Z}^{-1}\mathbf{Y} \tag{8}$$

for any positive definite matrix $\mathbf{Z}$.

1.2 Problem formulation

Consider the following descriptor fractional-order linear systems,

$$\left.\begin{aligned}\mathbf{E}D^{\alpha}x(t)&=(\mathbf{A}+\mathbf{\Delta}_{\mathrm{A}})x(t)+\mathbf{B}u(t)\\ y(t)&=\mathbf{C}x(t)\end{aligned}\right\} \tag{9}$$

where $1\leqslant\alpha<2$, $x(t)\in\mathrm{R}^{n}$ is the state, $u(t)\in\mathrm{R}^{m}$ is the control input, $y(t)\in\mathrm{R}^{p}$ is the measured output, $\mathbf{E}\in\mathrm{R}^{n\times n}$ is a singular square matrix, rank$(\mathbf{E})=r<n$, $\mathbf{A}\in\mathrm{R}^{n\times n}$, $\mathbf{B}\in\mathrm{R}^{n\times m}$, $\mathbf{C}\in\mathrm{R}^{p\times n}$, and $\mathbf{\Delta}_{\mathrm{A}}$ is a time-invariant matrix representing the norm-bounded parameter uncertainty of the following form:

$$\mathbf{\Delta}_{\mathrm{A}}=\mathbf{M}_{\mathrm{A}}\mathbf{\Delta}\mathbf{N}_{\mathrm{A}} \tag{10}$$

where $\mathbf{M}_{\mathrm{A}}$ and $\mathbf{N}_{\mathrm{A}}$ are known real constant matrices of appropriate dimensions, and the uncertain matrix $\mathbf{\Delta}$ satisfies

$$\mathbf{\Delta}\mathbf{\Delta}^{\mathrm{T}}\leqslant\mathbf{I} \tag{11}$$

In this work we design the predictive and memoryless static feedback control for system (9) as follows:

$$u(t)=-\mathbf{L}D^{\alpha}y(t)+\mathbf{H}y(t) \tag{12}$$

where $\mathbf{L}\in\mathrm{R}^{m\times n}$ and $\mathbf{H}\in\mathrm{R}^{m\times n}$ are gain matrices. The feedback controller $-\mathbf{L}D^{\alpha}y(t)$ is referred to as a regularizing controller. We try to determine the conditions of existence of gain matrix $\mathbf{L}$ such that system (9) is regularized, and then design the gain matrix $\mathbf{H}$ such that the obtained regularized system is robust asymptotically stable.

2 Regularization of uncertain descriptor fractional-order systems

Under the above feedback controller, we obtain the closed loop system as follows:

$$\mathbf{E}_1D^{\alpha}x(t)=(\mathbf{A}+\mathbf{BHC}+\mathbf{\Delta}_{\mathrm{A}})x(t) \tag{13}$$

where $\mathbf{E}_1=\mathbf{E}+\mathbf{BLC}$.

Lemma 2.1[17] System (9) is normalizable if and only if

$$\mathrm{rank}[\mathbf{E}\quad\mathbf{B}]=n \tag{14}$$

That is, Eq. (13) is regular if and only if $\mathbf{E}_1=\mathbf{E}+\mathbf{BLC}$ is invertible.

Theorem 2.1 If there exists a matrix $\mathbf{L}\in\mathrm{R}^{m\times p}$ and $0<\varepsilon\leqslant1$ such that the the following

LMI holds

$$\begin{pmatrix} \mathbf{E}^{T}\mathbf{E}+\mathbf{E}^{T}\mathbf{BLC}+\mathbf{C}^{T}\mathbf{L}^{T}\mathbf{B}^{T}\mathbf{E}+\mathbf{C}^{T}\mathbf{L}^{T}\mathbf{B}^{T}+\mathbf{BLC} & \mathbf{I} \\ \mathbf{I} & \varepsilon\mathbf{I} \end{pmatrix} > 0 \tag{15}$$

then, the controller

$$u(t) = -\mathbf{L}D^{\alpha}y(t) + \mathbf{H}y(t) \tag{16}$$

regularizes system (9) and the matrix $\mathbf{E}_1 = \mathbf{E} + \mathbf{BLC}$ is of full rank.

Proof The invertibility condition of matrix $\mathbf{E}_1 = \mathbf{E} + \mathbf{BLC}$ is equivalent to the inequality $\mathbf{E}_1^{\,T}\mathbf{E}_1 > 0$, that is

$$(\mathbf{E} + \mathbf{BLC})^{T}(\mathbf{E} + \mathbf{BLC}) > 0 \tag{17}$$

which can be rewritten as:

$$\mathbf{E}^{T}\mathbf{E}+\mathbf{E}^{T}\mathbf{BLC}+\mathbf{C}^{T}\mathbf{L}^{T}\mathbf{B}^{T}\mathbf{E}+\mathbf{C}^{T}\mathbf{L}^{T}\mathbf{B}^{T}\mathbf{BLC} > 0 \tag{18}$$

Obviously, for any $0 < \varepsilon \leqslant 1$, we have

$$\mathbf{C}^{T}\mathbf{L}^{T}\mathbf{B}^{T}\mathbf{BLC} \geqslant \varepsilon\mathbf{C}^{T}\mathbf{L}^{T}\mathbf{B}^{T}\mathbf{BLC} \tag{19}$$

According to Lemma 1.4, we obtain

$$\varepsilon\mathbf{C}^{T}\mathbf{L}^{T}\mathbf{B}^{T}\mathbf{BLC} + \frac{1}{\varepsilon}\mathbf{I} \geqslant \mathbf{C}^{T}\mathbf{L}^{T}\mathbf{B}^{T} + \mathbf{BLC} \tag{20}$$

which means

$$\mathbf{E}^{T}\mathbf{E}+\mathbf{E}^{T}\mathbf{BLC}+\mathbf{C}^{T}\mathbf{L}^{T}\mathbf{B}^{T}\mathbf{E}+\mathbf{C}^{T}\mathbf{L}^{T}\mathbf{B}^{T}+\mathbf{BLC}-\frac{1}{\varepsilon}\mathbf{I} > 0 \tag{21}$$

and thus Eq. (18) is true. Using the Schur complement lemma, Eq. (21) is equivalent to Eq. (15). This completes the proof.

3 Robust stabilization of uncertain descriptor fractional-order systems

We now design the gain matrix $\mathbf{H}$ such that the uncertain descriptor fractional-order systems is robust asymptotically stable for the fractional order $1 \leqslant \alpha < 2$.

It is known that $\mathbf{E}_1$ is invertible, then Eq. (13) is equivalent to

$$D^{\alpha}x(t) = \mathbf{E}_1^{-1}(\mathbf{A} + \mathbf{BHC} + \mathbf{\Delta}_A)x(t) = (\mathbf{A}_1 + \mathbf{B}_1\mathbf{HC} + \mathbf{E}_1^{-1}\mathbf{\Delta}_A)x(t) = \tilde{\mathbf{A}}x(t) \tag{22}$$

where $\mathbf{A}_1 = \mathbf{E}_1^{-1}\mathbf{A}$, $\mathbf{B}_1 = \mathbf{E}_1^{-1}\mathbf{B}$, and

$\tilde{\mathbf{A}} = \bar{\mathbf{A}} + \mathbf{E}_1^{-1}\mathbf{\Delta}_A = \mathbf{A}_1 + \mathbf{B}_1\mathbf{HC} + \mathbf{E}_1^{-1}\mathbf{\Delta}_A$.

Theorem 2.2 Suppose there exists a gain matrix $\mathbf{L}$ such that system (9) can be regularized. Then, the uncertain descriptor fractional-order system (13) with fractional order $1 \leqslant \alpha < 2$ under the output feedback control is robust stable if there exist a matrix $\mathbf{N} \in \mathrm{R}^{m\times p}$; a nonsingular $\mathbf{M} \in \mathrm{R}^{p\times p}$; $\mathbf{P}_0 = \mathbf{P}_0^{\,T} > 0$, $\in \mathrm{R}^{n\times n}$, and a real scalar $\delta > 0$, such that

$$\begin{pmatrix} \mathbf{\Pi}_{11} & * & * & * \\ \mathbf{\Pi}_{21} & \mathbf{\Pi}_{22} & * & * \\ \mathbf{N}_A\mathbf{P}_0 & 0 & -\delta\mathbf{I} & * \\ 0 & \mathbf{N}_A\mathbf{P}_0 & 0 & -\delta\mathbf{I} \end{pmatrix} < 0 \tag{23}$$

$$\mathbf{MC} = \mathbf{CP}_0 \tag{24}$$

where

$\mathbf{\Pi}_{11} = (\mathbf{A}_1\mathbf{P}_0 + \mathbf{P}_0\mathbf{A}_1^{\,T} + \mathbf{B}_1\mathbf{NC} + \mathbf{C}^{T}\mathbf{N}^{T}\mathbf{B}_1^{\,T})\sin\theta + \delta^{-1}\mathbf{E}_1^{-1}\mathbf{M}_A(\mathbf{E}_1^{-1}\mathbf{M}_A)^{T} = \mathbf{\Pi}_{22}$,

$\mathbf{\Pi}_{21} = (\mathbf{P}_0\mathbf{A}_1^{\,T} - \mathbf{A}_1\mathbf{P}_0 + \mathbf{C}^{T}\mathbf{N}^{T}\mathbf{B}_1^{\,T} - \mathbf{B}_1\mathbf{NC})\cos\theta$,

$\mathbf{A}_1 = (\mathbf{E} + \mathbf{BLC})^{-1}\mathbf{A}$,

$\mathbf{B}_1 = (\mathbf{E} + \mathbf{BLC})^{-1}\mathbf{B}$.

Moreover, the gain matrix $\mathbf{H}$ is given by

$$\mathbf{H} = \mathbf{NM}^{-1} \tag{25}$$

Proof The uncertain descriptor fractional-order system (9) with the static feedback (12) is equivalent to system (22). According to Lemmas 2 and 3, we have the asymptotical stability condition of system (22) as follows:

$$|\arg(\mathrm{spec}(\tilde{\mathbf{A}}))| > \frac{\alpha\pi}{2} \tag{26}$$

that is, there exists $\mathbf{P}_0 = \mathbf{P}_0^{\,T} > 0$ such that

$$\begin{pmatrix} (\tilde{\mathbf{A}}\mathbf{P}_0 + \mathbf{P}_0\tilde{\mathbf{A}}^{T})\sin\theta & (\tilde{\mathbf{A}}\mathbf{P}_0 - \mathbf{P}_0\tilde{\mathbf{A}}^{T})\cos\theta \\ (\mathbf{P}_0\tilde{\mathbf{A}}^{T} - \tilde{\mathbf{A}}\mathbf{P}_0)\cos\theta & (\tilde{\mathbf{A}}\mathbf{P}_0 + \mathbf{P}_0\tilde{\mathbf{A}}^{T})\sin\theta \end{pmatrix} = \begin{pmatrix} (\bar{\mathbf{A}}\mathbf{P}_0 + \mathbf{P}_0\bar{\mathbf{A}}^{T})\sin\theta & (\bar{\mathbf{A}}\mathbf{P}_0 - \mathbf{P}_0\bar{\mathbf{A}}^{T})\cos\theta \\ (\mathbf{P}_0\bar{\mathbf{A}}^{T} - \bar{\mathbf{A}}\mathbf{P}_0)\cos\theta & (\bar{\mathbf{A}}\mathbf{P}_0 + \mathbf{P}_0\bar{\mathbf{A}}^{T})\sin\theta \end{pmatrix} + \mathrm{Sym}\left\{\begin{pmatrix} \mathbf{E}_1^{-1}\mathbf{\Delta}_A\mathbf{P}_0\sin\theta & \mathbf{E}_1^{-1}\mathbf{\Delta}_A\mathbf{P}_0\cos\theta \\ -\mathbf{E}_1^{-1}\mathbf{\Delta}_A\mathbf{P}_0\cos\theta & \mathbf{E}_1^{-1}\mathbf{\Delta}_A\mathbf{P}_0\sin\theta \end{pmatrix}\right\} \tag{27}$$

where $\tilde{\mathbf{A}} = \bar{\mathbf{A}} + \mathbf{E}_1^{-1}\mathbf{\Delta}_A$.

Replace $\mathbf{\Delta}_A = \mathbf{M}_A\mathbf{\Delta}\mathbf{N}_A$ of Eq. (27), additionally, for any matrices $\mathbf{X}$ and $\mathbf{Y}$ with appropriate dimensions, we have

$$\mathbf{X}^{T}\mathbf{Y} + \mathbf{Y}^{T}\mathbf{X} \leqslant \delta\mathbf{X}^{T}\mathbf{X} + \delta^{-1}\mathbf{Y}^{T}\mathbf{Y} \tag{28}$$

for any $\delta > 0$.

Therefore,

$$\mathrm{Sym}\left\{\begin{pmatrix} \mathbf{E}_1^{-1}\mathbf{\Delta}_A\mathbf{P}_0\sin\theta & \mathbf{E}_1^{-1}\mathbf{\Delta}_A\mathbf{P}_0\cos\theta \\ -\mathbf{E}_1^{-1}\mathbf{\Delta}_A\mathbf{P}_0\cos\theta & \mathbf{E}_1^{-1}\mathbf{\Delta}_A\mathbf{P}_0\sin\theta \end{pmatrix}\right\}$$

which is equivalent to

$$\mathrm{Sym}\left\{\begin{pmatrix} \mathbf{E}_1^{-1}\mathbf{M}_A\mathbf{\Delta}\mathbf{N}_A\mathbf{P}_0\sin\theta & \mathbf{E}_1^{-1}\mathbf{M}_A\mathbf{\Delta}\mathbf{N}_A\mathbf{P}_0\cos\theta \\ -\mathbf{E}_1^{-1}\mathbf{M}_A\mathbf{\Delta}\mathbf{N}_A\mathbf{P}_0\cos\theta & \mathbf{E}_1^{-1}\mathbf{M}_A\mathbf{\Delta}\mathbf{N}_A\mathbf{P}_0\sin\theta \end{pmatrix}\right\}$$

that is

$$\mathrm{Sym}\left\{\begin{pmatrix} \mathbf{E}_1^{-1}\mathbf{M}_A\sin\theta & \mathbf{E}_1^{-1}\mathbf{M}_A\cos\theta \\ -\mathbf{E}_1^{-1}\mathbf{M}_A\cos\theta & \mathbf{E}_1^{-1}\mathbf{M}_A\sin\theta \end{pmatrix}\cdot \begin{pmatrix} \boldsymbol{\Delta} & 0 \\ 0 & \boldsymbol{\Delta} \end{pmatrix}\cdot \begin{pmatrix} \mathbf{N}_A\mathbf{P}_0 & 0 \\ 0 & \mathbf{N}_A\mathbf{P}_0 \end{pmatrix}\right\} \leqslant$$
$$\delta \begin{pmatrix} \mathbf{E}_1^{-1}\mathbf{M}_A\sin\theta & \mathbf{E}_1^{-1}\mathbf{M}_A\cos\theta \\ -\mathbf{E}_1^{-1}\mathbf{M}_A\cos\theta & \mathbf{E}_1^{-1}\mathbf{M}_A\sin\theta \end{pmatrix} \times \begin{pmatrix} \mathbf{E}_1^{-1}\mathbf{M}_A\sin\theta & \mathbf{E}_1^{-1}\mathbf{M}_A\cos\theta \\ -\mathbf{E}_1^{-1}\mathbf{M}_A\cos\theta & \mathbf{E}_1^{-1}\mathbf{M}_A\sin\theta \end{pmatrix}^{\mathrm{T}} +$$
$$\delta^{-1}\begin{pmatrix} \mathbf{N}_A\mathbf{P}_0 & 0 \\ 0 & \mathbf{N}_A\mathbf{P}_0 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \mathbf{N}_A\mathbf{P}_0 & 0 \\ 0 & \mathbf{N}_A\mathbf{P}_0 \end{pmatrix} \tag{29}$$

Substituting the above into Eq. (27), and using $\boldsymbol{\Delta}\boldsymbol{\Delta}^{\mathrm{T}} \leqslant \mathbf{I}$, we have

$$\begin{pmatrix} (\tilde{\mathbf{A}}\mathbf{P}_0 + \mathbf{P}_0\tilde{\mathbf{A}}^{\mathrm{T}})\sin\theta & (\tilde{\mathbf{A}}\mathbf{P}_0 - \mathbf{P}_0\tilde{\mathbf{A}}^{\mathrm{T}})\cos\theta \\ (\mathbf{P}_0\tilde{\mathbf{A}}^{\mathrm{T}} - \tilde{\mathbf{A}}\mathbf{P}_0)\cos\theta & (\tilde{\mathbf{A}}\mathbf{P}_0 + \mathbf{P}_0\tilde{\mathbf{A}}^{\mathrm{T}})\sin\theta \end{pmatrix} \leqslant$$
$$\begin{pmatrix} (\bar{\mathbf{A}}\mathbf{P}_0 + \mathbf{P}_0\bar{\mathbf{A}}^{\mathrm{T}})\sin\theta & (\bar{\mathbf{A}}\mathbf{P}_0 - \mathbf{P}_0\bar{\mathbf{A}}^{\mathrm{T}})\cos\theta \\ (\mathbf{P}_0\bar{\mathbf{A}}^{\mathrm{T}} - \bar{\mathbf{A}}\mathbf{P}_0)\cos\theta & (\bar{\mathbf{A}}\mathbf{P}_0 + \mathbf{P}_0\bar{\mathbf{A}}^{\mathrm{T}})\sin\theta \end{pmatrix} +$$
$$\delta \begin{pmatrix} \mathbf{E}_1^{-1}\mathbf{M}_A\sin\theta & \mathbf{E}_1^{-1}\mathbf{M}_A\cos\theta \\ -\mathbf{E}_1^{-1}\mathbf{M}_A\cos\theta & \mathbf{E}_1^{-1}\mathbf{M}_A\sin\theta \end{pmatrix} \times \begin{pmatrix} \mathbf{E}_1^{-1}\mathbf{M}_A\sin\theta & \mathbf{E}_1^{-1}\mathbf{M}_A\cos\theta \\ -\mathbf{E}_1^{-1}\mathbf{M}_A\cos\theta & \mathbf{E}_1^{-1}\mathbf{M}_A\sin\theta \end{pmatrix}^{\mathrm{T}} +$$
$$\delta^{-1}\begin{pmatrix} \mathbf{N}_A\mathbf{P}_0 & 0 \\ 0 & \mathbf{N}_A\mathbf{P}_0 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \mathbf{N}_A\mathbf{P}_0 & 0 \\ 0 & \mathbf{N}_A\mathbf{P}_0 \end{pmatrix} < 0.$$

Substituting $\mathbf{N} = \mathbf{H}\mathbf{M}$, $\mathbf{M}\mathbf{C} = \mathbf{C}\mathbf{P}_0$ and $\bar{\mathbf{A}} = \mathbf{A}_1 + \mathbf{B}\mathbf{K}$ into the above inequality yields

$$\begin{pmatrix} \boldsymbol{\Pi}_{11} & \boldsymbol{\Pi}_{21}^{\mathrm{T}} \\ \boldsymbol{\Pi}_{21} & \boldsymbol{\Pi}_{22} \end{pmatrix} + \delta^{-1}\begin{pmatrix} \mathbf{N}_A\mathbf{P}_0 & 0 \\ 0 & \mathbf{N}_A\mathbf{P}_0 \end{pmatrix}^{\mathrm{T}} \times \begin{pmatrix} \mathbf{N}_A\mathbf{P}_0 & 0 \\ 0 & \mathbf{N}_A\mathbf{P}_0 \end{pmatrix} < 0 \tag{30}$$

By using the Schur complement, we complete the proof.

Remark　In this paper, firstly, we design a predictive controller $\mathbf{L}$ to regularize the uncertain descriptor fractional-order systems; secondly, we provide the resulting system by a controller $\mathbf{H}$ that achieves the asymptotical stabilization of the regularized uncertain descriptor fractional-order systems.

4　Numerical example

Consider an uncertain fractional-order system with the following parameters:

$$\mathbf{E} = \begin{pmatrix} 2 & 0 & 0.5 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 2.4 & 0.2 & 1.2 \\ 4 & 1.5 & 2 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 4 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix},$$

$$\mathbf{M}_A = \begin{pmatrix} 0.2 & 0 & 0 \\ 0.2 & 0.3 & 0.4 \\ 0 & 0.2 & 0 \end{pmatrix}, \mathbf{N}_A = \begin{pmatrix} 0 & 0 & 0 \\ 0.2 & 0 & 0.4 \\ 0 & 0.1 & 0 \end{pmatrix};$$

where $\alpha = 1.23$.

The pair $(\mathbf{E}, \mathbf{A})$ is not regular. To design a state feedback control law such that the resulting regularized fractional order system is asymptotically stable, we use Theorem 2.1 and obtain the following $\mathbf{L}$ ($\varepsilon = 0.7524$):

$$\mathbf{L} = \begin{pmatrix} 1.1507 & -3.1224 \\ -2.573 & 0.1749 \\ -1.4722 & 2.8590 \end{pmatrix}.$$

This means that the system can be regularized by the regularizing controller. Then, design the gain matrix $\mathbf{H}$ such that the obtained regularized system is robust asymptotically stable, and we obtain

$$\mathbf{A}_1 = \begin{pmatrix} 0.0082 & 0.3478 & 0.4672 \\ 0.5374 & -1.8458 & 0.4729 \\ 0.1673 & 0.5928 & -0.9533 \end{pmatrix},$$

$$\mathbf{B}_1 = \begin{pmatrix} -1.5721 & -1.2533 \\ 1.5881 & 0.9582 \\ -2.0013 & -0.3663 \end{pmatrix}.$$

A feasible solution to Eqs. (23) and (24) is as follows:

$$\mathbf{N} = \begin{pmatrix} 2.9537 & 1.1854 \\ -0.2957 & 4.5732 \end{pmatrix}, \mathbf{M} = \begin{pmatrix} 1.5683 & 0.4632 \\ 1.0034 & 0.5746 \end{pmatrix},$$

$$\mathbf{P}_0 = \begin{pmatrix} 7.6529 & -1.6528 & -14.7992 \\ -1.6528 & 1.1663 & 1.0552 \\ -14.7992 & 1.0552 & 9.7665 \end{pmatrix}$$

where $\delta = 0.3779$.

From theorem 3.1, the asymptotically stabilizing state feedback gain is obtained as:

$$\mathbf{H} = \begin{pmatrix} 0.0067 & 1.9220 \\ -2.8971 & -0.3873 \end{pmatrix}.$$

5　Conclusion

This paper proposed an LMI-based robust controller design for uncertain descriptor fractional-order systems with the fractional order $1 \leqslant \alpha < 2$. This method is especially useful if the system states are not completely accessible. In the future, the robust stability and stabilization of descriptor fractional-order interval systems will be studied.

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