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On partially s-permutable subgroups of finite groups

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Abstract: Let H be a subgroup of G. H is said to be partially s-permutable in G provided G has a subnormal subgroup T such that $G=HT$ and $H\cap T\leqslant H_{sT}$, where H_{sT} is the subgroup of H generated by all the subgroups of H which permute with all the Sylow subgroups of T. Here, partially s-permutable subgroups were used to study the structure of finite groups and some new criteria of p-nilpotent groups and p-supersoluble groups were obtained.

Key words: finite groups; partially s-permutable subgroups; p-supersoluble groups; p-nilpotent groups

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有限群的局部 s 置换子群

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摘要:设 G 是一个有限群.群 G 的子群 H 称为在 G 中局部 s 置换,如果存在 G 的次正规子群 T 使得 $G=HT$ 且 $H\cap T\leqslant H_{sT}$，H_{sT} 是由所有包含在 H 中的并与 T 的所有 Sylow 子群可置换的子群生成.利用局部 s 置换子群研究了有限群的结构,得到了一些关于 p 幂零群和 p 超可解群的新判别准则.

关键词:有限群;局部 s 置换子群;p 超可解群;p 幂零群

0 Introduction

Throughout this paper, all groups are supposed to be finite and our notation and terminology are standard, as in Refs. [1-2].

Recall that a subgroup H of a group G is said to be s-permutable (or s-quasinormal) in G if it permutes with every Sylow subgroup of G (that is, $HP=PH$, for an arbitrary Sylow subgroup P of G). This concept was introduced by Kegel[3] in 1962. The s-permutability of a subgroup of a finite group G often yields some information about group G itself. Some studies[4] have since been conducted. On the other hand, in 1996, Wang[5] introduced

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the so-called c-normal subgroup: a subgroup H of G *is said to be* c-normal in G if there is a normal subgroup K of G such that $G=HK$ and $H\cap K\leqslant H_G$, where H_G is the core of H in G. Also, many results were obtained by using this concept[6-7]. Later on, Skiba[8] gave the concept of weakly s-permutable subgroup: A subgroup H of G is called weakly s-permutable in G if G has a subnormal subgroup T such that $G=HT$ and $H\cap T\leqslant H_{sG}$, where H_{sG} denotes the subgroup of H generated by all subgroups of H which are s-permutable in G. Obviously, every s-permutable subgroup and every c-normal subgroup are weakly s-permutable in G, but the converse is not true[8, Example 1.2]. By using this conception, Skiba achieved many results about the structure of finite groups. In order to unify all the conceptions mentioned above, now we introduce the following notion:

Definition 0.1 A subgroup H of a group G is said to be partially s-permutable in G provided G has a subnormal subgroup T such that $G=HT$ and $H\cap T\leqslant H_{sT}$.

Here H_{sT} is the subgroup of H generated by all the subgroups of H which permute with all the Sylow subgroups of T.

It is easy to see that every weakly s-permutable subgroup is also partially s-permutable. However the converse is not true in general. Here comes an example.

Example 0.2 Considering

$$G=C_{A_8}((12)(34)(56)(78))=M\rtimes S_4,$$

where

$$M=\langle(12)(34)(56)(78),(12)(34),(12)(56)\rangle$$

is an elementary abelian subgroup. Let

$$H=\langle(135)(246),(12)(34),(34)(56)\rangle\cong A_4$$

and

$$T=\langle(12)(34)(56)(78),(12)(34),(34)(56),(13)(24)(57)(68),(15)(26)(37)(48)\rangle.$$

Then we can see that $T\trianglelefteq G$, $G=HT$ and H permutes with every Sylow subgroup of T. Hence $H\cap T=V_4=\langle(12)(34),(34)(56)\rangle\leqslant H_{sT}=H$. This means that H is partially s-permutable in G.

On the other hand, let $Q=\langle(357),(468)\rangle$, then Q is a Sylow 3-subgroup of G. Obviously, $HQ\neq QH$, and so H is not s-permutable in G. In fact, we can prove that $H_{sG}=1$. Hence, $H\cap T\not\leqslant H_{sG}$, and so H is not weakly s-permutable in G.

This example shows that in general the set of all partially s-permutable subgroups is wider than the set of all weakly s-permutable subgroups, and of course, also wider than the set of all s-permutable subgroups and the set of all c-normal subgroups. In this paper, we characterize p-nilpotency and p-supersolubility of finite groups under the assumption that some subgroups are partially s-permutable. Some former results are generalized.

1 Preliminaries

Let p be a prime. Recall that a group G is said to be p-nilpotent if it has a normal p-complement and G is called p-supersoluble provided every p-chief factor of it is cyclic. It is easy to see that a p-nilpotent group is also p-supersoluble.

Besides, we use $\mathcal{U}_p$ to denote the class of all p-supersoluble groups. The symbol $G^{\mathcal{U}_p}$ denotes the $\mathcal{U}_p$-residual of G, that is, the intersection of all normal subgroups of G whose relative quotient groups belong to $\mathcal{U}_p$.

We also need the following Lemmas:

Lemma 1.1 Let $A\leqslant K\leqslant G$, $B\leqslant G$. Then:

① Suppose that A is normal in G. Then K/A is subnormal in G/A if and only if K is subnormal in G[9, A, 14.1].

② If A is subnormal in G, then $A\cap B$ is subnormal in B[9, A, 14.1].

③ If A and B are both subnormal in G, then so is $\langle A,B\rangle$[9, A, 14.4].

④ If A is subnormal in G and B is a minimal normal subgroup of G, then $B\leqslant N_G(A)$[9, A, 14.5].

⑤ If A is subnormal in G and B is a Hall π-subgroup of G, then $A\cap B$ is a Hall π-subgroup of A[10].

⑥ If A is subnormal in G and A is π-subgroup of G, then $A\leqslant O_\pi(G)$[10].

⑦ If A is subnormal in G and the index $|G:A|$ is a p'-number, then A contains all Sylow p-subgroups of G[10].

⑧ If A is subnormal in G and A is soluble, then A is contained in some normal soluble subgroup of G[10].

Lemma 1.2 Assume that H is s-permutable in G, $X\leqslant G$ and $N\trianglelefteq G$. Then:

① H is subnormal in G[3].

② HN/N is s-permutable in G/N[3].

③ $H\cap X$ is s-permutable in X[3].

④ H/H_G is nilpotent[4].

Lemma 1.3 Suppose that A and B are subgroups of G. Then:

① If H is a p-subgroup of G for some prime p, then H is s-permutable in G if and only if $O^p(G)\leqslant N_G(H)$[4, Lemma A].

② If A and B are both s-permutable in G, so are $\langle A, B\rangle$ and $A\cap B$[4, Corollary 1].

Now we provide some basic properties of partially s-permutable subgroups:

Lemma 1.4 Assume that $H\leqslant G$ and H is partially s-permutable in G, $K\leqslant G$ and $N\trianglelefteq G$.

① If $H\leqslant K$, then H is partially s-permutable in K.

② If $N\leqslant H$, then H/N is partially s-permutable in G/N.

③ If $(|H|, |N|)=1$, then HN/N is partially s-permutable in G/N.

Proof If H is partially s-permutable in G, then by the hypothesis, there exists a subnormal subgroup T of G such that

$$G=HT,\ H\cap T\leqslant H_{sT}.$$

① Obviously, $K\cap T$ is subnormal in K by Lemma 1.1 ② and

$$K=K\cap HT=H(K\cap T).$$

Besides,

$$H\cap(K\cap T)=H\cap T\leqslant H_{sT}\leqslant H_{s(K\cap T)}.$$

Hence, H is partially s-permutable in K.

② It is easy to see that TN/N is subnormal in G/N by Lemma 1.1 ① and

$$G/N=HT/N=(H/N)(TN/N).$$

Besides,

$$H/N\cap TN/N=(H\cap TN)/N=(H\cap T)N/N\leqslant H_{sT}N/N=H_{sT}/N\leqslant (H/N)_{s(TN/N)}.$$

Thus, H/N is partially s-permutable in G/N.

③ Since $(|G:T|, |N|)=1$, we have $N\leqslant T$. Then similarly as in ②, T/N is subnormal in G/N and

$$G/N=HT/N=(HN/N)(T/N).$$

On the other hand,

$$(HN/N)\cap(T/N)=(H\cap T)N/N\leqslant H_{sT}N/N\leqslant (HN/N)_{s(T/N)}.$$

Hence, ③ holds. □

Lemma 1.5[11, Lemma 2.6] Let G be a group. Assume that N is a non-trivial normal subgroup of G and $N\cap\Phi(G)=1$, then the fitting subgroup $F(N)$ of N lies in $Soc(G)$ and therefore $F(N)$ is the direct product of the minimal normal subgroups of G which are contained in $F(N)$.

Lemma 1.6[12, Theorem 2.1.6] If G is p-supersoluble and $O_{p'}(G)=1$, then the Sylow p-subgroup of G is normal in G.

Lemma 1.7[2, Ⅲ 3.5 (Gaschütz)] Let G be a group and $N\trianglelefteq M\trianglelefteq G$ be two normal subgroups of G satisfying $N\leqslant\Phi(G)$. If M/N is nilpotent, so is M.

Lemma 1.8[13, Chap 6, Theorem 3.2] If G is π-separable and $\overline{G}=G/O_{\pi'}(G)$, then $C_{\overline{G}}(O_\pi(\overline{G}))\subseteq O_\pi(\overline{G})$. In particular, if $O_{\pi'}(G)=1$, then $C_G(O_\pi(G))\subseteq O_\pi(G)$.

Lemma 1.9[2, Ⅲ 4.5 (Gaschütz)] Let G be a group, then $F(G)/\Phi(G)=F(G/\Phi(G))$ is a direct product of abelian minimal normal subgroups of $G/\Phi(G)$.

Lemma 1.10[13, Chap 8, Theorem 3.1 (Glauberman-Thompson)] If P is a Sylow p-subgroup of G, p odd, and if $N_G(Z(J(P)))$ has a normal p-complement, so does G (Here, $J(P)$ is the Thompson subgroup of P).

2 Main results

Theorem 2.1 Let p be a prime dividing the order of a group G and L a p-soluble normal subgroup of G such that G/L is p-supersoluble. If there exists a Sylow p-subgroup P of L such that every maximal subgroup of P is partially

s-permutable in G, then G is p-supersoluble.

Proof Assume that the assertion is false and take G as a counterexample with minimal order. We derive a contradiction via the following steps:

① G is p-soluble.

This is obvious since L is p-soluble and G/L is p-supersoluble.

② $O_{p'}(G)=1$.

Suppose that $D:=O_{p'}(G)\neq 1$. We consider $(G/D, LD/D)$. First of all,

$$(G/D)/(LD/D)\cong G/LD\cong (G/L)/(LD/L)$$

is p-supersoluble since G/L is p-supersoluble. Besides, let M/D be a maximal subgroup of PD/D, then we can find a maximal subgroup P_1 of P such that $M=P_1D$. By the hypothesis, P_1 is partially s-permutable in G, then M/D is partially s-permutable in G/D by Lemma 1.4 ③. Hence G/D is p-supersoluble by the choice of G, and so is G, a contradiction.

③ $N=O_p(L)$ is the unique minimal normal subgroup of G contained in L such that $G/O_p(L)$ is p-supersoluble and

$$G=NM=O_p(L)M,$$

where M is a maximal subgroup of G.

Let N be a minimal normal subgroup of G contained in L. Then by ① and ②, N is an elementary abelian p-group, and so $N\leqslant O_p(L)$. On the other hand, obviously, $N\leqslant P$ and P/N is a Sylow p-subgroup of L/N. Let P_1/N be a maximal subgroup of P/N. Then P_1 is a maximal subgroup of P. By the hypothesis, P_1 is partially s-permutable in G, and so P_1/N is also partially s-permutable in G/N by Lemma 1.4 ②. Besides,

$$(G/N)/(L/N)\cong G/L$$

is p-supersoluble. Hence $(G/N, L/N)$ satisfies the hypothesis of our theorem. By the choice of G, G/N is p-supersoluble. Since the class of all p-supersoluble groups is a saturated formation, N is the unique minimal normal subgroup of G contained in L.

If N is contained in all maximal subgroups of G, then $N\leqslant \Phi(G)$ and so G is p-supersoluble, a contradiction. This contradiction shows that there is a maximal subgroup of G, M say, such that

$$G=NM=O_p(L)M,\ O_p(L)\cap M=1.$$

This means that

$$O_p(L)=O_p(L)\cap NM=N(O_p(L)\cap M)=N.$$

Hence ③ holds.

④ $|O_p(L)|>p$ (This is clear by ③).

⑤ Final contradiction.

Let G_p be a Sylow p-subgroup of G containing P. Then by ③,

$$G_p=G_p\cap NM=N(G_p\cap M),$$

and $G_p\cap M<G_p$. This means that we can find a maximal subgroup G_1 of G_p such that $G_p\cap M\leqslant G_1$. Setting $P_1=P\cap G_1$. Then

$$|P:P_1|=|P:P\cap G_1|=|G_1P:G_1|=|G_p:G_1|=p,$$

and so P_1 is a maximal subgroup of P. Since $P=G_p\cap L\trianglelefteq G_p$, we have

$$P_1=P\cap G_1=(G_p\cap L)\cap G_1=L\cap G_1\trianglelefteq G_p.$$

By the hypothesis, P_1 is partially s-permutable in G. Hence, there is a subnormal subgroup T of G such that

$$G=P_1T,\ P_1\cap T\leqslant (P_1)_{sT}.$$

Since $|G:T|$ is a power of p, we get $O^p(G)\leqslant T$ by Lemma 1.1 ⑦. We know that $G/O^p(G)$ is a p-group, and so it is p-supersoluble. Therefore

$$G/N\cap O^p(G)\lesssim G/N\times G/O^p(G)$$

is also p-supersoluble. By the minimality of N, we get $N\leqslant O^p(G)\leqslant T$. Now we try to derive a contradiction. Firstly, we assume that $P_1\cap T=1$. Then $P_1\cap N=1$, too. However, this means that

$$|N|=|N:P_1\cap N|=|NP_1:P_1|=|P:P_1|=p,$$

contradicts ④.

Hence we can assume that $P_1\cap T\neq 1$. This means that

$$1\neq P_1\cap T=(P_1)_{sT}\cap T$$

is s-permutable in T by Lemma 1.3 ②. Then by Lemma 1.2 ① and Lemma 1.1 ⑥, it is easy to see that

$$P_1\cap T\leqslant O_p(L)=N.$$

On the other hand, taking G_q as an arbitrary Sylow q-subgroup of G, where q is a prime divisor of $|G|$

with $q\neq p$. Since $O^p(G)\leqslant T$, $G_q\leqslant T$. Hence
$$(P_1\cap T)G_q=G_q(P_1\cap T)$$
is a subgroup of G. Therefore
$$P_1\cap T=N\cap(P_1\cap T)G_q\trianglelefteq(P_1\cap T)G_q.$$
This means that $G_q\leqslant N_G(P_1\cap T)$. By the arbitrarity of q, $O^p(G)\leqslant N_G(P_1\cap T)$. Hence
$$N=(P_1\cap T)^G=(P_1\cap T)^{G_pO^p(G)}=$$
$$(P_1\cap T)^{G_p}\leqslant(P_1)^{G_p}=P_1.$$
However, this means that
$$P=P\cap G_p=P\cap NG_1=$$
$$N(P\cap G_1)=P_1,$$
a contradiction. The final contradiction completes the proof. □

If we choose L to be some special subgroups of G, for example, G or $G^{\mathcal{U}_p}$, then we can get some corollaries from Theorem 2.1.

Corollary 2.1 Let P be a Sylow p-subgroup of a p-soluble group G, where p is a fixed prime dividing the order of G. If every maximal subgroup of P is partially s-permutable in G, then G is p-supersoluble.

Corollary 2.2 A p-soluble group G is p-supersoluble if all maximal subgroups of any Sylow p-subgroup of $G^{\mathcal{U}_p}$ are partially s-permutable in G.

Remark 2.1 We point out that the assumption that L is p-soluble in Theorem 2.1 cannot be omitted. Taking A_5, the alternating group of degree 5, as an example. Obviously, the maximal subgroups of the Sylow 5-subgroups of L are trivial, and so are partially s-permutable in G. However, G is not p-nilpotent, and hence not p-supersoluble.

Now we use the p-fitting subgroup $F_p(G)$ ($F_p(G)=O_{p'p}(G)$) of a group G to describe the structure of G.

Theorem 2.2 Let p be a fixed prime dividing the order of G and L a p-soluble normal subgroup of G such that G/L is p-supersoluble. If all maximal subgroups of $F_p(L)$ containing $O_{p'}(L)$ are partially s-permutable in G, then G is p-supersoluble.

Proof Suppose that the assertion is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps:

① $O_{p'}(L)=1$.

Suppose that $O_{p'}(L)\neq 1$, then
$$(G/O_{p'}(L))/(L/O_{p'}(L))\cong G/L$$
is p-supersoluble. Obviously, $O_{p'}(L/O_{p'}(L))=1$, and so
$$F_p(L/O_{p'}(L))=F_p(L)/O_{p'}(L).$$
Now let $M/O_{p'}(L)$ be a maximal subgroup of $F_p(L/O_{p'}(L))$. Then M is a maximal subgroup of $F_p(L)$, which contains $O_{p'}(L)$, respectively. By the hypothesis, M is partially s-permutable in G, hence by Lemma 1.4 ②, $M/O_{p'}(L)$ is partially s-permutable in $G/O_{p'}(L)$, too. Thus $G/O_{p'}(L)$ is p-supersoluble by the choice of G, and so G is also p-supersoluble, a contradiction. Hence ① holds.

② $L\cap\Phi(G)=1$.

We put $R=L\cap\Phi(G)$ and suppose that it is not trivial. By Lemma 1.7 and ①, we have that
$$F(L/R)=F(L)/R=O_p(L)/R.$$
Putting $K/R=O_{p'}(L/R)$ and letting S be a Hall p'-subgroup of K, then $K=SR$. Hence, by Frattini argument,
$$G=KN_G(S)=SRN_G(S)=N_G(S),$$
so $S\trianglelefteq G$. This means that $S=1$ and so $O_{p'}(L/R)=1$. Therefore
$$F_p(L/R)=O_p(L/R)=$$
$$O_p(L)/R=F_p(L)/R.$$
Now let P_1/R be a maximal subgroup of $F_p(L/R)$, then P_1 is maximal in $F_p(L)$. By the hypothesis, it is partially s-permutable in G. Then P_1/R is also partially s-permutable in G/R by Lemma 1.4 ②. By the minimal choice of G, G/R is p-supersoluble and so is G, a contradiction.

③ $G/F_p(L)$ is p-supersoluble.

Since L is p-soluble and $O_{p'}(L)=1$, we have
$$C_L(O_p(L))\leqslant O_p(L)$$
by Lemma 1.8. By ②, $\Phi(L)=1$. This implies that $F(L)=F_p(L)=O_p(L)$ is a nontrivial elementary abelian p-group by Lemma 1.9. Thus $C_L(F(L))=F(L)$. On the other hand, in view of Lemma 1.5 and ②, $F_p(L)=N_1\times N_2\times\cdots\times N_r$, where N_i is a minimal normal subgroup of G,

where $1\leqslant i\leqslant r$. Let N_i be any of them. We prove that the hypothesis holds for $(G/N_i, L/N_i)$. First,

$$(G/N_i)/(L/N_i)\cong G/L$$

is p-supersoluble. Besides, by ①, $O_{p'}(L/N_i)=1$. Finally, obviously,

$$F_p(L/N_i)=O_p(L/N_i)=O_p(L)/N_i=F_p(L)/N_i.$$

Let H/N_i be a maximal subgroup of $F_p(L/N_i)$, then H is also a maximal subgroup of $F_p(L)$. By the hypothesis, H is partially s-permutable in G. Then by Lemma 1.4 ②, H/N_i is also partially s-permutable in G/N_i. Hence, $(G/N_i, L/N_i)$ satisfies the hypothesis of the theorem, and so G/N_i is p-supersoluble by the minimal choice of G. Since the class of all p-supersoluble groups is a saturated formation, we get $i=1$ and so $G/F_p(L)$ is p-supersoluble. Besides, $F_p(L)=O_p(L)$ is the only minimal normal subgroup of G contained in L.

④ $|F_p(L)|=|O_p(L)|=p$.

Putting $F_p(L)=O_p(L)=P$. Then there is a maximal subgroup of G, M say, such that $G=PM$. Let G_p be a Sylow p-group of G containing P. Then

$$G_p=G_p\cap PM=P(G_p\cap M)$$

and

$$G_p\cap M<G_p.$$

This means that we can find a maximal subgroup G_1 of G_p such that $G_p\cap M\leqslant G_1$. Setting $P_1=P\cap G_1$. Then

$$|P:P_1|=|P:P\cap G_1|=|G_1P:G_1|=|G_p:G_1|=p,$$

and so P_1 is a maximal subgroup of P. Since $P\trianglelefteq G_p$, we have $P_1=P\cap G_1\trianglelefteq G_p$. If $P_1=1$, then we are done. Hence we assume that $P_1\neq 1$. By the hypothesis, P_1 is partially s-permutable in G. So, there is a subnormal subgroup T of G such that

$$G=P_1T,\ P_1\cap T\leqslant (P_1)_{sT}.$$

Since $|G:T|$ is a power of p, we get $O^p(G)\leqslant T$ by Lemma 1.1 ⑦. We know that $G/O^p(G)$ is a p-group, and so it is p-supersoluble. By ③,

$$G/(P\cap O^p(G))\lesssim G/P\times G/O^p(G)$$

is also p-supersoluble. By the minimality of P, we get $P\leqslant O^p(G)\leqslant T$. This means that $T=G$ is the only supplement of P_1 in G, and so P_1 is s-permutable in G. By Lemma 1.3 ①, $O^p(G)\leqslant N_G(P_1)$. Hence

$$P=(P_1)^G=(P_1)^{G_pO^p(G)}=(P_1)^{G_p}=P_1,$$

a contradiction. The contradiction shows that $P_1=1$ and so ④ holds.

⑤ Final contradiction.

By ③ and ④, one can easily see that G is p-supersoluble, a contradiction. □

Corollary 2.3 Let G be a p-soluble group, where p is a fixed divisor of $|G|$. If all maximal subgroups of $F_p(G)$ containing $O_{p'}(G)$ are partially s-permutable in G, then G is p-supersoluble.

Corollary 2.4 Let G be a p-soluble group, where p is a fixed divisor of $|G|$. If all maximal subgroups of $F_p(G^{\mathcal{U}_p})$ containing $O_{p'}(G^{\mathcal{U}_p})$ are partially s-permutable in G, then G is p-supersoluble.

Remark 2.2 The hypothesis that L is p-soluble in Theorem 2.2 cannot be omitted, either. Considering the same example of the group $G=A_5$ as in Remark 2.1. Then clearly, the maximal subgroups of any Sylow 5-subgroup of $F_5(L)$ (since $F_5(G)=1$) are trivial, and thus partially s-permutable in G. However, G is not 5-supersoluble.

Theorem 2.3 Let p be an odd prime dividing the order of G and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and all maximal subgroups of P are partially s-permutable in G, then G is p-nilpotent.

Proof We suppose that the conclusion is false and take G as a counterexample of minimal order. Then:

① If M is a proper subgroup of G satisfying $P\leqslant M<G$, then M is p-nilpotent.

Firstly, it is obvious to see that $N_M(P)\leqslant N_G(P)$, and so it is p-nilpotent. Besides by Lemma 1.4 ①, all maximal subgroups of P are partially s-permutable in M. Hence M satisfies the hypothesis of our theorem. The minimal choice of G implies that M is p-nilpotent.

② $O_{p'}(G)=1$.

Suppose that this is false. We consider the quotient group $G/O_{p'}(G)$. It is easy to see that $PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of

$G/O_{p'}(G)$. By Lemma 1.4 ③, we can see that all maximal subgroups of $PO_{p'}(G)/O_{p'}(G)$ are partially s-permutable in $G/O_{p'}(G)$. Since

$$N_{G/O_{p'}(G)}(PO_{p'}(G)/O_{p'}(G)) = N_G(P)O_{p'}(G)/O_{p'}(G)$$

is p-nilpotent, $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. The minimal choice of G yields that $G/O_{p'}(G)$ is p-nilpotent and so is G, a contradiction.

③ $O_p(G) \neq 1$.

Put $J(P)$ as the Thompson subgroup of P. Then $N_G(P) \leqslant N_G(Z(J(P))) \leqslant G$. If $N_G(Z(J(P))) < G$, then in view of ①, $N_G(Z(J(P)))$ is p-nilpotent and so G is p-nilpotent by Lemma 1.10, a contradiction. Hence $N_G(Z(J(P))) = G$. This means that $Z(J(P))$ is a normal p-subgroup of G, and so $1 < O_p(G) < G$.

④ G is p-soluble.

By ③, we can easily see that $G/O_p(G)$ satisfies the hypothesis of the theorem. Hence $G/O_p(G)$ is p-nilpotent by the minimal choice of G. Therefore it is also p-soluble and so is G.

⑤ Final contradiction.

Applying Corollary 2.1 and ④, G is p-supersoluble. Then by Lemma 1.6, P is normal in G. Therefore $G = N_G(P)$ is p-nilpotent, a contradiction. □

Corollary 2.5 Let p be a prime dividing $|G|$ and L a normal subgroup of G such that G/L is p-nilpotent. Suppose that there is a Sylow p-subgroup P of L such that all maximal subgroups of P are partially s-permutable in G and $N_G(P)$ is p-nilpotent. Then G is p-nilpotent.

Proof It is obvious that $N_L(P)$ is p-nilpotent. Besides all maximal subgroups of P are partially s-permutable in L by Lemma 1.4 ①. By Theorem 2.3, L is p-nilpotent. Let $L_{p'}$ be the normal Hall p'-subgroup of L. Then $L_{p'}$ is also normal in G. Suppose that $L_{p'} \neq 1$. Now we consider the quotient group $G/L_{p'}$. Firstly,

$$(G/L_{p'})/(L/L_{p'}) \cong G/L$$

is p-nilpotent. Besides, it is easy to see that every maximal subgroup of $PL_{p'}/L_{p'}$ is partially s-permutable in $G/L_{p'}$ by Lemma 1.4 ③. Finally,

$$N_{G/L_{p'}}(PL_{p'}/L_{p'}) = N_G(P)L_{p'}/L_{p'}$$

is p-nilpotent. This means that $G/L_{p'}$ satisfies the hypothesis of our corollary. By induction, $G/L_{p'}$ is p-nilpotent and so is G. So we can suppose that $L_{p'} = 1$. Then $L = P$ and so $N_G(P) = N_G(L) = G$ is p-nilpotent, as desired. □

Remark 2.3 The assumption that $N_G(P)$ is p-nilpotent in the proof of Theorem 2.3 cannot be omitted. To illustrate this, one can also consider the example of $G = A_5$ with prime 5 as in Remark 2.1.

3 Applications

As we have mentioned in Introduction, a subgroup, be it c-normal, s-permutable or weakly s-permutable, is partially s-permutable. Hence the following results are special cases of our theorems in Section 2.

Corollary 3.1[7, Theorem 3.1] Let p be a prime, G a p-soluble group and H a normal subgroup of G such that G/H is p-supersoluble. If there is a Sylow p-subgroup P of H such that every maximal subgroup of P is c-normal in G, then G is p-supersoluble.

Corollary 3.2[6, Theorem 3.1] Let p be an odd prime dividing the order of G and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and all maximal subgroups of P are c-normal in G, then G is p-nilpotent.

Corollary 3.3[14, Theorem 3.1] Let p be an odd prime dividing the order of G and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and all maximal subgroups of P are weakly s-permutable in G, then G is p-nilpotent.

Corollary 3.4[14, Theorem 3.3] Let G be a p-soluble group and P a Sylow p-subgroup of G. If every maximal subgroup of P is weakly s-permutable in G, then G is p-supersoluble.

Corollary 3.5[14, Theorem 3.5] Let G be a p-soluble group and p a prime divisor of $|G|$. If every maximal subgroup of $F_p(G)$ containing $O_{p'}(G)$ is weakly s-permutable in G, then G is p-supersoluble.

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