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Edge fault-tolerance of super restricted edge-connected Cartesian product graphs

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Abstract: A subset F of edges in a connected graph G is a restricted edge-cut if $G-F$ is disconnected and every component has at least two vertices. A graph G is super restricted edge-connected (super-λ' for short) if every minimum restricted edge-cut of G isolates at least one edge. The edge fault-tolerance $\rho'(G)$ of a super-λ' graph G is the maximum integer m for which $G-F$ is still super-λ' for any subset $F\subseteq E(G)$ with $|F|\leqslant m$. It was shown that

$$\min\{k_1+k_2-1, \upsilon_1 k_2-2k_1-2k_2+1, \upsilon_2 k_1-2k_1-2k_2+1\}\leqslant \rho'(G_1\times G_2)\leqslant k_1+k_2-1,$$

where G_i is a k_i-regular k_i-edge-connected graph of order υ_i with $k_i\geqslant 4$ for each $i\in\{1,2\}$ and $G_1\times G_2$ is the Cartesian product graph of G_1 and G_2. And some sufficient conditions such that $\rho'(G_1\times G_2)=k_1+k_2-1$ were presented.

Key words: graphs; connectivity; fault tolerance; super restricted edge-connected; Cartesian product; regular graphs; networks

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超限制边连通笛卡尔乘积图的边容错性

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摘要：如果 $G-F$ 不连通且每个连通分支至少含有两个顶点，则连通图 G 的边子集 F 称为限制边割. 如果图 G 的每个最小限制边割都孤立 G 中的一条边，则称 G 是超限制边连通的（简称超 λ'）. 对于满足 $|F|\leqslant m$ 的任意子集 $F\subseteq E(G)$，超 λ'图 G 的边容错性 $\rho'(G)$ 是使得 $G-F$ 仍是超 λ'的最大整数 m. 这里给出了

$$\min\{k_1+k_2-1, \upsilon_1 k_2-2k_1-2k_2+1, \upsilon_2 k_1-2k_1-2k_2+1\}\leqslant \rho'(G_1\times G_2)\leqslant k_1+k_2-1,$$

其中，对每个 $i\in\{1,2\}$，G_i 是阶为 υ_i 的 k_i 正则 k_i 边连通图且 $k_i\geqslant 4$，$G_1\times G_2$ 是 G_1 和 G_2 的笛卡尔乘积. 并给

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出了使得 $\rho'(G_1\times G_2)=k_1+k_2-1$ 的一些充分条件.

关键词：图；连通度；容错性；超限制边连通；笛卡尔乘积；正则图；网络

0 Introduction

We follow Ref. [1] for graph-theoretical terminology and notation not defined here. Let $G=(V, E)$ be a simple connected graph, where $V=V(G)$ is the vertex-set of G and $E=E(G)$ is the edge-set of G. It is well known that when the underlying topology of an interconnection network is modeled by a connected graph G, the edge-connectivity $\lambda(G)$ of G is an important measurement for reliability and fault tolerance of the network. In general, the larger $\lambda(G)$ is, the more reliable the network is. Because the connectivity has some shortcomings, Fàbrega and Fiol[2-3] generalized the concept of the edge-connectivity to the h-extra edge-connectivity for a graph.

Definition 0.1 Let $h\geqslant 0$ be an integer. A subset $F\subseteq E(G)$ is an h-extra edge-cut if $G-F$ is disconnected and every component of $G-F$ has more than h vertices. The h-extra edge-connectivity of G, denoted by $\lambda^{(h)}(G)$, is defined as the minimum cardinality of an h-extra edge-cut of G.

Clearly, $\lambda^{(0)}(G)=\lambda(G)$ and $\lambda^{(1)}(G)=\lambda'(G)$ for any graph G, the latter is called the restricted edge-connectivity proposed by Esfahanian and Hakimi[4], who proved that for a connected graph G of order $n\geqslant 4$, $\lambda'(G)$ exists if and only if G is not the star $K_{1,n-1}$. A graph G is said to be $\lambda^{(h)}$-connected if $\lambda^{(h)}(G)$ exists, and to be not $\lambda^{(h)}$-connected otherwise. For a $\lambda^{(h)}$-connected graph G, an h-extra edge-cut F is called a $\lambda^{(h)}$-cut if $|F|=\lambda^{(h)}(G)$.

For two disjoint subsets X and Y in $V(G)$, use $[X,Y]$ to denote the set of edges between X and Y in G. In particular, $E_G(X)=[X,\overline{X}]$ and let $d_G(X)=|E_G(X)|$, where $\overline{X}=V(G)\setminus X$. For a subset $X\subset V(G)$, use $G[X]$ to denote the subgraph of G induced by X. For a $\lambda^{(h)}$-connected graph G, there is certainly a subset $X\subset V(G)$ with $|X|\geqslant h+1$ such that $E_G(X)$ is a $\lambda^{(h)}$-cut and, both $G[X]$ and $G[\overline{X}]$ are connected. Such an X is called a $\lambda^{(h)}$-fragment of G. Let

$$\xi_h(G)=\min\{d_G(X): X\subset V(G),\ |X|=h+1 \text{ and } G[X] \text{ is connected}\}.$$

Clearly, $\xi_0(G)=\delta(G)$, the minimum vertex-degree of G, and $\xi_1(G)=\xi(G)$, the minimum edge-degree of G defined as $\min\{d_G(x)+d_G(y)-2: xy\in E(G)\}$. For a $\lambda^{(h)}$-connected graph G, Whitney's inequality shows $\lambda^{(0)}(G)\leqslant\xi_0(G)$; Esfahanian and Hakimi[4] showed $\lambda^{(1)}(G)\leqslant\xi_1(G)$; Bonsma et al.[5] showed $\lambda^{(2)}(G)\leqslant\xi_2(G)$. For $h\geqslant 3$, Bonsma et al.[5] found that the inequality $\lambda^{(h)}(G)\leqslant\xi_h(G)$ is no longer true in general. Zhang and Yuan[6] proved that $\lambda^{(h)}(G)$ exists and $\lambda^{(h)}(G)\leqslant\xi_h(G)$ for any graph G with $h\leqslant\delta(G)$ except for a class of special graphs.

Definition 0.2 A $\lambda^{(h)}$-connected graph G is super h-extra edge-connected (super-$\lambda^{(h)}$ for short), if every $\lambda^{(h)}$-cut of G isolates at least one connected subgraph of order $h+1$.

Definition 0.3 The edge fault-tolerance of a super-$\lambda^{(h)}$ graph G on super-$\lambda^{(h)}$ property, denoted by $\rho^{(h)}(G)$, is the maximum integer m for which $G-F$ is still super-$\lambda^{(h)}$ for any subset $F\subseteq E(G)$ with $|F|\leqslant m$.

It is clear that the edge fault-tolerance $\rho^{(h)}(G)$ is a measurement for vulnerability of a super-$\lambda^{(h)}$ graph G when its edge failures appear. For convenience, we write λ, λ', λ'', ρ and ρ' for $\lambda^{(0)}$, $\lambda^{(1)}$, $\lambda^{(2)}$, $\rho^{(0)}$ and $\rho^{(1)}$, respectively. In this paper, we only focus on $\rho'(G)$ for a super-λ' Cartesian product graph G.

Very recently, Hong et al.[7] have shown

$$\min\{\delta_1+\delta_2-1,\ \upsilon_1\delta_2-\delta_1-\delta_2-1,\ \upsilon_2\delta_1-\delta_1-\delta_2-1\}\leqslant \rho(G_1\times G_2)\leqslant\delta_1+\delta_2-1,$$

where G_i is a δ_i-edge-connected graph of order υ_i with $\delta(G_i)=\delta_i\geqslant 2$ for each $i\in\{1,2\}$ and $G_1\times G_2$ is the Cartesian product graph of G_1 and G_2. The edge fault-tolerance ρ of super edge-connectivity of three families of networks has been discussed by Wang and Lu[8].

In this paper, we consider ρ' for Cartesian product of regular graphs and obtain

$$\min\{k_1+k_2-1, v_1k_2-2k_1-2k_2+1, v_2k_1-2k_1-2k_2+1\}\leqslant \rho'(G_1\times G_2)\leqslant k_1+k_2-1,$$

where G_i is a k_i-regular k_i-edge-connected graph of order v_i with $k_i\geqslant 4$ for each $i\in\{1,2\}$. We also show that the bounds on $\rho'(G_1\times G_2)$ are best possible and give some sufficient conditions such that $\rho'(G_1\times G_2)=k_1+k_2-1$.

1 Preliminaries

The Cartesian product is a very effective method for constructing a larger graph from several specified graphs, and plays an important role in designing and analysing large-scale interconnection networks[9, Section 2.2].

The Cartesian product of graphs G_1 and G_2 is the graph $G_1\times G_2$ with vertex-set $V(G_1)\times V(G_2)$, two vertices x_1x_2 and y_1y_2, where $x_1, y_1\in V(G_1)$ and $x_2, y_2\in V(G_2)$, being adjacent in $G_1\times G_2$ if and only if either $x_1=y_1$ and $x_2y_2\in E(G_2)$, or $x_2=y_2$ and $x_1y_1\in E(G_1)$. For any $y\in V(G_2)$ (resp. $x\in V(G_1)$), we use G_1^y (resp. G_2^x) to denote the subgraph of $G_1\times G_2$ induced by $V(G_1)\times\{y\}$ (resp. $\{x\}\times V(G_2)$). Clearly, G_1^y (resp. G_2^x) is isomorphic to G_1 (resp. G_2).

To investigate the edge fault-tolerance ρ' for Cartesian product of regular graphs, the following lemmas are needed.

Lemma 1.1[10, Corollary 4] Let G_i be a k_i-regular k_i-edge-connected graph for each $i=1,2$. Then

$$\lambda(G_1\times G_2)=k_1+k_2.$$

The following result gives a necessary and sufficient condition for a graph to be super-λ'.

Lemma 1.2[11, Theorem 2.3] Let G be a λ'-connected graph and $\lambda''(G)$ exists. Then G is super-λ' if and only if $\lambda''(G)>\xi(G)$.

In Ref. [12], Ou obtained a sufficient and necessary condition for the Cartesian product of regular graphs to be super-λ'.

Lemma 1.3[12, Theorem 2.2.1] Let G_i be a k_i-regular k_i-edge-connected graph with $k_i\geqslant 2$ for each $i=1, 2$. Then $G_1\times G_2$ is super-λ' if and only if it is not isomorphic to the Cartesian product of a complete graph and a cycle.

Lemma 1.4 Let G_i be a k_i-regular k_i-edge-connected graph with $k_i\geqslant 4$ for each $i=1, 2$ and $G=G_1\times G_2$. Then $d_{G-F}(X)>\xi(G-F)$ holds for any $F\subseteq E(G)$ and any $X\subseteq V(G)$ such that $|F|\leqslant k_1+k_2-1$, $|X|\geqslant 3$, $G[X]-F$ is connected, and $X\subset V(G_1^{y_0})$ with $y_0\in V(G_2)$ or $X\subset V(G_2^{x_0})$ with $x_0\in V(G_1)$.

Proof Suppose, to the contrary, that there exist a subset F of $E(G)$ and a subset X of $V(G)$ satisfying the specified conditions, but

$$d_{G-F}(X)\leqslant \xi(G-F) \tag{1}$$

By Lemma 1.1, $\lambda(G)=k_1+k_2$. Since

$$|F|\leqslant k_1+k_2-1=\lambda(G)-1,$$

we have $G-F$ is connected and

$$\xi(G-F)\leqslant \xi(G) \tag{2}$$

Since G is (k_1+k_2)-regular, $\xi(G)=2k_1+2k_2-2$. By (1) and (2),

$$2k_1+2k_2-2\geqslant d_{G-F}(X)\geqslant d_G(X)-|F|\geqslant d_G(X)-(k_1+k_2-1).$$

Thus

$$d_G(X)\leqslant 3k_1+3k_2-3 \tag{3}$$

Let $|X|=a$. Without loss of generality, assume that $X\subset V(G_1^{y_0})$ with $y_0\in V(G_2)$. Then

$$d_G(X)=d_{G_1^{y_0}}(X)+|[X,V(G)\backslash V(G_1^{y_0})]|= d_{G_1^{y_0}}(X)+\sum_{z=(x,y_0)\in X} d_{G_2^x}(z)=d_{G_1^{y_0}}(X)+ak_2.$$

Combining this with (3) and noting that $a\geqslant 3$ and $k_2\geqslant 4$, we have

$$d_{G_1^{y_0}}(X)\leqslant 3k_1-(a-3)k_2-3\leqslant 3k_1-4a+9 \tag{4}$$

Since

$$d_{G_1^{y_0}}(X)=\sum_{z\in X}(d_{G_1^{y_0}}(z)-d_{G_1^{y_0}[X]}(z))\geqslant a(k_1-a+1),$$

by (4), we have

$$(a-3)k_1\leqslant a^2-5a+9 \tag{5}$$

Since $G_1^{y_0}$ is isomorphic to G_1, $|X| \geqslant 3$ and

$$X \subset V(G_1^{y_0}),\ d_{G_1^{y_0}}(X) \geqslant \lambda(G_1) = k_1.$$

By (4), $k_1 \leqslant 3k_1 - 4a + 9$ and so $2k_1 \geqslant 4a - 9$. Combining this with (5) and $a \geqslant 3$, we have $(a-1)(2a-9) \leqslant 0$ and so $3 \leqslant a \leqslant 4$. Consider the following two cases.

Case 1 $a = 4$.

Note that $k_1 + k_2 \geqslant 8$. Since G is $(k_1 + k_2)$-regular,

$$d_G(X) = \sum_{z \in X} (d_G(z) - d_{G[X]}(z)) = 4(k_1 + k_2) - 2\,|E(G[X])|.$$

Therefore, $d_G(X)$ is even and $d_G(X) \geqslant 4k_1 + 4k_2 - 12$. If $d_G(X) > 4k_1 + 4k_2 - 11$, then, by $k_1 + k_2 \geqslant 8$, $d_G(X) > 3k_1 + 3k_2 - 3$, a contradiction to (3). Thus

$$4k_1 + 4k_2 - 12 \leqslant d_G(X) \leqslant 4k_1 + 4k_2 - 11.$$

Since $d_G(X)$ is even, we have

$$d_G(X) = 4k_1 + 4k_2 - 12.$$

Thus $G[X] \cong K_4$.

If

$$|F \cap [X, \overline{X}]| < k_1 + k_2 - 2,$$

then, by $k_1 + k_2 \geqslant 8$,

$$d_{G-F}(X) \geqslant d_G(X) - |F \cap [X, \overline{X}]| > 3k_1 + 3k_2 - 10 \geqslant 2k_1 + 2k_2 - 2 = \xi(G) \geqslant \xi(G - F),$$

a contradiction to (1).

If $k_1 + k_2 - 2 \leqslant |F \cap [X, \overline{X}]| \leqslant k_1 + k_2 - 1$, then

$$d_{G-F}(X) \geqslant 3k_1 + 3k_2 - 11.$$

Since

$$|F \cap [X, \overline{X}]| \geqslant k_1 + k_2 - 2 \geqslant 6,$$

$|F \cap E(G[X])| \leqslant 1$ and $G[X] \cong K_4$, there exists an edge in $G[X] - F$, say uv, such that

$$|F \cap [\{u, v\}, \overline{X}]| \geqslant 3.$$

Therefore,

$$\xi(G - F) \leqslant d_G(u) + d_G(v) - 2 - |F \cap [\{u, v\}, \overline{X}]| \leqslant 2(k_1 + k_2) - 2 - 3 < 3k_1 + 3k_2 - 11 \leqslant d_{G-F}(X),$$

a contradiction to (1).

Case 2 $a = 3$.

Since G is $(k_1 + k_2)$-regular,

$$d_G(X) = 3(k_1 + k_2) - 2\,|E(G[X])|.$$

Since $G[X]$ is connected, $2 \leqslant |E(G[X])| \leqslant 3$ and so $d_G(X)$ is equal to $3k_1 + 3k_2 - 6$ or $3k_1 + 3k_2 - 4$.

Subcase 2.1 $d_G(X) = 3k_1 + 3k_2 - 6$.

In this case, $G[X] \cong K_3$. If

$$|F \cap [X, \overline{X}]| < k_1 + k_2 - 4,$$

then

$$d_{G-F}(X) \geqslant d_G(X) - |F \cap [X, \overline{X}]| > 2k_1 + 2k_2 - 2 = \xi(G) \geqslant \xi(G - F),$$

a contradiction to (1). Next suppose that

$$k_1 + k_2 - 4 \leqslant |F \cap [X, \overline{X}]| \leqslant k_1 + k_2 - 1.$$

If

$$k_1 + k_2 - 4 \leqslant |F \cap [X, \overline{X}]| \leqslant k_1 + k_2 - 2,$$

then

$$d_{G-F}(X) \geqslant d_G(X) - |F \cap [X, \overline{X}]| \geqslant 2k_1 + 2k_2 - 4.$$

Since $G[X] - F$ is connected,

$$0 \leqslant |E(G[X]) \cap F| \leqslant 1.$$

Note that

$$|F \cap [X, \overline{X}]| \geqslant k_1 + k_2 - 4 \geqslant 4$$

and $G[X] \cong K_3$. If

$$|E(G[X]) \cap F| = 0,$$

then there exists an edge in $G[X]$, say uv, such that

$$|F \cap [\{u, v\}, \overline{X}]| \geqslant 3.$$

Thus,

$$\xi(G - F) \leqslant d_G(u) + d_G(v) - 2 - |F \cap [\{u, v\}, \overline{X}]| \leqslant 2k_1 + 2k_2 - 2 - 3 < 2k_1 + 2k_2 - 4 \leqslant d_{G-F}(X),$$

a contradiction to (1). If

$$|E(G[X]) \cap F| = 1,$$

say

$$E(G[X]) \cap F = \{uv\},$$

then there exists an edge in $G[X] - uv$, say uw, such that

$$|F \cap [\{u, w\}, \overline{X}]| \geqslant 2.$$

Thus,

$$\xi(G - F) \leqslant d_G(u) + d_G(w) - 2 - |F \cap [\{u, v\}, \overline{X}]| - |\{uv\}| \leqslant 2k_1 + 2k_2 - 2 - 2 - 1 < 2k_1 + 2k_2 - 4 \leqslant d_{G-F}(X),$$

a contradiction to (1).

If

$$|F \cap [X, \overline{X}]| = k_1 + k_2 - 1,$$

then $F \subseteq [X, \overline{X}]$ and

$$d_{G-F}(X) = d_G(X) - |F \cap [X, \overline{X}]| = 2k_1 + 2k_2 - 5.$$

Since $|F \cap [X, \overline{X}]| \geqslant 7$ and $G[X] \cong K_3$, there exists an edge in $G[X]$, say uv, such that

$$|F \cap [\{u, v\}, \overline{X}]| \geqslant 5.$$

Thus,

$$\xi(G-F) \leqslant d_G(u) + d_G(v) - 2 - |F \cap [\{u, v\}, \overline{X}]| \leqslant 2k_1 + 2k_2 - 2 - 5 < 2k_1 + 2k_2 - 5 = d_{G-F}(X),$$

a contradiction to (1).

Subcase 2.2 $d_G(X) = 3k_1 + 3k_2 - 4$.

In this case, $G[X] \cong P_3$ and $F \cap E(G[X]) = \emptyset$. If

$$|F \cap [X, \overline{X}]| < k_1 + k_2 - 2,$$

then

$$d_{G-F}(X) = d_G(X) - |F \cap [X, \overline{X}]| > 2k_1 + 2k_2 - 2 = \xi(G) \geqslant \xi(G-F),$$

a contradiction to (1). Next suppose that

$$k_1 + k_2 - 2 \leqslant |F \cap [X, \overline{X}]| \leqslant k_1 + k_2 - 1.$$

Then

$$d_{G-F}(X) = d_G(X) - |F \cap [X, \overline{X}]| \geqslant 2k_1 + 2k_2 - 3.$$

Since

$$|F \cap [X, \overline{X}]| \geqslant k_1 + k_2 - 2 \geqslant 6$$

and $G[X] \cong P_3$, there exists an edge in $G[X]$, say uv, such that $|F \cap [\{u, v\}, \overline{X}]| \geqslant 3$. Thus,

$$\xi(G-F) \leqslant d_G(u) + d_G(v) - 2 - |F \cap [\{u, v\}, \overline{X}]| \leqslant 2k_1 + 2k_2 - 2 - 3 < 2k_1 + 2k_2 - 3 \leqslant d_{G-F}(X),$$

a contradiction to (1). □

Lemma 1.5 Let G_i be a k_i-regular k_i-edge-connected graph with $k_i \geqslant 4$ for $i = 1, 2$ and $G = G_1 \times G_2$. Then

$$d_{G-F}(X_1 \cup X_2) > \xi(G-F)$$

holds for any $F \subseteq E(G)$ and any $X_1, X_2 \subseteq V(G)$ such that $|F| \leqslant k_1 + k_2 - 1$, $X_1 \cap X_2 = \emptyset$, $|X_1| \geqslant 1$, $|X_2| \geqslant 1$, $|X_1 \cup X_2| \geqslant 3$, $G[X_1 \cup X_2] - F$ is connected, $X_1 \subset V(G_1^{y_1})$ (resp. $X_1 \subset V(G_2^{x_1})$) and $X_2 \subset V(G_1^{y_2})$ (resp. $X_2 \subset V(G_2^{x_2})$) with $\{y_1, y_2\} \subseteq V(G_2)$ (resp. $\{x_1, x_2\} \subseteq V(G_1)$).

Proof Suppose, to the contrary, that there exist a subset F of $E(G)$ and two subsets X_1 and X_2 of $V(G)$ satisfying the specified conditions, but

$$d_{G-F}(X_1 \cup X_2) \leqslant \xi(G-F) \tag{6}$$

By Lemma 1.1, $\lambda(G) = k_1 + k_2$. Since

$$|F| \leqslant k_1 + k_2 - 1 = \lambda(G) - 1,$$

we have $G-F$ is connected and

$$\xi(G-F) \leqslant \xi(G) \tag{7}$$

Since G is $(k_1 + k_2)$-regular, $\xi(G) = 2k_1 + 2k_2 - 2$. By (6) and (7),

$$2k_1 + 2k_2 - 2 \geqslant d_{G-F}(X_1 \cup X_2) \geqslant d_G(X_1 \cup X_2) - |F| \geqslant d_G(X_1 \cup X_2) - (k_1 + k_2 - 1).$$

Thus

$$d_G(X_1 \cup X_2) \leqslant 3k_1 + 3k_2 - 3 \tag{8}$$

Let $|X_1| = a_1$ and $|X_2| = a_2$. Then $a_1, a_2 \geqslant 1$ and $a_1 + a_2 \geqslant 3$. Without loss of generality, assume that $X_1 \subset V(G_1^{y_1})$ and $X_2 \subset V(G_1^{y_2})$ with $\{y_1, y_2\} \subseteq V(G_2)$. Then

$$d_G(X_1 \cup X_2) \geqslant d_{G_1^{y_1}}(X_1) + d_{G_1^{y_2}}(X_2) + |[X_1 \cup X_2, V(G) \setminus (V(G_1^{y_1}) \cup (V(G_1^{y_2}))]| \geqslant d_{G_1^{y_1}}(X_1) + d_{G_1^{y_2}}(X_2) + |X_1 \cup X_2|(k_2 - 1) = d_{G_1^{y_1}}(X_1) + d_{G_1^{y_2}}(X_2) + (a_1 + a_2)(k_2 - 1).$$

Combining this with (8), we obtain

$$d_{G_1^{y_1}}(X_1) + d_{G_1^{y_2}}(X_2) \leqslant 3k_1 - (a_1 + a_2 - 3)(k_2 - 1) \tag{9}$$

Since $a_1 + a_2 \geqslant 3$ and $k_2 \geqslant 4$, by (9) we have

$$d_{G_1^{y_1}}(X_1) + d_{G_1^{y_2}}(X_2) \leqslant 3k_1 - 3(a_1 + a_2) + 9 \tag{10}$$

Note that

$$d_{G_1^{y_1}}(X_1) = \sum_{z \in X_1} (d_{G_1^{y_1}}(z) - d_{G_1^{y_1}[X_1]}(z)) \geqslant a_1(k_1 - a_1 + 1)$$

and $d_{G_1^{y_2}}(X_2) \geqslant a_2(k_1 - a_2 + 1)$. Combining this with (10), we have

$$(a_1 + a_2 - 3)k_1 \leqslant a_1^2 + a_2^2 - 4(a_1 + a_2) + 9 \tag{11}$$

Since $d_{G_1^{y_1}}(X_1) \geqslant \lambda(G_1) = k_1$ and $d_{G_1^{y_2}}(X_2) \geqslant$

$\lambda(G_1)=k_1$, by (10), $2k_1\leqslant 3k_1-3(a_1+a_2)+9$ and so $k_1\geqslant 3(a_1+a_2-3)$. Combining this with (11), we get

$$(a_1+a_2)^2-7(a_1+a_2)+9+a_1a_2\leqslant 0,$$

that is,

$$(a_1+a_2-1)(a_1+a_2-6)+a_1a_2+3\leqslant 0 \quad (12)$$

Since a_1, $a_2\geqslant 1$ and $a_1+a_2\geqslant 3$, by (12), we have

$$(a_1+a_2-1)(a_1+a_2-6)<0$$

and so $3\leqslant a_1+a_2\leqslant 5$.

If $a_1+a_2=5$, then, by (12), we have $a_1a_2\leqslant 1$, a contradiction to $a_1+a_2\geqslant 3$.

If $a_1+a_2=4$, then $\{a_1,a_2\}$ equals $\{1,3\}$ or $\{2,2\}$. Since $G[X_1\cup X_2]$ is a connected subgraph of $G=G_1\times G_2$, $|E(G[X_1\cup X_2])|\leqslant 4$. By

$$k_1+k_2\geqslant 8,$$

$$d_G(X_1\cup X_2)= 4(k_1+k_2)-2|E(G[X_1\cup X_2])|\geqslant 4k_1+4k_2-8>3k_1+3k_2-3,$$

a contradiction to (8).

If $a_1+a_2=3$, then $\{a_1,a_2\}=\{1,2\}$. Since $G[X_1\cup X_2]$ is connected, $G[X_1\cup X_2]$ is isomorphic to P_3. Let $X=X_1\cup X_2$, then

$$d_G(X)=3k_1+3k_2-4.$$

Since $G[X]-F$ is connected, $F\cap E(G[X])=\emptyset$. If

$$|F\cap[X,\overline{X}]|<k_1+k_2-2,$$

then

$$d_{G-F}(X)=d_G(X)-|F\cap[X,\overline{X}]|> 2k_1+2k_2-2=\xi(G)\geqslant\xi(G-F),$$

a contradiction to (6). Next suppose that

$$k_1+k_2-2\leqslant|F\cap[X,\overline{X}]|\leqslant k_1+k_2-1.$$

Then

$$d_{G-F}(X)\geqslant 2k_1+2k_2-3.$$

Since

$$|F\cap[X,\overline{X}]|\geqslant k_1+k_2-2\geqslant 6$$

and $G[X]\cong P_3$, there exists an edge in $G[X]$, say uv, such that $|F\cap[\{u,v\},\overline{X}]|\geqslant 3$. Thus,

$$\xi(G-F)\leqslant d_G(u)+d_G(v)-2-|F\cap[\{u,v\},\overline{X}]|\leqslant 2k_1+2k_2-2-3< 2k_1+2k_2-3\leqslant d_{G-F}(X),$$

a contradiction to (6). □

2 Main results

In this section, we give the lower and upper bounds on $\rho'(G)$ for Cartesian product of regular graphs and show that the bounds are best possible.

Theorem 2.1 Let G_i be a k_i-regular k_i-edge-connected graph of order υ_i with $k_i\geqslant 4$ for $i=1$, 2. Then

$$\min\{k_1+k_2-1,\upsilon_1k_2-2k_1-2k_2+1,\upsilon_2k_1-2k_1-2k_2+1\}\leqslant \rho'(G_1\times G_2)\leqslant k_1+k_2-1.$$

Proof Let $G=G_1\times G_2$. Then G is super-λ' by Lemma 1.3. Note that G is (k_1+k_2)-regular and $k_1+k_2\geqslant 8$. Let F be a set of edges incident with some vertex of degree k_1+k_2. Since $G-F$ is disconnected, $G-F$ is not super-λ'. By definition of $\rho'(G)$, we have

$$\rho'(G)\leqslant k_1+k_2-1.$$

Let

$$m=\min\{k_1+k_2-1,\upsilon_1k_2-2k_1-2k_2+1,\upsilon_2k_1-2k_1-2k_2+1\}.$$

To show that $\rho'(G)\geqslant m$, we only need to show that for any $F\subseteq E(G)$ with $|F|\leqslant m$, $G'=G-F$ is super-λ'. By Lemma 1.1, $\lambda(G)=k_1+k_2$. Since

$$|F|\leqslant m\leqslant k_1+k_2-1=\lambda(G)-1,$$

G' is connected. It is easy to find that G' is not a star, since G' is a spanning subgraph of $G_1\times G_2$ with $\upsilon_i>4$. Thus, G' is λ'-connected. If G' is not λ''-connected, then every λ'-cut of G' isolates one edge and so G' is super-λ'. Next, suppose that G' is λ''-connected. Then, by Lemma 1.2, we need to show $\lambda''(G')>\xi(G')$. Let X be any λ''-fragment of G' such that $|\overline{X}|\geqslant|X|\geqslant 3$. By $d_{G'}(X)=\lambda''(G')$ and the arbitrary of X, it suffices to prove

$$d_{G'}(X)>\xi(G') \quad (13)$$

Since

$$d_{G'}(X)\geqslant d_G(X)-|F|\geqslant d_G(X)-m$$

and $\xi(G')\leqslant\xi(G)$, if $d_G(X)>\xi(G)+m$, then $d_{G'}(X)>\xi(G')$. In the following, we assume that

$$d_G(X)\leqslant\xi(G)+m \quad (14)$$

and prove that (13) holds.

Let

$$I_1=\{x: x\in V(G_1) \text{ and}$$

$G_2^x - [X, \overline{X}]$ is disconnected},

$I_2 = \{y: y \in V(G_2)$ and

$G_1^y - [X, \overline{X}]$ is disconnected}.

Claim 1 $|I_i| < \upsilon_i$ for $i = 1, 2$.

Suppose, without loss of generality, that $|I_1| = \upsilon_1$. Then $G_2^x - [X, \overline{X}]$ is disconnected for any $x \in V(G_1)$ and so $d_G(X) \geqslant \upsilon_1 \lambda(G_2) = \upsilon_1 k_2$. Combining this with (14), we have

$$\upsilon_1 k_2 \leqslant \xi(G) + m \leqslant 2k_1 + 2k_2 - 2 + \upsilon_1 k_2 - 2k_1 - 2k_2 + 1 = \upsilon_1 k_2 - 1,$$

a contradiction.

Claim 2 $|I_i| \geqslant 1$ for $i = 1, 2$.

Suppose, without loss of generality, that $|I_1| = 0$. Then $G_2^x - [X, \overline{X}]$ is connected for any $x \in V(G_1)$. By Claim 1, there exists at least one vertex $y \in V(G_2)$ such that $G_1^y - [X, \overline{X}]$ is connected. Thus we have that $G - [X, \overline{X}]$ is connected, a contradiction.

Claim 3 $|I_1| \leqslant 2$ or $|I_2| \leqslant 2$.

Suppose that $|I_1| \geqslant 3$ and $|I_2| \geqslant 3$. Then

$$d_G(X) \geqslant 3k_1 + 3k_2 = k_1 + k_2 - 1 + 2k_1 + 2k_2 - 2 + 3 \geqslant m + \xi(G) + 3,$$

a contradiction to (14).

By Claim 2 and Claim 3, we assume, without loss of generality, that $1 \leqslant |I_2| \leqslant 2$. Since $G'[X]$ and $G'[\overline{X}]$ are connected, $G[X]$ and $G[\overline{X}]$ are also connected. We consider the following two cases.

Case 1 $|I_2| = 1$.

Let $I_2 = \{y_0\}$. Then $G_1^{y_0} - [X, \overline{X}]$ is disconnected and $G_1^y - [X, \overline{X}]$ is connected for each $y \in V(G_2) \setminus \{y_0\}$. By Claim 1, there exists at least one vertex $x \in V(G_1)$ such that $G_2^x - [X, \overline{X}]$ is connected. Thus $G_2^x \cup (\bigcup_{y \in V(G_2) \setminus \{y_0\}} G_1^y) - [X, \overline{X}]$ is connected and it is completely contained in $G[\overline{X}]$ since $|\overline{X}| \geqslant |X|$. It follows that $X \subseteq V(G_1^{y_0})$. Since $G_1^{y_0} - [X, \overline{X}]$ is disconnected, $X \subset V(G_1^{y_0})$. Note that $G'[X] = G[X] - F$ is connected. By Lemma 1.4, we have $d_{G-F}(X) > \xi(G-F)$. Hence, (13) is proved.

Case 2 $|I_2| = 2$.

Let $I_2 = \{y_1, y_2\}$. Then $G_1^{y_j} - [X, \overline{X}]$ is disconnected for each $j = 1, 2$ and $G_1^y - [X, \overline{X}]$ is connected for each $y \in V(G_2) \setminus \{y_1, y_2\}$. By Claim 1, there exists at least one vertex $x \in V(G_1)$ such that $G_2^x - [X, \overline{X}]$ is connected. Thus $G_2^x \cup (\bigcup_{y \in V(G_2) \setminus \{y_1, y_2\}} G_1^y) - [X, \overline{X}]$ is connected and it is completely contained in $G[\overline{X}]$ since $|\overline{X}| \geqslant |X|$. It follows that $X \subseteq V(G_1^{y_1} \cup G_1^{y_2})$. Let $X_1 = X \cap V(G_1^{y_1})$ and $X_2 = X \cap V(G_1^{y_2})$. Then $X = X_1 \cup X_2$ with $X_1 \cap X_2 = \emptyset$ and $|X_1 \cup X_2| \geqslant 3$. Since $G_1^{y_j} - [X, \overline{X}]$ is disconnected, $|X_j| \geqslant 1$ and $X_j \subset V(G_1^{y_j})$ for each $j = 1, 2$. Note that $G'[X_1 \cup X_2] = G[X_1 \cup X_2] - F$ is connected. By Lemma 1.5, we have

$$d_{G-F}(X_1 \cup X_2) > \xi(G - F).$$

Hence, (13) is proved. □

The following result presents some sufficient conditions for ρ' of a Cartesian product graph to attain the maximum value. This result also shows that the lower and upper bounds on ρ' given in Theorem 2.1 are best possible.

Corollary 2.2 Let G_i be a k_i-regular k_i-edge-connected graph of order υ_i with $k_i \geqslant 4$ for $i = 1, 2$. Then

$$\rho'(G_1 \times G_2) = k_1 + k_2 - 1,$$

if one of the following conditions holds:

① G_1 and G_2 are not complete graphs;

② $k_1 + k_2 \geqslant 10$;

③ $k_1 = 4$, $k_2 = 5$ and G_2 is not complete.

Proof By Theorem 2.1, it suffices to prove that

$$\upsilon_1 k_2 - 2k_1 - 2k_2 + 1 \geqslant k_1 + k_2 - 1,$$
$$\upsilon_2 k_1 - 2k_1 - 2k_2 + 1 \geqslant k_1 + k_2 - 1,$$

that is

$$\upsilon_1 k_2 \geqslant 3k_1 + 3k_2 - 2 \tag{15}$$
$$\upsilon_2 k_1 \geqslant 3k_1 + 3k_2 - 2 \tag{16}$$

If G_1 is not complete, then $\upsilon_1 \geqslant k_1 + 2$. Combining this with $k_1, k_2 \geqslant 4$, we have

$$\upsilon_1 k_2 \geqslant (k_1 + 2)k_2 = 3k_1 + 3k_2 - 2 + (k_1 - 1)(k_2 - 3) - 1 > 3k_1 + 3k_2 - 2.$$

Similarly, if G_2 is not complete, then (16) holds. Therefore, (15) and (16) hold under

Condition ①.

If $k_1+k_2\geqslant 10$, then $(k_1-2)(k_2-3)\geqslant 4$ and $(k_1-3)(k_2-2)\geqslant 4$ and so

$$\begin{aligned} \upsilon_1 k_2 &\geqslant (k_1+1)k_2 = \\ &3k_1+3k_2-2+(k_1-2)(k_2-3)-4\geqslant \\ &3k_1+3k_2-2 \end{aligned} \tag{17}$$

By substituting υ_2 for υ_1 and exchanging k_1 with k_2 in (17), we can get (16). Therefore, (15) and (16) hold under Condition ②.

If $k_1=4$ and $k_2=5$, then

$$\upsilon_1 k_2 \geqslant (k_1+1)k_2 = 3k_1+3k_2-2.$$

If G_2 is not complete, then (16) holds. Thus, (15) and (16) hold under Condition ③. □

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