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Two results on the signless Laplacian matrix of a graph

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Abstract: Let G be a simple connected graph with n vertices and m edges and $Q(G)$ its signless Laplacian matrix. The spectral radius and the entries of the principal vector of $Q(G)$ were investigated.

Key words: graph; signless Laplacian matrix; spectral radius; principal eigenvector

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关于图的无符号拉普拉斯矩阵的两个结果

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摘要：设 G 是具有 n 个顶点和 m 条边的简单无向图，$Q(G)$是图 G 的无符号拉普拉斯矩阵. 讨论了 $Q(G)$的谱半径和与谱半径对应的特征向量的分量.

关键词：图；无符号拉普拉斯矩阵；谱半径；主特征向量

0 Introduction

Let $G=(V,E)$ be a simple connected graph with order n and size m, where $V=\{v_1,v_2,\cdots,v_n\}$ and $E=\{e_1,e_2,\cdots,e_m\}$. The adjacency matrix of G is denoted by $A(G)=(a_{ij})$, where $a_{ij}=1$ if v_i and v_j are adjacent and $a_{ij}=0$ otherwise. The degree diagonal matrix of G is denoted by $D(G)=\mathrm{diag}(d_1(G),d_2(G),\cdots,d_n(G))$, where $d_i(G)$ is the degree of v_i. The signless Laplacian matrix of G is $Q(G)=D(G)+A(G)$, and its largest eigenvalue is called the spectral radius of $Q(G)$, denoted by $\rho(Q)$.

Motivated by Refs. [1-2], we will give two results on the signless Laplacian matrix $Q(G)$. The first one is to show which graph has the largest spectral radius of $Q(G)$, among all simple connected graphs with n vertices and m edges; and the second one is on the upper bounds of the entries of the corresponding eigenvector of the spectral radius of $Q(G)$.

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1 Main results

Let $\phi(n, m)$ be the set of signless Laplacian matrices of the simple connected graphs having n vertices and m edges. Let $\phi^*(n, m)$ denote the subset of $\phi(n,m)$ consisting of those matrices $Q=[b_{ij}]$ such that whenever $i<j$ and $b_{ij}=1$, then $b_{kl}=1$ for all $k<l$ with $k\leqslant i$ and $l\leqslant j$. For instance, the matrix

$$\begin{pmatrix} 5 & 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 & 1 & 0 \\ 1 & 1 & 4 & 1 & 1 & 0 \\ 1 & 1 & 1 & 4 & 1 & 0 \\ 1 & 1 & 1 & 1 & 4 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is in $\phi^*(6,11)$.

Let $g(n, m)=\max\{\rho(Q): Q\in\phi(n, m)\}$ and $g^*(n,m)=\max\{\rho(Q): Q\in\phi^*(n,m)\}$.

Theorem 1.1 Let $Q\in\phi(n,m)$. Then $\rho(Q)\leqslant g^*(n,m)$, where equality holds only if there exists a permutation matrix P such that $PQP^{\mathrm{T}}\in\phi^*(n,m)$. In particular, $g^*(n,m)=g(n,m)$.

Proof Let $Q=(q_{ij})\in\phi(n,m)$ with $\rho(Q)=g(n,m)$. From the theory of nonnegative matrices we know that $\rho(Q)$ has an associated positive eigenvector $x=(x_1,\cdots,x_n)^{\mathrm{T}}$ with $x^{\mathrm{T}}x=1$. Because the simultaneous row and column operations do not change the eigenvalues, we may assume that $x_1\geqslant x_2\geqslant\cdots\geqslant x_n>0$. Suppose $Q\notin\phi^*(n,m)$. First suppose there exist integers s and t with $s<t$ such that $q_{st}=0$ and $q_{s,t+1}=1$. Let $C=(c_{ij})$ be the matrix obtained from Q by switching q_{st} and $q_{s,t+1}$ and switching q_{ts} and $q_{t+1,s}$. Then

$$x^{\mathrm{T}}Cx-x^{\mathrm{T}}Qx=2x_s(x_t-x_{t+1})+(x_t^2-x_{t+1}^2)\geqslant 0.$$

We know that Q is nonnegative symmetrical. From $\rho(Q)=\max x^{\mathrm{T}}Qx$ it follows that $\rho(C)\geqslant\rho(Q)$ with strict inequality if $x_s(x_t-x_{t+1})+(x_t^2-x_{t+1}^2)\neq 0$. Since $\rho(Q)=g(n,m)$, we conclude that

$$x_s(x_t-x_{t+1})+(x_t^2-x_{t+1}^2)=0.$$

Since $x_s\neq 0$, then $x_t=x_{t+1}$ and $x^{\mathrm{T}}Cx=\rho(Q)$. Hence $Cx=\rho(Q)x$. But then

$$\rho(Q)(x_t)=(Cx)_t=$$
$$(Qx)_t+x_s+x_t>(Qx)_t=\rho(Q)x_t,$$

a contradiction.

Now suppose there exist integers s and t with $s+1<t$ such that $q_{st}=0$ and $q_{s+1,t}=1$. Let $C=(c_{ij})$ be the matrix obtained from Q by switching q_{st} and $b_{s+1,t}$ and switching q_{ts} and $q_{t,s+1}$. Then

$$x^{\mathrm{T}}Cx-x^{\mathrm{T}}Qx=2x_t(x_s-x_{s+1})+(x_s^2-x_{s+1}^2)\geqslant 0.$$

It follows as above that $Cx=\rho(Q)x$, and

$$2x_t(x_s-x_{s+1})+(x_s^2-x_{s+1}^2)=0.$$

Thus $x_s=x_{s+1}$, and then

$$\rho(Q)x_s=(Cx)_s=$$
$$(Qx)_s+x_s+x_t>(Qx)_s=\rho(Q)x_s.$$

This results in a contradiction as above, and the theorem follows. □

Since G is a connected graph, the signless Laplacian matrix Q(G) is a nonnegative irreducible symmetric matrix. By the well-known Perron-Frobenius Theorem on the nonegative matrices, we know that the matrix Q(G) has an unit positive eigenvector $x=(x_1,x_2,\cdots,x_n)^{\mathrm{T}}$ corresponding to the eigenvalue $\rho(G)$. We call the unique positive vector x the principal eigenvector of Q(G).

Theorem 1.2 Let $Q=(q_{ij})$ be the signless Laplacian matrix of a simple connected graph G with n vertices. Let $x=(x_1,x_2,\cdots,x_n)^{\mathrm{T}}$ be the principal eigenvector of Q corresponding to spectral radius $\rho=\rho(Q)$. Then, for $1\leqslant i\leqslant n$, we have

$$x_i<\sqrt{\frac{\rho}{4(\rho-d_i)}} \tag{1}$$

Proof By the AM-GM inequalities,

$$\rho=x^{\mathrm{T}}Qx=\sum_{i\neq j}x_iq_{ij}x_j+\sum_{i=1}^n d_ix_i^2=$$
$$\sum_{i<j}2x_iq_{ij}x_j+\sum_{i=1}^n d_ix_i^2\leqslant$$
$$\sum_{i<j}(x_i^2+x_j^2)q_{ij}+\sum_{i=1}^n d_ix_i^2=$$
$$2\sum_{i=1}^n d_ix_i^2 \tag{2}$$

It implies that

$$\sum_{i=1}^n d_ix_i^2\geqslant\frac{\rho}{2} \tag{3}$$

Since

$$Qx = \rho x, \sum_{j=1}^{n} q_{ij} x_j = \rho x_i, \ i = 1, 2, \cdots, n,$$

we have

$$\rho x_i^2 = \sum_{j=1}^{n} q_{ij} x_i x_j = \sum_{j \neq i} q_{ij} x_i x_j + d_i x_i^2 \quad (4)$$

Notice that $q_{ij} = q_{ji} \geqslant 0$ and $x_i > 0$. Without loss of generality, we consider x_1, hence

$$\begin{aligned}
&\rho x_1^2 - \rho \sum_{i=2}^{n} x_i^2 = \\
&\sum_{j=2}^{n} q_{1j} x_1 x_j + d_1 x_1^2 - \\
&\sum_{i=2}^{n} \Big(\sum_{j \neq i}^{n} q_{ij} x_i x_j + d_i x_i^2 \Big) = \\
&\sum_{i=2}^{n} q_{1i} x_1 x_i + d_1 x_1^2 - \\
&\sum_{i=2}^{n} \Big(\sum_{j \neq i} q_{ij} x_i x_j + d_i x_i^2 \Big) = \\
&\sum_{i=2}^{n} \Big(q_{1i} x_1 x_i - \sum_{j \neq i} q_{ij} x_i x_j \Big) + d_1 x_1^2 - \sum_{i=2}^{n} d_i x_i^2 = \\
&\sum_{i=2}^{n} \Big(- \sum_{j=2, j \neq i}^{n} q_{ij} x_i x_j \Big) + d_1 x_1^2 - \sum_{i=2}^{n} d_i x_i^2 \leqslant \\
&d_1 x_1^2 - \sum_{i=2}^{n} d_i x_i^2 \quad (5)
\end{aligned}$$

By (4), it follows that

$$\rho x_1^2 - \rho(1 - x_1^2) \leqslant 2 d_1 x_1^2 - \frac{\rho}{2} \quad (6)$$

So we have

$$x_1 \leqslant \sqrt{\frac{\rho}{4(\rho - d_1)}}.$$

As for x_i, we can conclude the same conclusion as above, which implies (1).

It is obvious that if the equalities in (4), (5) and (6) hold, then we must have

$$x_1 = x_2 = \cdots = x_n$$

and

$$\sum_{i=2}^{n} \Big(- \sum_{j=2, j \neq i}^{n} q_{ij} x_i x_j \Big) = 0.$$

It is easy to see this will never happen. Then (5) and hence (1) is a strict inequality. □

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