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Some energy properties of Yang-Mills connections

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Abstract: E is a vector bundle over a compact Riemannian manifold $M=M^n$, $n\geqslant 4$, and A is a Yang-Mills connection with $L^{\frac{n}{2}}$ curvature F_A on E. Through a mean value inequality of the density $|F_A|^{\frac{n}{2}}$, an energy concentrate principle of sequences of solutions that have bounded energy is proved. Unless E is a flat bundle, the energy must be bounded from below by some positive constant.

key words: Yang-Mills connection; energy concentrate; energy gap

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关于杨-米尔斯联络的一些能量性质

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摘要:$M=M^n$ 是一个紧致的黎曼流形,E 是 M 上的向量丛,A 是 E 上具有 $L^{\frac{n}{2}}$ 曲率的杨-米尔斯联络.通过证明一个关于$|F_A|^{\frac{n}{2}}$的平均值不等式,得到了一列能量有限解具有能量集中的性质.还得到 E 是一个平坦丛,否则能量一定具有非零下界.

关键词:杨-米尔斯联络;能量集中;能量间隙

0 Introduction

M is a compact n-dimensional Riemannian manifold, E is a vector bundle of rank r over M with structure group G, where G is a compact Lie group. And A is a connection on E, whose Yang-Mills energy is

$$YM(A) := \| F_A \|^2,$$

where F_A is the curvature of A, $\|\cdot\|$ denotes the L^2 norm. The critical points of $YM(A)$ are called Yang-Mills connections, which satisfy the Yang-Mills equation:

$$d_A^* F_A = 0 \tag{1}$$

We consider energy gap phenomena and energy concentrate phenomena about the Yang-Mills connections. The energy gap phenomena is considered by Bourguignon and Lawson in Ref. [2]

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firstly. In Ref. [2, Theorem C], they proved if the curvature of any Yang-Mills connection which is over $S^n(n\geqslant 3)$, satisfies the pointwise estimate

$$F^2 = -\operatorname{tr}(F_{\mu\lambda}F^{\mu\lambda}) < n(n-1)/2 \qquad (2)$$

the connection is flat. In Ref. [4], Gerhardt considered a compact Riemannian manifold, M, with a metric which satisfies the condition

$$R_{\alpha\beta}\Lambda^{\alpha}_{\lambda}\Lambda^{\beta\lambda} - \frac{1}{2}R_{\alpha\beta\mu\lambda}\Lambda^{\alpha\beta}\Lambda^{\mu\lambda} \geqslant c_0\Lambda_{\alpha\beta}\Lambda^{\alpha\beta} \qquad (3)$$

for all skew-symmetric $\Lambda_{\alpha\beta}\in T^{0,2}(M)$, where $R_{\alpha\beta}$ is Ricci curvature tensor, $R_{\alpha\beta\mu\lambda}$ is Riemann curvature tensor, and $c_0>0$. Then he proved the following theorem.

Theorem 0.1[4, Theorem1.2] Let M be a compact manifold. When condition (3) and $c_0>0$ holds, the Yang-Mills connections over M with compact, semi-simple Lie group either are flat or satisfy

$$\left(\int_M |F|^{\frac{n}{2}}\right)^{\frac{2}{n}} \geqslant k_0 \qquad (4)$$

for some constant $k_0>0$, which only depends on the Sobolev constants of M, n, c_0 and the dimension of the Lie group G.

In this paper, we provide an alternative proof of Theorem 0.1. When considering an arbitrary compact Riemannian manifold, we can not obtain a similar result. However, we can prove either any Yang-Mills connection satisfies (4), or the vector bundle E is a flat bundle, i.e. there exists a flat connection over the bundle.

Theorem 0.2 Let $M=M^n$, $n\geqslant 4$, be a compact Riemannian manifold, and E is a vector bundle over M. Either any Yang-Mills connection over M with compact, semi-simple Lie group satisfies (4) for some constant $k_0>0$ depending on M, n, or the bundle E is smoothly isomorphic to a flat bundle.

Influenced by our work, Feehan goes forward about this problem and posted his work in Ref. [5].

Theorem 0.3[5, Theorem 1.1] Let G be a compact Lie group, and P is a principal G-bundle over a closed, smooth manifold M endowed with a smooth Riemannian metric g, whose dimension $n\geqslant 2$. Then there is a positive constant, $\varepsilon=\varepsilon(n,g,G)$, if A is a smooth Yang-Mills connection on P with respect to the metric, and its curvature F_A obeys

$$\|F_A\|_{L^{n/2}(X)} \leqslant \varepsilon,$$

then A is a flat connection.

1 Preliminaries and basic estimates

First, we recall some standard notations and definitions.

Let T^*M be the cotangent bundle of M. And for $1\leqslant p\leqslant n$, let $\Lambda^p(M)$ be the p-form bundles on M with $T^*M=\Lambda^1 M$. $E\otimes\Lambda^p$ is the associated bundle, $\Omega^p(E)$ is the set of sections of $E\otimes\Lambda^p$. Let $\mathfrak{g}$ be the Lie algebra of G, $Ad: G\to Aut(\mathfrak{g})$ is the adjoint representation, and adE is the associated adjoint vector bundle.

Denote

$$\Omega^p(ad(E)) = \Gamma(adE\otimes\Lambda^p(M)).$$

For a connection A on E, we have exterior derivatives

$$d_A: \Omega^p(adE)\to\Omega^{p+1}(adE).$$

They are uniquely determined by the properties (see Ref. [3, p. 35]):

① $d_A=\nabla_A$ on $\Omega^0(adE)$;

② $d_A(\alpha\wedge\beta)=d_A\alpha\wedge\beta+(-1)^p\alpha\wedge d_A\beta$;

for any $\alpha\in\Omega^p(adE)$, $\beta\in\Omega^q(adE)$.

The curvature $F_A\in\Omega^2(ad(E))$ of the connection A is defined by

$$d_A d_A u = F_A u$$

for any section $u\in\Gamma(E)$. If A is a connection on E, we can define covariant derivatives

$$\nabla_A: \Omega^p(E)\to\Gamma(\Lambda^p T^*M\otimes T^*M\otimes E).$$

For ∇_A and d_A, we have adjoint operators ∇_A^* and d_A^*. We also have Weitzenböck formula[2, Theorem 3.10]

$$(d_A d_A^* + d_A^* d_A)\varphi = \nabla_A^*\nabla_A\varphi + \varphi\circ(Ric\wedge g+2R) + \mathscr{R}^A(\varphi) \qquad (5)$$

where $\varphi\in\Omega^2(ad(E))$, Ric is the Ricci tensor and R is the Riemannian curvature tensor.

The operator of $Ric\wedge g+2R$ and $\varphi\circ(Ric\wedge g+2R)$ are defined by Bourguignon and Lawson [2]. They are

$$(Ric \wedge g)_{X,Y} = Ric(X) \wedge Y + X \wedge Ric(Y)$$

and

$$\varphi \circ (Ric \wedge g + 2R)(X,Y) = \frac{1}{2}\sum_{j=1}^{n}\varphi(e_j, (Ric \wedge g + 2R)_{X,Y}(e_j)).$$

In a local orthonormal frame $(e_1, \cdots, e_n)$ of TM, the quadratic term $\mathcal{R}^A(F_A) \in \Omega^2(ad(E))$ can be expressed as

$$\mathcal{R}^A(F_A)(X,Y) = 2\sum_{j=1}^{n}[F_A(e_j, X), F_A(e_j, Y)].$$

Lemma 1.1 Let M be a compact Riemannian manifold and λ be the minimal eigenvalue of the operator $Ric \wedge g + 2R$. We assume that λ is positive. If A is a Yang-Mills connection and $\| F_A \|_{L^\infty}$ is sufficiently small, A is flat.

Proof From the Weitzenböck formula (5), we have

$$(d_A d_A^* + d_A^* d_A)F_A = \nabla_A^* \nabla_A F_A + F_A \circ (Ric \wedge g + 2R) + \mathcal{R}^A(F_A).$$

The left hand side vanishes by (1) and the Bianchi identity $d_A F_A = 0$. Taking inner product with F_A in L^2 norm, we get

$$0 = \| \nabla_A F_A \|_{L^2}^2 + \langle F_A, F_A \circ (Ric \wedge g + 2R)\rangle + \langle F_A, \mathcal{R}^A(F_A)\rangle \geqslant \| \nabla_A F_A \|_{L^2}^2 + (\lambda - 4\| F_A \|_{L^\infty}) \| F_A \|_{L^2}^2 \quad (6)$$

If $\| F_A \|_{L^\infty}$ is sufficiently small, then A is flat. Here we have used our assumption

$$\langle F_A, F_A \circ (Ric \wedge g + 2R)\rangle \geqslant \lambda \| F_A \|_{L^2}^2$$

and the fact

$$| \langle F_A, \mathcal{R}^A(F_A)\rangle | \leqslant 4\| F_A \|_{L^\infty} \| F_A \|_{L^2}^2.$$

In fact, the condition (3) with $c_0 > 0$ is equivalent to the positivity of λ which is the minimal eigenvalue of the operator $Ric \wedge g + 2R$. Thanks to Uhlenbeck's work[8, Theorem 3.5], one can control the $L^\infty(M)$ norms of the curvature F_A by the $L^{\frac{n}{2}}$ norms. Then, we provided another proof of Theorem 0.1 by Lemma 1.1.

Remark 1.1 There is an M such that $Ric \wedge g + 2R$ is a positive operator, for example,

① S^n, where $\lambda \equiv 2(n-1)$;

② M with the positive curvature operator, $Ric \wedge g + 2R$ must be positive;

③ M with the section curvature $\overline{R}$ which satisfy

$$\alpha \overline{R}_{\max} \leqslant \overline{R} \leqslant \overline{R}_{\max} (\alpha \geqslant 1 - \frac{3}{2n-2})$$

(see Ref. [1, p. 79]).

According to the Weitzenböck formula[2, Theorem 3.10], we can also obtain a differential inequality for $|F_A|^{\frac{n}{2}}$, and the proof is similar to the case $n=4$ (see Ref. [3]).

Lemma 1.2 Let M be a compact n-dimensional Riemannian manifold, $n \geqslant 4$, and A is a Yang-Mills connection, then $|F_A|^{\frac{n}{2}}$ satisfies

$$\Delta |F_A|^{\frac{n}{2}} \leqslant C_1 |F_A|^{\frac{n}{2}} + c |F_A|^{\frac{n+2}{2}} \quad (7)$$

where C_1, c only depend on the metric on M.

Proof Form the Weitzenböck formula (5), we have

$$(d_A d_A^* + d_A^* d_A)F_A = \nabla_A^* \nabla_A F_A + F_A \circ (Ric \wedge g + 2R) + \mathcal{R}^A(F_A).$$

The left hand side vanishes form 1.1 and $d_A F_A = 0$. The quadratic term $\mathcal{R}^A(F_A) \in \Omega^2(ad(E))$ can be expressed as

$$\mathcal{R}^A(F_A)(X,Y) = 2\sum_{j=1}^{n}[F_A(e_j, X), F_A(e_j, Y)]$$

with the help of a local orthonormal frame $(e_1, \cdots, e_n)$ of TM. The estimate of the Laplacian follows from

$$\begin{aligned} -\nabla_A^* \nabla_A |F_A|^{\frac{n}{2}} = & -\frac{n}{2}\langle \nabla_A^* \nabla_A F_A, F_A\rangle \langle F_A, F_A\rangle^{\frac{n}{4}-1} - \\ & \frac{n}{2}\langle \nabla_A F_A, \nabla_A F_A\rangle \langle F_A, F_A\rangle^{\frac{n}{4}-1} - \\ & \frac{n}{2}\left(\frac{n-4}{2}\right)\langle \nabla_A F_A, F_A\rangle^2 \langle F_A, F_A\rangle^{\frac{n}{4}-2} \leqslant \\ & -\frac{n}{2}\langle \nabla_A^* \nabla_A F_A, F_A\rangle \langle F_A, F_A\rangle^{\frac{n}{4}-1} \leqslant \\ & \frac{n}{2}(\langle F_A, F_A \circ (Ric \wedge g + 2R)\rangle + \\ & \langle F_A, \mathcal{R}^A(F_A)\rangle)\langle F_A, F_A\rangle^{\frac{n}{4}-1} \leqslant \\ & C |F_A|^{\frac{n}{2}} + c |F_A|^{\frac{n+2}{2}} \end{aligned} \quad (8)$$

Here the constant C depends on the Ricci transform Ric and the scalar curvature R of the metric on M. The constant c only depends on the metric.

Theorem 1.1[10, Theorem 7] Let M be a compact Riemannian n-manifold. $\mathscr{U}=\{U_\alpha\}_{\alpha\in I}$ is a finite open covering of M, any two point x,y in a nonempty intersection $U_\alpha\cap U_\beta$ can be connected by a C^1 curve in $U_\alpha\cap U_\beta$, whose length$\leqslant l$, where l is a uniform constant. And $\{g_{\alpha\beta}\}$ is a set of smooth transition function with respect to $\mathscr{U}$, then there exists a constant

$$\varepsilon_1=\varepsilon_1(M,l,\mathscr{U})>0,$$

if

$$\sup_{x\in U_{\alpha\beta}}|\nabla g_{\alpha\beta}(x)|\leqslant\varepsilon_1,\ \forall\alpha,\beta\in I,$$

the bundle defined by $\{g_{\alpha\beta}\}$ is smoothly isomorphic to a flat bundle.

2 Proof of the main theorem

Let $\left\{\frac{\partial}{\partial x_i}\right\}_{i=1}^n$ and $\{dx_i\}_{i=1}^n$ denote respectively the basis of the tangent bundle TM and cotangent bundle T^*M on B_r, $r\leqslant i(M)$, where $i(M)$ is the injectivity radius of each point $x\in M$. Let (g_{ij}) be a Riemannian metric of M by

$$\left\langle\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right\rangle=g_{ij},\ \langle dx_i,dx_j\rangle=g^{ij},$$

where $(g^{ij})=(g_{ij})^{-1}$. For any $x_0\in M$, there exists a normal coordinate in the geodesic ball $B_{r_0}(x_0)$, which is at the center x_0 with radius $r_0\leqslant i(M)$, and for some constant C, we have

$$\left.\begin{array}{r}|g_{ij}-\delta_{ij}|\leqslant C|x|^2,\ \left|\frac{\partial g_{ij}}{\partial x_k}\right|\leqslant C|x|,\\ \forall\ x\in B_{i(M)}\end{array}\right\}\quad(9)$$

Proposition 2.1 Every $n\in\mathbb{N}$, there exist constants C_0 and $\delta>0$ such that the following holds for all $0<r\leqslant 1$ and all metrics g on $\mathbb{R}^n$ with $\|g_{ij}-\delta_{ij}\|_{W^{1,\infty}}\leqslant\delta$. If $v\in C^2(B_r(0))$ and $v\geqslant 0$ satisfies $\Delta v\leqslant 0$, then

$$v(0)\leqslant C_0r^{-n}\int_{B_r(0)}v.$$

The above proposition is a special case of Theorem 2.1 in Ref.[6]. The starting point of the proof is Morrey's [7] mean value inequality for subharmonic functions.

Lemma 2.1 Let $B_r(0)$ be a geodesic ball of radius r, $0<r\leqslant 1$, which is sufficiently small. Then there exist constant C_2 and $\mu>0$, we have either

$$\int_{B_r(0)}|F_A|^{\frac{n}{2}}\geqslant\mu c^{-\frac{n}{2}}$$

or

$$|F_A|^{\frac{n}{2}}(0)\leqslant C_2(C_1^{\frac{n}{2}}+r^{-n})\int_{B_r(0)}|F_A|^{\frac{n}{2}},$$

where c and C_1 are the same constants as in Lemma 1.2.

Proof We denote $e=|F_A|^{\frac{n}{2}}$. Considering the function $f(\rho)=(1-\rho)^n\sup_{B_{\rho r}(0)}e$ for $\rho\in[0,1]$, it attains its maximum at some $\bar\rho<1$. Let

$$\bar a=\sup_{B_{\bar\rho r}(0)}e=e(\bar x)$$

and

$$\delta=\frac{1}{2}(1-\bar\rho)<\frac{1}{2},$$

then

$$e(0)=f(0)\leqslant f(\bar\rho)=2^n\delta^n\bar a.$$

Moreover, for all $x\in B_{\delta r}(\bar x)\subset B_r(0)$, we have

$$e(x)\leqslant\sup_{B_{(\bar\rho+\delta)r}(0)}e=(1-\bar\rho-\delta)^{-n}f(\bar\rho+\delta)\leqslant$$
$$2^n(1-\bar\rho)^{-n}f(\bar\rho)=2^n\bar a.$$

From Lemma 1.2,

$$\Delta e\leqslant C_1e+ce^{\frac{n+2}{n}},$$

we have

$$\Delta e\leqslant 2^nC_1\bar a+2^{n+2}c\bar a^{\frac{n+2}{n}}.$$

Now, we define the function

$$v(x):=e(x)+\frac{1}{n}(2^n\bar a(C_1+4c\bar a^{\frac{2}{n}}))|x-\bar x|^2$$

with the Euclidean norm $|x-\bar x|$. It is nonnegative and subharmonic on $B_{\delta r}(\bar x)$ if the metric g_{ij} is sufficiently C^1-close to δ_{ij}. This can be seen as follows,

$$\Delta_0|x-\bar x|^2=-2n,$$

where $\Delta_0=-\sum_{i=1}^n\partial_i{}^2$, and $|x-\bar x|\leqslant\delta r\leqslant 1$ is bounded, so $\Delta|x-\bar x|^2\leqslant -n$ whenever

$$\|g_{ij}-\delta_{ij}\|_{W^{1,\infty}}\leqslant\epsilon$$

is sufficiently small. If not, we can choose a smaller radial from (9), so it is true. The control of the metric also ensures that the integral $\int_{B_{\rho r}(\bar{x})} |x-\bar{x}|^2$ is bounded by the following integral over the Euclidean ball $B^0_{2\rho r}(\bar{x})$: with the constant $C_3 = 2^{n+3}\mathrm{Vol}S^{n-1}/(n+2)$,

$$2\int_{B^0_{2\rho r}(\bar{x})} |x-\bar{x}|^2 = 2\int_0^{2\rho r} t^{n+1}\mathrm{Vol}S^{n-1}\mathrm{d}t = C_2(\rho r)^{n+2}$$

So from (9), about function $v(x)$, we have

$$v(\bar{x}) \leqslant C_0(\rho r)^{-n}\int_{B_{\rho r}(\bar{x})} v \tag{10}$$

Let

$$C_4 = \max\left\{C_0, \frac{1}{n}2^n C_0 C_2\right\},$$

for all $0<\rho\leqslant\delta$, from (10), we get

$$\bar{a} = v(\bar{x}) \leqslant C_4\bar{a}(C_1 + 4c\bar{a}^{-\frac{2}{n}})(\rho r)^2 + C_4(\rho r)^{-n}\int_{B_{\rho r}(\bar{x})} e \tag{11}$$

If

$$C_4(C_1 + 4c\bar{a}^{-\frac{2}{n}})(\rho r)^2 \leqslant \frac{1}{2},$$

then (11) implies

$$\bar{a} \leqslant 2C_4(\rho r)^{-n}\int_{B_r(0)} e.$$

So if

$$C_4(C_1 + 4c\bar{a}^{-\frac{2}{n}})(\delta r)^2 \leqslant \frac{1}{2},$$

then $\rho=\delta$ proves the assertion

$$e(0) \leqslant 2^n\delta^n\bar{a} \leqslant 2^{n+1}C_4 r^{-n}\int_{B_r(0)} e.$$

Otherwise, we can choose $0<\rho<\delta$ such that $(\rho r)^{-2} = 2C_4(C_1 + 4c\bar{a}^{-\frac{2}{n}})$. And we obtain

$$e(0) \leqslant \bar{a} \leqslant C_5(C_1 + 4c\bar{a}^{-\frac{2}{n}})^{\frac{n}{2}}\int_{B_{\rho r}(\bar{x})} e$$

with $C_5 = (2C_4)^{1+\frac{n}{2}}$. We have to distinguish two cases: Firstly, if $4c\bar{a}^{-\frac{2}{n}} \leqslant C_1$, this yields

$$e(0) \leqslant C_5(2C_1)^{\frac{n}{2}}\int_{B_{\rho r}(\bar{x})} e.$$

Secondly, if $C_1 < 4c\bar{a}^{-\frac{2}{n}}$, then

$$\bar{a} < \bar{a}C_5(8c)^{\frac{n}{2}}\int_{B_{\rho r}(\bar{x})} e,$$

with $\mu = 8^{-\frac{n}{2}}C_5^{-1} > 0$, we have

$$\int_{B_r(0)} e > \mu c^{-\frac{n}{2}}.$$

So we either have the above or with some constant $C_2 = \max\{2^{n+1}C_4, 2^{\frac{n}{2}}C_5\}$,

$$e(0) \leqslant C_2(C_1^{\frac{n}{2}} + r^{-n})\int_{B_r(0)} e.$$

Remark 2.1 By using local geodesic coordinates, the above lemma also implies a mean value inequality on closed Riemannian manifolds with uniform constants C_2, μ, and all geodesic balls whose radius are less than a uniform constant.

Theorem 2.1 Let $M = M^n$, $n \geqslant 4$, be a compact Riemannian manifold, and A_i is a sequence of Yang-Mills connections, we denote $e_i = |F_{A_i}|^{\frac{n}{2}}$. Assuming that there a unform bounded $\int_M e_i \leqslant E < \infty$, there exist finitely many points, $x_1, x_2, \cdots, x_N \in M$ (with $N \leqslant E/\nu$) and a sequence of connections such that the e_i are uniformly bounded on every compact subset of $M\backslash\{x_1, x_2, \cdots, x_N\}$. And there is a concentration of energy ν at each x_j: For every $r>0$, there exists $N_{j,r} \in \mathbb{N}$ such that

$$\int_{B_r(x_j)} e_i \geqslant \nu, \ \forall\ i \geqslant N_{j,r} \tag{12}$$

where ν is a constant only depending on n, M.

Proof Supposing there exist some points $x_i \in M$, e_i is uniformly bounded on the neighbourhood of x_i. Then, there is a subsequence e_i (again denoted e_i) and $M \ni y_i \to x$, such that $e_i(y_i) = R_i^n$ with $R_i^n \to \infty$. Then, we can apply the Lemma 2.1 on the balls $B_{r_i}(y_i)$, whose radius $r_i = R_i^{-\frac{1}{2}} > 0$. For a sufficiently large $i \in \mathbb{N}$, there lies an appropriate coordinates charts of M, and according to the Lemma 2.1, there are uniform constant C_2 and $\nu = \mu c^{-\frac{n}{2}} > 0$ such that for every $i \in \mathbb{N}$, either

$$\int_{B_{r_i}(y_i)} e_i > \nu \tag{13}$$

or $\int_{B_{r_i}(y_i)} e_i \leqslant \nu$. Hence,

$$R_i^n = e(y_i) \leqslant C_2(C_1^{\frac{n}{2}} + r_i^{-n})\int_{B_{r_i}(y_i)} e_i.$$

In the latter case, multiplication by $r_i^n = R_i^{-\frac{n}{2}}$ implies

$$R_i^{\frac{n}{2}} \leqslant C_2\nu(C_1^{\frac{n}{2}} R_i^{-\frac{n}{2}} + 1) \tag{14}$$

As $i \to \infty$, the left hand side diverges to ∞, where the right hand side converges to $C_2\nu$. Thus the alternative (13) must hold for all sufficiently large $i \in \mathbb{N}$. In particular, this implies the energy concentration (12) at $x_j = x$.

Now we can go through the same argument for any other point $x \in M$, where the present subsequence e_i is not locally uniformly bounded. That way we iteratively find points $x_j \in M$ such that iteration yields $N \leqslant E/\nu$ distinct points $x_1, x_2, \cdots, x_N$ (and might not even terminate after that). Then we have a subsequence e_i for which at least energy $\nu > 0$ concentrates near each x_j. Since the points are distinct, this contradicts the energy bound $\int_M e_i \leqslant E$. Hence this iteration must stop after at most E/ν steps, when the present subsequence e_i is locally uniformly bounded in the complement of the finitely many points, where we found the energy concentration before.

To prove Theorem 0.2, we need only consider the case when A is a Yang-Mills connection on E with $\| F_A \|_{L^{\frac{n}{2}}}$ sufficiently small. We have the following theorem proved by Uhlenbeck.

Theorem 2.2[8, Theorem 3.5] There exists a constant ε_1 such that if F_A is Yang-Mills field in $B_{2a_0}(x_0)$ and $\int_{B_{2a_0}(x_0)} | F_A |^{\frac{n}{2}} < \varepsilon_2$, then $|F_A(x)|$ is uniformly bounded in the interior of $B_{2a_0}(x_0)$ and

$$| F_A(x) |^2 \leqslant C_6(a^{-n}\int_{B_a(x)} | F_A |^{\frac{n}{2}})^{\frac{4}{n}}$$

for all $B_a(x) \subset B_{a_0}(x_0)$.

Assuming $\| F_A \|_{L^{\frac{n}{2}}(M)}$ is sufficiently small, from the above Theorem 2.2, we have

$$\| F_A \|_{L^\infty} = \sup_{x\in M} | F_A | (x) \leqslant C_6\rho^{-4} \| F_A \|_{L^{\frac{n}{2}}},$$

here ρ is the injectivity radius of M.

Lemma 2.2 Let M be a compact n-dimensional Riemannian manifold, and E is a smooth vector bundle over M. Let A be a smooth connection on E, then there exists $\varepsilon_3 = \varepsilon_3(M) > 0$, such that if

$$\| F_A \|_{L^\infty} \leqslant \varepsilon_3,$$

E is smoothly isomorphic to a flat bundle.

Proof We cover M with coordinate balls $\{U_\alpha\}$, and any two points x, y in a nonempty intersection $U_\alpha \cap U_\beta$ can be connected by a C^1 curve in $U_\alpha \cap U_\beta$, whose length $\leqslant$ diam(M). Let ϕ_α: $E|_{U_\alpha} \to B_1(0) \times \mathbb{R}^r$ be trivializations on U_α and A_α be the $\mathfrak{g}$-value 1-form on U_α corresponding to A under ϕ_α.

Let x_α be the center of the U_α. For any point $x \in U_\alpha$, we let γ_α^x be the shortest geodesic from x_α to x inside U_α, $h_\alpha(x) \in G$ is the parallel transport of the bundle from x_α to x along γ_α^x using the trivialization of the bundle.

Note that $h_\alpha(x_\alpha) = Id$, we regard h_α^{-1} as gauge transformations on U_α, and denote $h_\alpha^{-1}(A)$ by $\widetilde{A}_\alpha$. We use the normal spherical coordinates $\{r, \theta^j\}_{j=1,\cdots,n-1}$. Let us assume that

$$\widetilde{A}_\alpha = \widetilde{A}_{\alpha,r}dr + \widetilde{A}_{\alpha,j}d\theta^j, \text{ on } U_\alpha$$

and

$$F_{\widetilde{A}_\alpha} = F_{\alpha,rj}dr \wedge d\theta^j + F_{\alpha,ij}d\theta^i \wedge d\theta^j, \text{ on } U_\alpha$$

Then by the definition of h_α, we have $\widetilde{A}_{\alpha,r} \equiv 0$ on U_α. Hence

$$\partial_r(\widetilde{A}_{\alpha,j}) = F_{\alpha,rj},\ j = 1, \cdots, n-1 \tag{15}$$

By integrating (15) and $\widetilde{A}_\alpha(0) = 0$, we have

$$| \widetilde{A}_\alpha | (x) \leqslant | x | \int_0^1 | F_{\widetilde{A}_\alpha} | (tx)\mathrm{d}t \leqslant \varepsilon_3 r_\alpha \tag{16}$$

We define $h_{\alpha\beta} = h_\alpha^{-1}(\phi_\alpha \cdot \phi_\beta^{-1})h_\beta$ on $U_\alpha \cap U_\beta$, and we can check that $\{h_{\alpha\beta}\}$ is a set of transition functions. Now we have

$$\begin{aligned} dh_{\alpha\beta} = {} & dh_\alpha^{-1}(\phi_\alpha \cdot \phi_\beta^{-1})h_\beta + \\ & h_\alpha^{-1}d(\phi_\alpha \cdot \phi_\beta^{-1})h_\beta + h_\alpha^{-1}(\phi_\alpha \cdot \phi_\beta^{-1})dh_\beta = \\ & dh_\alpha^{-1}h_\alpha h_{\alpha\beta} + h^{-\alpha}(A_\alpha\phi_\alpha \cdot \phi_\beta^{-1})h_\beta - \\ & h^{-\alpha}(\phi_\alpha \cdot \phi_\beta^{-1})A_\beta h_\beta + h_\alpha^{-1}A_\alpha(\phi_\alpha \cdot \phi_\beta^{-1})dh_\beta = \\ & h_\alpha^{-1}(A_\alpha) \circ h_{\alpha\beta} - h_{\alpha\beta} \circ h_\beta^{-1}(A_\beta) \end{aligned} \tag{17}$$

where we using $h_\alpha^{-1}(A_\alpha)=h_\alpha^{-1}A_\alpha h_\alpha+h_\alpha^{-1}dh_\alpha$, $h_\beta^{-1}(A_\beta)=h_\alpha^{-1}A_\alpha h_\alpha+h_\alpha^{-1}dh_\alpha$ and $d(\phi_\alpha\cdot\phi_\beta^{-1})=A_\alpha(\phi_\alpha\cdot\phi_\beta^{-1})-(\phi_\alpha\cdot\phi_\beta^{-1})A_\beta$. Hence from (16), we have

$$|\nabla h_{\alpha\beta}|\leqslant\varepsilon_3,\ \text{on } U_\alpha\cup U_\beta.$$

By taking ε_3 sufficiently small, we establish the lemma from Lemma 1.1.

From Lemma 2.2 and Theorem 2.2, we give a proof of Theorem 0.2.

Remark 2.2 We cannot conclude that the Yang-Mills connection, A on E, in Lemma 2.2 with $L^{\frac{n}{2}}$-small curvature, F_A, itself is flat, but rather just that E supports some flat connections and thus is a flat bundle.

In Ref. [9], Uhlenbeck proved there exist a constant, ε, if $\|F_A\|_{L^p(X)}\leqslant\varepsilon$ $(2p>n)$, then there exist a flat connection Γ and a constant C such that

$$\|A-\Gamma\|_{L_1^p(X)}\leqslant C\|F_A\|_{L^p}.$$

By the Lajasiewicz-Simon gradient inequality on a Sobolev neighborhood of a flat connection (see Ref. [5, Theorem 3.2]), Feehan proved the Theorem 0.3.

References

[1] BOURGUIGNON J P, KARCHER H. Curvature operators: Pinching estimates and geometric examples [J]. Ann Sci École Norm Sup(4), 1978, 11(1): 71-92.

[2] BOURGUIGNON J P, LAWSON H B. Stability and isolation phenomena for Yang-Mills fields[J]. Comm Math Phys, 1981, 79(2): 189-230.

[3] DONALDSON S K, KRONHEIMER P B. The Geometry of Four-Manifolds[M]. New York: Oxford University Press, 1990.

[4] GERHARDT C. An energy gap for Yang-Mills connections[J]. Comm Math Phys, 2010, 298(2): 515-522.

[5] FEEHAN P M N. Energy gap for Yang-Mills connections, Ⅱ: Arbitrary closed Riemannian manifolds[DB/OL]. arXiv:1502.00668v2.

[6] LI P, SCHOEN R. L^p and mean value properties of subharmonic functions on Riemannian manifolds[J]. Acta Mathematica, 1984, 153(1): 279-301.

[7] MORREY C B. Multiple Integrals in the Calculus of Variations[M]. Berlin: Springer, 1966.

[8] UHLENBECK K K. Removable singularities in Yang-Mills fields[J]. Comm Math Phys, 1982, 83(1): 11-29.

[9] UHLENBECK K K. The Chern classes of Sobolev connections[J]. Comm Maht Phys, 1985, 101: 445-457.

[10] YANG B. Removable singularities for Yang-Mills connections in higher dimensions[J]. Pacific J Math, 2003, 209(2): 381-398.