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n-color 1-2 compositions of positive integers

GUO Yuhong

(*School of Mathematics and Statistics, Hexi University, Zhangye 734000, China*)

Abstract: An n-color 1-2 composition is defined as an n-color composition with only parts of size 1 or 2 of positive integer. An n-color 1-2 palindromic composition is an n-color 1-2 composition in which the parts are ordered such that they are read the same forward and backwards. Here the generating function, explicit formulas and recurrence relations for n-color 1-2 compositions and n-color 1-2 palindromic compositions were obtained. In addition, a relation between the number of n-color 1-2 compositions of ν and the number of n-color 1-2 palindromic compositions of ν was given.

Key words: compositions of positive integer; n-color 1-2 composition; n-color 1-2 palindromic composition; generating function; explicit formulas; recurrence relations

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正整数 n-color 1-2 有序分拆

郭育红

(河西学院数学与统计学院,甘肃张掖 734000)

摘要:正整数的 n-color 1-2 有序分拆是指正整数的只含有分部量是 1 或者 2 的 n-color 有序分拆,而正整数的回文的 n-color 1-2 有序分拆是指只含有分部量是 1 或者 2 的 n-color 有序分拆且分部量从前往后读与从后往前读是相等的. 这里给出了正整数的 n-color 1-2 有序分拆数和回文的 n-color 1-2 有序分拆数的生成函数、显式公式以及递推公式. 而且还给出了正整数的 n-color 1-2 有序分拆数和回文的 n-color 1-2 有序分拆数之间的一个关系式.

关键词:正整数的有序分拆;n-color 1-2 有序分拆;回文的 n-color 1-2 有序分拆;生成函数;显式公式;递推公式

0 Introduction

A composition of positive integer ν is a sequence of positive integers called parts that sum to ν. In recent years, there has been considerable interest in compositions with restrictions on the

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Biography: GUO Yuhong, female, born in 1970, master/Prof. Research field: combinatorial number theory. E-mail: gyh7001@163.com

size of parts (see Refs. [1-4]). For example, odd compositions of ν are the compositions having only odd parts, 1-free compositions of ν are the compositions having no parts of size 1 and so on. Recently there has been interest in n-color compositions defined as compositions of ν for which a part of size n can take n colors (see Refs. [5-11]). As a brief example, there are 8 n-color compositions of 3. Viz.,

$$3_1, 3_2, 3_3, 2_1+1_1, 2_2+1_1, 1_1+2_1,$$
$$1_1+2_2, 1_1+1_1+1_1.$$

In 2012, Shapcott[12] studied C-color compositions which are a generalization of n-color compositions.

A palindromic composition or palindrome[1,13] also referred to as a self-inverse composition (see Refs. [9, 11, 14]) is a composition whose part sequence is the same whether it is read from left to right or right to left. For example, there are 4 palindromic compositions of 4. Viz.,

$$4, 2+2, 1+2+1, 1+1+1+1.$$

And there are nine n-color palindromic compositions of 4. Viz.,

$$4_1, 4_2, 4_3, 4_4, 2_1+2_1, 2_2+2_2, 1_1+2_1+1_1,$$
$$1_1+2_2+1_1, 1_1+1_1+1_1+1_1.$$

Ref. [1] studied the compositions with ones and twos of positive integer, referred to here as 1-2 compositions.

Thus, for example, there are five 1-2 compositions of 4:

$$2+2, 1+2+1, 2+1+1, 1+1+2, 1+1+1+1.$$

And Ref. [1] also gave the generating function of the number of the 1-2 compositions as

$$\frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n+1} x^n,$$

where F_n is the $(n)^{st}$ Fibonacci number. That is the number of 1-2 compositions of n is F_{n+1}.

We denote the number of 1-2 compositions of ν by $C_{1\text{-}2}(\nu)$, the result of Ref. [1] is

$$C_{1\text{-}2}(1) = 1,\ C_{1\text{-}2}(2) = 2,$$
$$C_{1\text{-}2}(\nu) = C_{1\text{-}2}(\nu-1) + C_{1\text{-}2}(\nu-2)$$

when $\nu > 2$.

In this paper, we will study n-color 1-2 compositions, n-color 1-2 palindromic compositions. And we shall give some properties of these compositions. We first give the following definitions.

Definition 0.1 An n-color 1-2 composition is an n-color composition having only parts of size 1 or 2.

For example, there are 11 n-color 1-2 compositions of 4. Viz.,

$$2_1+2_1, 2_1+2_2, 2_2+2_1, 2_2+2_2, 2_1+1_1+1_1,$$
$$2_2+1_1+1_1, 1_1+1_1+2_1, 1_1+1_1+2_2,$$
$$1_1+2_1+1_1, 1_1+2_2+1_1, 1_1+1_1+1_1+1_1.$$

Definition 0.2 An n-color 1-2 composition whose parts read from left to right are identical with when read from right to left is called an n-color 1-2 palindromic composition.

For example, there are 5 n-color 1-2 palindromic compositions of 3. Viz.,

$$2_1+2_1, 2_2+2_2,\ 1_1+2_1+1_1,\ 1_1+2_2+1_1,$$
$$1_1+1_1+1_1.$$

In Section 1 we will give the generating function, recurrence formulas and explicit formulas of n-color 1-2 compositions of positive integers.

1 n-color 1-2 compositions

We denote the number of n-color 1-2 compositions of ν by $A_{1\text{-}2}(\nu)$, and the number of n-color 1-2 compositions of ν into m parts by $A_{1\text{-}2}(m,\nu)$, respectively. In this section we first prove the following theorem.

Theorem 1.1 Let $A_{1\text{-}2}(m,q)$ and $A_{1\text{-}2}(q)$ denote the generating functions for $A_{1\text{-}2}(m,\nu)$ and $A_{1\text{-}2}(\nu)$, respectively. Then

$$A_{1\text{-}2}(m,q) = (q+2q^2)^m \tag{1}$$

$$A_{1\text{-}2}(q) = \frac{q+2q^2}{1-q-2q^2} \tag{2}$$

$$A_{1\text{-}2}(m,\nu) = 2^{\nu-m}\binom{m}{\nu-m} \tag{3}$$

$$A_{1\text{-}2}(\nu) = \sum_{m\leqslant\nu} 2^{\nu-m}\binom{m}{\nu-m} \tag{4}$$

Proof Following Agarwal's proof in Ref. [5], we have

$$A_{1\text{-}2}(m,q) = \sum_{\nu=1}^{\infty} A_{1\text{-}2}(m,\nu) q^{\nu} = (q+2q^2)^m.$$

This proves (1).

And

$$A_{1\text{-}2}(q) = \sum_{m=1}^{\infty} A_{1\text{-}2}(m,q) =$$

$$\sum_{m=1}^{\infty} (q+2q^2)^m = \frac{q+2q^2}{1-q-2q^2},$$

which proves (2).

Equating the coefficients of q^{ν} in (1), we have

$$A_{1\text{-}2}(m,\nu) = 2^{\nu-m}\binom{m}{\nu-m}.$$

This proves (3).

Obviously, $m \leqslant \nu$, so (4) is also proven.

Thus we complete the proof. □

From the generating function of the number of n-color 1-2 compositions, we have the following recurrence formula.

Theorem 1. 2 Let $A_{1\text{-}2}(\nu)$ denote the number of n-color 1-2 compositions of ν. Then

$$A_{1\text{-}2}(1) = 1,\ A_{1\text{-}2}(2) = 3,$$

$$A_{1\text{-}2}(\nu) = A_{1\text{-}2}(\nu-1) + 2A_{1\text{-}2}(\nu-2)$$

for $\nu > 2$.

Although this recurrence relation is the straight result of the formula (2), we still present the two proofs of this theorem.

We first give the combinatorial proof of this theorem.

Proof (combinatorial) We split the n-color 1-2 compositions of ν into three classes:

(A) compositions having 1_1 on the right,

(B) compositions having 2_1 on the right,

(C) compositions having 2_2 on the right.

We transform the n-color 1-2 compositions in Class (A) by deleting 1_1 on the right of compositions. This produces n-color 1-2 compositions enumerated by $A_{1\text{-}2}(\nu-1)$. Conversely, for any n-color 1-2 composition enumerated by $A_{1\text{-}2}(\nu-1)$, we add 1_1 to the right of compositions to produce the elements of Class (A). In this way, we prove that there are exactly $A_{1\text{-}2}(\nu-1)$ elements in Class (A).

Similarly, we can produce n-color 1-2 compositions enumerated by $A_{1\text{-}2}(\nu-2)$ by deleting 2_1 on the right of the compositions in Class (B). And we also get $A_{1\text{-}2}(\nu-2)$ compositions by deleting 2_2 on the right of compositions in Class (C).

Hence, we have

$$A_{1\text{-}2}(\nu) = A_{1\text{-}2}(\nu-1) + 2A_{1\text{-}2\text{-}3}(\nu-2).$$

Thus, we complete the proof. □

We also give another proof of this theorem.

Second Proof From Theorem 1. 1 we have

$$A_{1\text{-}2}(\nu) = \sum_{m\leqslant\nu} 2^{\nu-m}\binom{m}{\nu-m} =$$

$$\sum_{m\leqslant\nu} 2^{\nu-m}\left(\binom{m-1}{\nu-m} + \binom{m-1}{\nu-m-1}\right)$$

(by the binomial identity

$$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m})$$

$$= \sum_{m\leqslant\nu} 2^{\nu-m}\binom{m-1}{\nu-m} + \sum_{m\leqslant\nu} 2^{\nu-m}\binom{m-1}{\nu-m-1} =$$

$$\sum_{m\leqslant\nu} 2^{\nu-m}\binom{m-1}{\nu-m} + \sum_{m\leqslant -1} 2^{\nu-m-1}\binom{m}{\nu-2-m} =$$

$$\sum_{m\leqslant\nu-1} 2^{\nu-m-1}\binom{m}{\nu-m-1} +$$

$$2\sum_{m\leqslant\nu-2} 2^{\nu-m-2}\binom{m}{\nu-2-m} =$$

$$A_{1\text{-}2}(\nu-1) + 2A_{1\text{-}2}(\nu-2).$$

We complete the proof. □

2 n-color 1-2 palindromic compositions

Ref. [13] studied the palindromic compositions with parts of size being less than 3, being referred to here as 1-2 palindromic compositions. We denote the number of 1-2 palindromic compositions of ν by $P_{1\text{-}2}(\nu)$. Ref. [13] gave the generating function of the number of the 1-2 palindromic compositions as

$$\sum_{n=0}^{\infty} P_{1\text{-}2}(n)x^n = \frac{1+x+x^2}{1-x^2-x^4}.$$

This is the generating function for interleaved Fibonacci sequence 1, 1, 2, 1, 3, 2, 5, 3, 8, 5, 13, 8, 21… i. e.

$$P_{1\text{-}2}(2\nu) = F_{\nu+2},\ \text{for } \nu \geqslant 1 \quad (5)$$

$$P_{1\text{-}2}(2\nu-1) = F_{\nu},\ \text{for } \nu \geqslant 1 \quad (6)$$

where F_n is the $(n)^{st}$ Fibonacci number.

From the generating function of the number of 1-2 palindromic compositions we get easily the recurrence relation of $P_{1\text{-}2}(\nu)$.

Theorem 2.1 Let $P_{1\text{-}2}(\nu)$ denote the number of 1-2 palindromic compositions of ν. Then

$$P_{1\text{-}2}(1)=1,\ P_{1\text{-}2}(2)=2,$$
$$P_{1\text{-}2}(3)=1,\ P_{1\text{-}2}(4)=3,$$
$$P_{1\text{-}2}(\nu)=P_{1\text{-}2}(\nu-2)+P_{1\text{-}2}(\nu-4)$$

for $\nu>4$.

We give the combinatorial proof of this recurrence relation.

Proof We split the 1-2 palindromic compositions of ν into two classes:

(A) compositions with 1 on both extremes,

(B) compositions with 2 on both extremes.

We transform the 1-2 palindromic compositions in Class (A) by deleting 1 on both extremes of compositions. This produces the 1-2 palindromic compositions enumerated by $P_{1\text{-}2}(\nu-2)$. Conversely, give any 1-2 palindromic composition enumerated by $P_{1\text{-}2}(\nu-2)$ and add 1 to both extremes of composition to produce the compositions of Class (A). In this way we establish the fact that there are exactly $P_{1\text{-}2}(\nu-2)$ elements in Class (A).

Similarly, we can produce $P_{1\text{-}2}(\nu-4)$ elements in Class (B) by deleting 2 on both extremes of compositions.

Hence, we have

$$P_{1\text{-}2}(\nu)=P_{1\text{-}2}(\nu-2)+P_{1\text{-}2}(\nu-4).$$

We complete the proof. □

Next, we consider the n-color 1-2 palindromic compositions of ν. We denote the number of n-color 1-2 palindromic compositions of ν by $S_{1\text{-}2}(\nu)$. We have the following results.

Theorem 2.2 Let $S_{1\text{-}2}(\nu)$ and $A_{1\text{-}2}(\nu)$ denote the number of n-color 1-2 palindromic compositions of ν and the number of n-color 1-2 compositions of ν, respectively. Then

$$S_{1\text{-}2}(2\nu)=A_{1\text{-}2}(\nu)+2A_{1\text{-}2}(\nu-1) \tag{7}$$
$$S_{1\text{-}2}(2\nu-1)=A_{1\text{-}2}(\nu-1) \tag{8}$$

where ν is positive integer.

Proof Obviously, there are two cases for an even number 2ν with n-color 1-2 palindromic compositions.

Case 1: the number of parts is odd and the central part is 2 in n-color 1-2 palindromic compositions of 2ν. In this case, an n-color 1-2 palindromic composition of even number 2ν can be read as the central part of size 2, and two identical n-color 1-2 compositions of $\nu-1$ on each side of the central part of size 2. Because the central part of size 2 has two colors, 2_1 and 2_2, we have $2A_{1\text{-}2}(\nu-1)$ compositions in Case 1.

Case 2: the number of parts is even in n-color 1-2 palindromic compositions of 2ν. In this case, an n-color 1-2 palindromic composition of even number 2ν has two identical n-color 1-2 compositions of ν on each side and has no central part. Thus we get $A_{1\text{-}2}(\nu)$ compositions in Case 2.

Hence, we have

$$S_{1\text{-}2}(2\nu)=A_{1\text{-}2}(\nu)+2A_{1\text{-}2}(\nu-1).$$

We prove (7).

Because an odd number $2\nu-1$ have n-color 1-2 palindromic compositions only when the number of parts is odd and the central part is 1. An n-color 1-2 palindromic composition of odd number $2\nu-1$ can be read as the central part of size 1, and two identical n-color 1-2 compositions of $\nu-1$ on each side of the central part of size 1. So we have $A_{1\text{-}2}(\nu-1)$ compositions as the central part of size 1 with only one color, that is $S_{1\text{-}2}(2\nu-1)=A_{1\text{-}2}(\nu-1)$. Especially, we set $A_{1\text{-}2}(0)=1$ when $\nu=1$. We also prove (8). □

From Theorem 1.2 we get the following corollary easily.

Corollary 2.1 Let $S_{1\text{-}2}(\nu)$ and $A_{1\text{-}2}(\nu)$ denote the number of n-color 1-2 palindromic compositions of ν and the number of n-color 1-2 compositions of ν, respectively. Then

$$S_{1\text{-}2}(2\nu)=A_{1\text{-}2}(\nu+1) \tag{9}$$
$$S_{1\text{-}2}(2\nu-1)=A_{1\text{-}2}(\nu-1) \tag{10}$$

where ν is positive integer.

From (4) in Theorem 1.1 we also have the explicit formulas of n-color 1-2 palindromic compositions easily.

Corollary 2.2 Let $S_{1\text{-}2}(\nu)$ denote the number of n-color 1-2 palindromic compositions of ν. Then

$$S_{1\text{-}2}(2\nu) = \sum_{m\leqslant \nu+1} 2^{\nu-m+1}\binom{m}{\nu-m+1} \quad (11)$$

$$S_{1\text{-}2}(2\nu-1) = \sum_{m\leqslant \nu-1} 2^{\nu-m-1}\binom{m}{\nu-m-1} \quad (12)$$

where ν is a positive integer.

Now we give the recurrence relation of $S_{1\text{-}2}(\nu)$.

Theorem 2.3 Let $S_{1\text{-}2}(\nu)$ denote the number of n-color 1-2 palindromic compositions of ν. Then

$S_{1\text{-}2}(1) = 1$, $S_{1\text{-}2}(2) = 3$,

$S_{1\text{-}2}(3) = 1$, $S_{1\text{-}2}(4) = 5$,

$S_{1\text{-}2}(\nu) = S_{1\text{-}2}(\nu-2) + 2S_{1\text{-}2}(\nu-4)$

for $\nu > 4$.

Proof We split the 1-2 palindromic compositions of ν into three classes:

(a) compositions with 1_1 on both extremes,

(b) compositions with 2_1 on both extremes,

(c) compositions with 2_2 on both extremes.

We transform the n-color 1-2 palindromic compositions in Class (a) by deleting 1_1 on both extremes of compositions. This produces the n-color 1-2 palindromic compositions enumerated by $S_{1\text{-}2}(\nu-2)$. Conversely, give any n-color 1-2 palindromic composition enumerated by $S_{1\text{-}2}(\nu-2)$ and add part of size 1_1 to both extremes of the composition to produce the compositions in Class (a). In this way we establish the fact that there are exactly $S_{1\text{-}2}(\nu-2)$ elements in Class (a).

Similarly, we can produce $S_{1\text{-}2}(\nu-4)$ n-color 1-2 palindromic compositions by deleting 2_1 on both extremes of the compositions in Class (b), and we also produce $S_{1\text{-}2}(\nu-4)$ n-color 1-2 palindromic compositions by deleting 2_2 on both extremes of the compositions in Class (c). Therefore we proved that there are exactly $S_{1\text{-}2}(\nu-4)$ compositions in both Classes (b) and (c).

Hence, we have

$$S_{1\text{-}2}(\nu) = S_{1\text{-}2}(\nu-2) + 2S_{1\text{-}2}(\nu-4),$$

for $\nu>4$.

So we complete the proof. □

Next, we give the generating function for $S_{1\text{-}2}(\nu)$ from recurrence relation of $S_{1\text{-}2}(\nu)$.

Theorem 2.4 Let $S_{1\text{-}2}(q)$ denote the generating function of $S_{1\text{-}2}(\nu)$. Then

$$S_{1\text{-}2}(q) = \frac{q+3q^2+2q^4}{1-q^2-2q^4}.$$

Proof From Theorem 2.3 we have

$$\begin{aligned}
S_{1\text{-}2}(q) &= \sum_{\nu=1}^{\infty} S_{1\text{-}2}(\nu)q^{\nu} = \\
&q+3q^2+q^3+5q^4+\sum_{\nu=5}^{\infty} S_{1\text{-}2}(\nu)q^{\nu} = \\
&q+3q^2+q^3+5q^4+ \\
&\sum_{\nu=5}^{\infty}(S_{1\text{-}2}(\nu-2)+2S_{1\text{-}2}(\nu-4))q^{\nu} = \\
&q+3q^2+q^3+5q^4+ \\
&\sum_{\nu=5}^{\infty} S_{1\text{-}2}(\nu-2)q^{\nu}+2\sum_{\nu=5}^{\infty} S_{1\text{-}2}(\nu-4)q^{\nu} = \\
&q+3q^2+q^3+5q^4+\sum_{\nu=1}^{\infty} S_{1\text{-}2}(\nu)q^{\nu+2} - \\
&q^3-3q^4+2\sum_{\nu=1}^{\infty} S_{1\text{-}2}(\nu)q^{\nu+4} = \\
&q+3q^2+2q^4+q^2\sum_{\nu=1}^{\infty} S_{1\text{-}2}(\nu)q^{\nu}+2q^4\sum_{\nu=1}^{\infty} S_{1\text{-}2}(\nu)q^{\nu}.
\end{aligned}$$

Then

$$(1-q^2-2q^4)\sum_{\nu=1}^{\infty} S_{1\text{-}2}(\nu)q^{\nu} = q+3q^2+2q^4.$$

So

$$S_{1\text{-}2}(q) = \frac{q+3q^2+2q^4}{1-q^2-2q^4}.$$

We complete the proof. □

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(下转第 1002 页)