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A new proof of the classification of solutions to some linearized fractional Yamabe type equations

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Abstract: The classifications of all the solutions to the linearized Yamabe equations and fractional Yamabe type equations are crucial to the proof of the compactness of the scalar curvature problems and the fractional scalar curvature problems respectively. These classifications, though having been proved in an analytical way before, have been proved by adopting some new geometric approaches from the perspective of conformal geometry.

Key words: fractional Laplacian; fractional Yamabe type equations; scattering operators; spherical harmonic functions

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线性化的分数阶 Yamabe 型问题解的分类的新证明

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摘要:线性化的 Yamabe 问题以及分数阶 Yamabe 型问题解的分类分别在数量曲率问题与分数阶数量曲率问题的解集的紧性证明中起了很重要的作用.上述的两种分类已有分析的证明方法,而这里则尝试从共形几何的角度,给出几何化的新证明.

关键词:分数阶拉普拉斯算子;分数阶 Yamabe 型问题;散射算子;球调和函数

0 Introduction

The classical Sobolev inequality in $\mathbb{R}^n$ says that there exists a constant $K(n)>0$ such that

$$\left(\int_{\mathbb{R}^n}|u|^{\frac{2n}{n-2}}\mathrm{d}x\right)^{\frac{n-2}{n}}\leqslant K(n)\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x$$

holds for all $u\in C_0^\infty(\mathbb{R}^n)$. The extremal functions with the equality holding were constructed separately by Aubin[1] and Talenti[2] in 1976, with the form

$$u_{a,\lambda}(x)=(n(n-2))^{\frac{n-2}{4}}\left(\frac{\lambda}{\lambda^2+|x-a|^2}\right)^{\frac{n-2}{2}}$$

for any $a\in\mathbb{R}^n$ and $\lambda>0$. Later in 1989, Caffarelli et al.[3] proved that these functions are exactly all

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the solutions to the following Yamabe type equation

$$\Delta u+u^{\frac{n+2}{n-2}}=0,\ u>0,\ \text{in } \mathbb{R}^n \tag{1}$$

These functions are so-called bubbles, which played an important role in solving the Yamabe problem[4-5]. For simplicity, here we denote by $u_0(x):=u_{0,1}(x)=(n(n-2))^{\frac{n-2}{4}}(1+|x|^2)^{-\frac{n-2}{2}}$. Here we call u_0 the standard bubble. The equation (1) is conformally invariant in the following sense: if u is a solution to (1), let

$$v(x)=\mu^{\frac{n-2}{2}}u(\mu(x-\xi)),$$

for any $\xi\in\mathbb{R}^n$ and $\mu>0$, then v still solves equation (1). For the linearized equation

$$\Delta\psi+\frac{n+2}{n-2}u_0^{\frac{4}{n-2}}\psi=0,\ \text{in } \mathbb{R}^n \tag{2}$$

of (1) at u_0, Chen and Lin[6] proved the following theorem.

Theorem 0.1[6] Suppose that ψ is a solution to (2). If $\lim\limits_{|x|\to+\infty}\psi(x)=0$, then there exist constants C_i, $i=0,1,\cdots,n$, such that

$$\psi=C_0\left(\frac{n-2}{2}u_0+\sum_{i=1}^n x_i\frac{\partial u_0}{\partial x_i}\right)+\sum_{i=1}^n C_i\frac{\partial u_0}{\partial x_i} \tag{3}$$

Remark 0.2 The elegant result played an important role in analyzing the concentration behavior of a family of bubbles derived from the scalar curvature problems (see Refs. [6-7]).

Recently, González et al.[8] initiated the research on the fractional Yamabe problem. A natural question is: does there exist a similar classification theorem on the solutions of the linearized fractional Yamabe equations? The answer is affirmative. In order to set up this problem, some notations are needed on the fractional Laplacian. For $\gamma\in(0,1)$, the fractional Laplacian operator $(-\Delta)^\gamma$ is defined as a pseudo-differential operator by Fourier transformation

$$\widehat{(-\Delta)^\gamma f}(\xi)=|\xi|^{2\gamma}\hat{f}(\xi)$$

for function $f:\mathbb{R}^n\to\mathbb{R}$. It follows that its principal symbol is $|\xi|^{2\gamma}$. And it can also be defined by singular integral

$$(-\Delta_x)^\gamma f(x)=C(n,\gamma)\ \text{p. v.}\int_{\mathbb{R}^n}\frac{f(x)-f(\xi)}{|x-\xi|^{n+2\gamma}}d\xi,$$

where $C(n,\gamma)$ is some normalized constant and p. v. stands for Cauchy principal value. Frank et al.[9] proved the so-called Hardy-Littlewood-Sobolev inequality: there exists constant $K(n,\gamma)>0$ such that

$$\left(\int_{\mathbb{R}^n}|w|^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{n}}\leqslant K(n,\gamma)\int_{\mathbb{R}^n}|(-\Delta_x)^{\frac{\gamma}{2}}w|^2dx$$

holds for all $w\in C_0^\infty(\mathbb{R}^n)$. And the functions, up to some constant factors,

$$w_{\lambda,a}(x)=\left(\frac{\lambda}{\lambda^2+|x-a|^2}\right)^{\frac{n-2\gamma}{2}}$$

are all the optimizers of the inequality with some $\lambda>0$ and $a\in\mathbb{R}^n$. And also these are exactly all the solutions of the fractional Yamabe equation

$$(-\Delta_x)^\gamma w=c_{n,\gamma}w^{\frac{n+2\gamma}{n-2\gamma}},\ w>0\ \text{in } \mathbb{R}^n \tag{4}$$

where $c_{n,\gamma}=2^{2\gamma}\frac{\Gamma\left(\frac{n}{2}+\gamma\right)}{\Gamma\left(\frac{n}{2}-\gamma\right)}$, where Γ is the Gamma function. This nonlocal equation is also conformally invariant like the previous Yamabe type equation (1). Here we denote by $w_0(x):=w_{1,0}(x)=(1+|x|^2)^{-(n-2\gamma)/2}$ and consider the linearized fractional Yamabe equation of (4) at w_0,

$$(-\Delta_x)^\gamma\phi=c_{n,\gamma}\frac{n+2\gamma}{n-2\gamma}w_0^{\frac{4\gamma}{n-2\gamma}}\phi,\ \text{in } \mathbb{R}^n \tag{5}$$

Then the similar classification theorem on the solutions holds for equation (5).

Theorem 0.3 Let ϕ be any bounded solution to (5) and $\gamma\in(0,1)$, then there exist constants C_i, $i=0,\cdots,n$, such that

$$\phi=C_0\left(\frac{n-2\gamma}{2}w_0+\sum_{i=1}^n x_i\frac{\partial w_0}{\partial x_i}\right)+\sum_{i=1}^n C_i\frac{\partial w_0}{\partial x_i}.$$

Remark 0.4 (Ⅰ) The functions $\frac{n-2\gamma}{2}w_0+\sum_{i=1}^n x_i\frac{\partial w_0}{\partial x_i}$ and $\frac{\partial w_0}{\partial x_i}$, $i=1,\cdots,n$, all are solutions to Eq. (5).

(Ⅱ) In fact, the solutions of (5) can be expressed directly by

$$\phi(x)=c_{n,\gamma}\frac{n+2\gamma}{n-2\gamma}\int_{\mathbb{R}^n}\frac{1}{|x-\xi|^{n-2\gamma}}w_0^{\frac{4\gamma}{n-2\gamma}}(\xi)\phi(\xi)d\xi.$$

Applying the above integral formula, Dávila et al.[10] had proved Theorem 0.3. In this paper, however, we use some scattering operators from conformal geometry to give a new proof of this classification theorem of the solutions. Moreover, using these geometric approaches, we can also give a geometric proof of Theorem 0.1.

1 Preliminary

Firstly, we recall the definition of asymptotically hyperbolic Riemannian manifolds. Suppose that X^{n+1} is a smooth manifold with smooth boundary M^n for $n\geqslant 3$. A function ρ is called a defining function of boundary M^n in X^{n+1} if

$$\rho>0 \text{ in } X^{n+1},\ \rho=0 \text{ on } M^n,\ d\rho\neq 0 \text{ on } M^n.$$

We say that metric g^+ is conformally compact if, for some defining function ρ, the metric $\bar{g}=\rho^2 g^+$ extends to $\bar{X}^{n+1}$ so that $(\bar{X}^{n+1}, \bar{g})$ is a compact Riemannian manifold. This induces a conformal class of metric $\hat{h}=\bar{g}|_{M^n}$ on M^n when defining functions vary. And the conformal manifold $(M^n,[\hat{h}])$ is called the conformal infinity of (X^{n+1}, g^+). Moreover, if the sectional curvature of conformally compact metric g^+ approaches -1 at infinity, then g^+ is called asymptotically hyperbolic.

It follows from Refs. [11-12], given any $f\in C^\infty(M^n)$, that $\mathrm{Re}(s)>\frac{n}{2}$ and $s(n-s)$ is not an L^2-eigenvalue for $-\Delta_{g^+}$, then the generalized eigenvalue problem

$$-\Delta_{g^+}u-s(n-s)u=0,\ \text{in } X^{n+1} \tag{6}$$

has a solution of the form

$$u=F\rho^{n-s}+G\rho^s,\ F,G\in C^\infty(X^{n+1}),\ F|_{\rho=0}=f.$$

Ref. [11] introduced the meromorphic family of scattering operators as

$$S(s)f=G|_{\rho=0}$$

which is a family of pseudo-differential operators, for the asymptotically hyperbolic manifold (X^{n+1},g^+) and a choice of representative $\hat{h}$ of the conformal infinity $(M^n,[\hat{h}])$. Instead one often considers the normalized scattering operators

$$P_\gamma[g^+,\hat{h}]=2^{2\gamma}\frac{\Gamma(\gamma)}{\Gamma(1-\gamma)}S\left(\frac{n}{2}+\gamma\right),$$

for $\gamma\in\left(0,\frac{n}{2}\right)$, $\gamma\notin\mathbb{N}$. And for $\gamma\in\mathbb{N}$, Graham-Jenne-Mason-Sparling[13] constructed the so-called GJMS operators as

$$P_k[g^+,\hat{h}]=c_k\mathrm{Re}\ s_{s=\frac{n}{2}+k}S(s),$$
$$c_k=(-1)^k 2^{2k}k!(k-1)!.$$

And when $k=1$, we get the conformal Laplacian

$$P_1[g^+,\hat{h}]=-\Delta_{\hat{h}}+\frac{n-2}{4(n-1)}R_{\hat{h}},$$

where $R_{\hat{h}}$ is the scalar curvature of metric $\hat{h}$.

Actually, for all $\gamma\in\left(0,\frac{n}{2}\right)$, the normalized scattering operators $P_\gamma[g^+,\hat{h}]$ are conformally covariant in the sense that, for any w, $\phi\in C^\infty(M^n)$, and $w>0$, it holds that

$$P_\gamma[g^+,w^{\frac{4}{n-2\gamma}}\hat{h}](\phi)=w^{-\frac{n+2\gamma}{n-2\gamma}}P_\gamma[g^+,\hat{h}](w\phi) \tag{7}$$

Secondly, using Caffarelli and Silvestre's extension method[14], we can express the nonlocal operator $(-\Delta_x)^\gamma$ on $\mathbb{R}^n$ with $\gamma\in(0,1)$ as a generalized Dirichlet-to-Neumann map for a weighted elliptic boundary-value problem with local differential operators defined in the upper half-space $\mathbb{R}^{n+1}_+=\{(x,y):x\in\mathbb{R}^n,\ y>0\}$, i.e. the nonlocal equation

$$(-\Delta_x)^\gamma f=h,\ \text{in } \mathbb{R}^n \tag{8}$$

is equivalent to

$$\left.\begin{aligned}
&-\mathrm{div}(y^{1-2\gamma}\nabla U)=0 && \text{in } \mathbb{R}^{n+1}_+,\\
&-d_\gamma\lim_{y\to 0}y^{1-2\gamma}\partial_y U=h && \text{on } \mathbb{R}^n,\\
&U=f && \text{on } \mathbb{R}^n
\end{aligned}\right\} \tag{9}$$

where $d_\gamma=2^{2\gamma-1}\frac{\Gamma(\gamma)}{\Gamma(1-\gamma)}$.

From now on, we assume that $\gamma\in(0,1)$ throughout the paper. If we choose $X^{n+1}=\mathbb{R}^{n+1}_+$, $g^+=g_{\mathbb{H}}$ and $\rho=y\in\mathbb{R}^+$, then the hyperbolic space $(\mathbb{R}^{n+1}_+,g_{\mathbb{H}})$, where $g_{\mathbb{H}}=y^{-2}(dy^2+|dx|^2)$, which is certainly asymptotically hyperbolic. It was proved in Ref. [15] that $P_\gamma[g_{\mathbb{H}},|dx|^2]$ agrees with $(-\Delta_x)^\gamma$ as defined on $\mathbb{R}^n$.

Theorem 1.1[15] If U is a solution of the extension problem (9) and $f=U|_{y=0}$, then $u=y^{n-s}U$ is a solution of the eigenvalue problem (6) for $s=\frac{n}{2}+\gamma$, and moreover

$$P_\gamma[g_{\mathbb{H}},|dx|^2](f)=-d_\gamma\lim_{y\to 0}y^{1-2\gamma}\partial_y U=(-\Delta_x)^\gamma f.$$

Let $\Pi:(\mathbb{S}^n\backslash\{0,\cdots,0,1\},g_0)\to(\mathbb{R}^n,|dx|^2)$ be the stereographic projection. Then the inverse map

$$\Pi^{-1}:x=(x_1,\cdots,x_n)\in\mathbb{R}^n\mapsto$$
$$\xi=(\xi_1,\cdots,\xi_n,\xi_{n+1})\in\mathbb{S}^n\backslash\{0,\cdots,0,1\}$$

is given by

$$\xi_i=\frac{2x_i}{|x|^2+1},\ i=1,\cdots,n,$$
$$\xi_{n+1}=\frac{|x|^2-1}{|x|^2+1}.$$

It is well-known that

$$(\Pi^{-1})^*g_0=w_1^{\frac{4}{n-2\gamma}}|dx|^2,$$

where

$$w_1(x)=\left(\frac{2}{|x|^2+1}\right)^{\frac{n-2\gamma}{2}}=2^{\frac{n-2\gamma}{2}}w_0(x).$$

Denote

$$\psi_0(x):=\frac{n-2\gamma}{2}w_0+\sum_{i=1}^n x_i\frac{\partial w_0}{\partial x_i},$$
$$\psi_i(x):=\frac{\partial w_0}{\partial x_i},\ i=1,\cdots,n.$$

It is easy to check that all the $\psi_k(k=0,\cdots,n)$ are solutions of Eq. (5). In order to prove Theorem 0.3, we only need to show that any solution of Eq. (5) can be the linear combination of $\psi_k(k=0,\cdots,n)$. The proof will be given in the next section.

2 Proof of Theorem 0.3

The fractional Laplacian on $\mathbb{S}^n$ can be defined as $P_\gamma^{\mathbb{S}^n}$, which is also called intertwining operator (see Ref. [16]), i.e. for any $\hat f\in C^\infty(\mathbb{S}^n)$, by stereographic projection,

$$(P_\gamma^{\mathbb{S}^n}\hat f)\circ\Pi^{-1}=\left(\frac{2}{|x|^2+1}\right)^{-\frac{n+2\gamma}{2}}(-\Delta_x)^\gamma\cdot\left(\left(\frac{2}{|x|^2+1}\right)^{\frac{n-2\gamma}{2}}\hat f\circ\Pi^{-1}\right)$$
$$=w_1^{-\frac{n+2\gamma}{n-2\gamma}}(-\Delta_x)^\gamma(w_1\hat f\circ\Pi^{-1})\qquad(10)$$

On the other hand, Branson[16] showed that $P_\gamma^{\mathbb{S}^n}$ can be rewritten in terms of $-\Delta_{\mathbb{S}^n}$ as

$$\left.\begin{aligned}P_\gamma^{\mathbb{S}^n}&=\frac{\Gamma\left(B+\gamma+\frac{1}{2}\right)}{\Gamma\left(B-\gamma+\frac{1}{2}\right)},\\ B&=\sqrt{-\Delta_{\mathbb{S}^n}+\left(\frac{n-1}{2}\right)^2}\end{aligned}\right\}\qquad(11)$$

Now let us turn to the proof of Theorem 0.3.

Proof of Theorem 0.3 For any $\hat f\in C^\infty(\mathbb{S}^n)$, by Theorem 1.1 and the conformal covariant property (7), we get from (10) that

$$(P_\gamma^{\mathbb{S}^n}\hat f)\circ\Pi^{-1}=w_1^{-\frac{n+2\gamma}{n-2\gamma}}(-\Delta_x)^\gamma(w_1\hat f\circ\Pi^{-1})=$$
$$w_1^{-\frac{n+2\gamma}{n-2\gamma}}P_\gamma[g_{\mathbb{H}},|dx|^2](w_1\hat f\circ\Pi^{-1})=$$
$$P_\gamma[g_{\mathbb{H}},w_1^{\frac{4}{n-2\gamma}}|dx|^2](\hat f\circ\Pi^{-1})\qquad(12)$$

Suppose that ϕ is a solution to the linearized equation (5). Then also by the conformal covariant property (7) and Theorem 1.1, we have

$$P_\gamma[g_{\mathbb{H}},w_1^{\frac{4}{n-2\gamma}}|dx|^2](w_1^{-1}\phi)=$$
$$w_1^{-\frac{n+2\gamma}{n-2\gamma}}P_\gamma[g_{\mathbb{H}},|dx|^2](\phi)=$$
$$w_1^{-\frac{n+2\gamma}{n-2\gamma}}(-\Delta_x)^\gamma\phi=$$
$$w_1^{-\frac{n+2\gamma}{n-2\gamma}}c_{n,\gamma}\frac{n+2\gamma}{n-2\gamma}(2^{-\frac{n-2\gamma}{2}}w_1)^{\frac{4\gamma}{n-2\gamma}}\phi=$$
$$\frac{\Gamma(n/2+\gamma+1)}{\Gamma(n/2-\gamma+1)}w_1^{-1}\phi\qquad(13)$$

Choosing $\hat f=w_1^{-1}\phi\circ\Pi$, it follows from (12) and (13) that

$$P_\gamma^{\mathbb{S}^n}\hat f=\frac{\Gamma(n/2+\gamma+1)}{\Gamma(n/2-\gamma+1)}\hat f,\ \text{on}\ \mathbb{S}^n\backslash\{0,\cdots,0,1\}.$$

With some modification to Ref. [17, Proposition 2.7], if ϕ is bounded in $\mathbb{R}^n$, we can prove $\hat f$ actually solves the equation on the whole sphere, i.e.

$$P_\gamma^{\mathbb{S}^n}\hat f=\frac{\Gamma(n/2+\gamma+1)}{\Gamma(n/2-\gamma+1)}\hat f,\ \text{on}\ \mathbb{S}^n.$$

Let $\Phi_k(x,y)$ be a homogeneous harmonic polynomial of degree k in $\mathbb{R}^{n+1}$, then

$$\Delta_{\mathbb{R}^{n+1}}\Phi_k(x,y)=0\ \text{and}\ \Phi_k(x,y)=r^k\phi_k(\xi),$$

where $r=|(x,y)|$ and $\xi\in\mathbb{S}^n$. It is easy to check that

$$\Delta_{\mathbb{S}^n}\phi_k(\xi)+k(n+k-1)\phi_k(\xi)=0.$$

In particular, for $k=1$, we have

$$\Delta_{\mathbb{S}^n}\phi_1(\xi)+n\phi_1(\xi)=0.$$

Then Proposition 1 of Ref. [18] implies $\phi_1(\xi)=\xi_i$, $i=1,\cdots,n+1$, which are the same as the previous ones, i.e. the coordinates functions restricted on $\mathbb{S}^n$. Then due to (11),

$$P_\gamma{}^{\mathbb{S}^n}\xi_i=\frac{\Gamma\left(\sqrt{n+\left(\frac{n-1}{2}\right)^2}+\gamma+\frac{1}{2}\right)}{\Gamma\left(\sqrt{n+\left(\frac{n-1}{2}\right)^2}-\gamma+\frac{1}{2}\right)}\xi_i=$$

$$\frac{\Gamma(n/2+\gamma+1)}{\Gamma(n/2-\gamma+1)}\xi_i,\ i=1,\cdots,n+1.$$

Then

$$\hat{f}=\xi_i,\ i=1,\cdots,n+1,$$

i. e.

$$\omega_1^{-1}\phi=\frac{2x^i}{|x|^2+1}=-\omega_1^{-1}\frac{2}{n-2\gamma}2^{\frac{n-2\gamma}{2}}\psi_i,\quad i=1,\cdots,n,$$

or

$$\omega_1^{-1}\phi=\frac{|x|^2-1}{|x|^2+1}=-\omega_1^{-1}\frac{2}{n-2\gamma}2^{\frac{n-2\gamma}{2}}\psi_0.$$

Hence ϕ can be achieved by $n+1$ different solutions $-\frac{2}{n-2\gamma}2^{\frac{n-2\gamma}{2}}\psi_i$, $i=1,\cdots,n$, and $-\frac{2}{n-2\gamma}2^{\frac{n-2\gamma}{2}}\psi_0$. Thanks to the linearity of equation (5), there exist some constants $C_i(i=0,1,\cdots,n)$ such that $\phi=\sum_{i=0}^{n}C_i\psi_i$. This completes the proof of Theorem 0. 3. □

Remark 2. 1 The same approach is valid for the proof of Chen-Lin's Theorem[6] by using $P_1[g_{\mathbb{H}},|dx|^2]$ instead of $P_\gamma[g_{\mathbb{H}},|dx|^2]$.

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