文章编号:0253-2778(2015)10-0871-10

Consensus of second-order multi-agent systems with directed topologies and time-varying delays

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Abstract: The consensus problem of second-order multi-agent systems with directed topologies and time-varying delays is investigated, and two cases for consensus problems of second-order multi-agent systems are analyzed. Firstly, the consensus problem for fixed directed network topology with time-varying delays is discussed, where the directed graph is assumed to have a spanning tree. Secondly, the consensus problem for switching directed network topologies and time-varying delays is studied. In both cases, delay-dependent asymptotical stability condition in terms of linear matrix inequalities (LMIs) is derived. Simulation examples illustrate the effectiveness of this method.

Key words: multi-agent systems, consensus, directed network topologies, time-varying delays, Lyapunov-Krasovskii functional, Linear matrix inequality.

CLC number: TP18 **Document code**: A doi: 10.3969/j.issn.0253-2778.2015.10.010

Citation: OU Meiying, GU Shengwei, DONG Kexiu. Consensus of second-order multi-agent systems with directed topologies and time-varying delays[J]. Journal of University of Science and Technology of China, 2015,45(10):871-880.
欧美英,谷胜伟,董可秀. 具有时变时延的二阶多智能体系统在有向拓扑下的一致性问题[J]. 中国科学技术大学学报,2015,45(10):871-880.

具有时变时延的二阶多智能体系统在有向拓扑下的一致性问题

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摘要:研究了具有时变时延的二阶多智能体系统的一致性问题,在通讯拓扑为有向图的前提下,讨论了两种情况的一致性问题. 首先,对于固定的有向网络拓扑,假设有向图含有一颗生成树的前提下,研究了具有时变时延的二阶多智能体系统的一致性问题;然后,对于切换的有向网络拓扑,讨论了具有时变时延的二阶多智能体系统的一致性问题.针对于上述两种情况,通过LMI的方法分别给出了系统渐进稳定的条件.仿真例子证明了该方法的有效性.

关键词:多智能体系统;一致性;有向网络拓扑;时变时延;Lyapunov-Krasovskii 函数;线性矩阵不等式

Received: 2014-12-19; **Revised**: 2015-05-07

Foundation item: Supported by Anhui Provincial Natural Science Foundation (1408085QF134), Natural Science Foundation of the Anhui Higher Education Institutions(KJ2015A252) and Initial Research Fund from Chuzhou University(2014qd022).

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0 Introduction

Recent years have witnessed a rapid development of distributed coordination control of multi-agent systems. A critical problem for distributed coordination is to design an appropriate algorithms or protocols to guarantee that all the agents can reach an agreement on certain quantities of interest, such as the formation center in formation control, the destination in the rendezvous problem, and so on. This problem is called consensus. In the past few years, as a fundamental problem in distributed cooperative control, consensus has been studied extensively [1-6].

In the reference, most of the existing work on the consensus problem considered the case which each agent is governed by single-integrator dynamics[1, 2]. However, when acceleration is considered as the control input, each agent should be modeled as a double-integrator dynamics[7-12]. So double-integrator dynamics can be used to model more complex processes in reality, and the insight into the second-order consensus problem is especially meaningful. In this case, the consensus problem becomes more challenging. Many consensus algorithms have also been proposed for the second-order multi-agent systems in the presence or absence of communication delays with directed or undirected network information flow. Ren et al. [13] analyzed the consensus problem for second-order multi-agent systems under directed and fixed network topology, necessary and sufficient conditions were derived under which consensus can be reached. In Ref. [14], on the basis of the finite-time control technique, the finite-time consensus tracking algorithms were given for double-integrator kinematics.

In practical situations, the disturbance of time-delay is usually unavoidable, which might make the multi-agent system to oscillate or diverge, and therefore it is important to investigate its effects on the behavior of the multi-agent system. Generally speaking, there are two kinds of delays in multi-agent systems. One is communication delays, which is related to communication from one agent to another. The other one is input delays, which occurs between actuators and controllers.

Compared with conventional control systems, dealing with delay-related problems in multi-agent systems is much more difficult and complex, since the closed-loop system matrices are usually singular. Recently, many results have been reported for multi-agent systems with the disturbance of time delay[15-20]. Bliman et al. focused on the average consensus problem and extended the results of Ref. [2] to the time-delay case[15]. Sun et al. investigated the average consensus problem for first-order multi-agent systems with switching topologies and multiple time-varying communication delays[16]. Ref. [17] considered consensus tracking problems for both first-order and second-order multi-agent systems with communication and input delays, timed-omain and frequency-domain approaches were used to derive the consensus algorithms under a fixed directed network topology. Ref. [18] investigated the consensus and robust consensus problems for first-order multi-agent systems with time-varying communication delays, in which the communication topology was fixed and assumed to have a spanning tree. In Ref. [19], the robust consensus problem of second-order multi-agent systems with constant input delays and time-varying communication delays was investigated, and the communication topology was assumed to have a spanning tree. Ref. [20] discussed the problem of leader-following consensus for second-order multi-agent system with time delay, and some sufficient conditions were given to make the system reach consensus.

In this paper, we will investigate the consensus problem for second-order continuous-time multi-agent systems with time-varying communication delays. For the communication topologies, both cases are discussed. Firstly, under the fixed directed network topology, the consensus problem for second-order multi-agent systems with time-varying delays are studied. Secondly, when the network topologies are directed arbitrary switching topologies, and each is assumed to have a spanning tree, the consensus problem with time-varying delays are discussed. In both cases, the consensus algorithms are given, sufficient conditions are obtained which guarantee that all agents asymptotically reach consensus. These conditions are expressed as linear matrix inequalities (LMIs), readily solvable by available numerical software.

1 Preliminaries

Throughout this paper, R represents the real number set. $\boldsymbol{A}^{\mathrm{T}}$ means the transpose of the matrix $\boldsymbol{A}$. $\boldsymbol{I}_{\mathrm{n}}$ is an $n \times n$ -dimensional identity matrix. We say $\boldsymbol{X} > \boldsymbol{Y}$ if $\boldsymbol{X} - \boldsymbol{Y}$ is positively definite, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are symmetric matrices of same dimensions. $\boldsymbol{a} = [a, \cdots, a]^{\mathrm{T}}$ is a column vector of appropriate dimension, where a is a constant.

1.1 Graph theory

Let $G = (V, \varepsilon, A)$ describe a directed graph of order n with the set of nodes $V = \{1, 2, \cdots, n\}$, $\varepsilon \subseteq V \times V$ is an edge set with element (i, j) that describes the communication from node i to node j. The node indexes belong to a finite index set $\Gamma = \{1, 2, \cdots n\}$. If the state of node i is available to node j, there will be an edge $(i, j) \in \varepsilon$, and we say node i is a neighbor of node j. The set of neighbors of node i is denoted by $N_i = \{j \in V: (i, j) \in \varepsilon\}$. A directed path is a sequence of ordered edges of the form $(i_1, i_2), (i_3, i_4), \cdots$, where $(i_k, i_{k+1}) \in \varepsilon$ in a directed graph. The weighted adjacency matrix is defined as $\boldsymbol{A} = [a_{ij}]$, the element a_{ij} associated with the arc of the digraph is positive, i.e., $a_{ij} > 0 \Leftrightarrow (i, j) \in \varepsilon$. Moreover, it is usually assumed that $a_{ii} = 0$ for all $i \in V$.

A digraph is strongly connected if any two distinct nodes of the graph can be connected via a directed path. A directed tree is a digraph, where every node has exactly one parent except for one node, called the root, which has no parent, and the root has a directed path to every other node. A (directed) spanning tree of a digraph is a directed tree formed by graph edges that connect all the nodes of the graph. We say that a graph has (or contains) a (directed) spanning tree if there exists a (directed) spanning tree that is a subset of the graph. Note that the condition that a digraph has a (directed) spanning tree is equivalent to the case where there exists at least one node having a directed path to all the other nodes.

The in-degree and out-degree of node i are, respectively, defined as follows: $\deg_{\mathrm{in}}(i) = \sum_{j \in N_i} a_{ij}$, $\deg_{\mathrm{out}}(i) = \sum_{j \in N_i} a_{ji}$, $i \in V$. A digraph is called balanced if all of its nodes are balanced, that is, $\deg_{\mathrm{in}}(i) = \deg_{\mathrm{out}}(i)$, $i \in V$. Then the Laplacian of the weighted digraph G is defined as $L = \mathrm{diag}\{\deg_{\mathrm{in}}(1), \cdots, \deg_{\mathrm{in}}(n)\} - \boldsymbol{A} \in R^{n \times n}$.

In contrast to a directed graph, the pairs of nodes in an undirected graph are unordered, where the edge (i, j) denotes that node i and j can obtain information from each other. An undirected graph is connected if there is an undirected path between every pair of distinct nodes. In this paper, we assume that graph G is directed.

To describe the variable interconnection topology, we define a switching signal

$$\sigma = s(t) : [0, \infty) \rightarrow \phi = \{1, 2, \cdots, N\}$$

($N \in Z^{+}$ denotes the total number of all possible directed graphs) is a switching signal that determines the network topology. The set $\phi = \{1, 2, \cdots, N\}$ is finite because at most a graph of order n is complete and has $n(n-1)$ edges. In this case, the Laplacian matrix of the graph G_σ can be represented by L_σ. If σ is a constant function, then the corresponding interconnection topology is fixed.

Example 1.1 Consider the directed graphs G_1, G_2 and G_3 in Fig. 1, the communication topology of each figure has a spanning tree, and the corresponding adjacency matrices are limited to

0-1 matrices. According to the above description, we can get that:

$$L_1 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1}$$

$$L_2 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

$$L_3 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

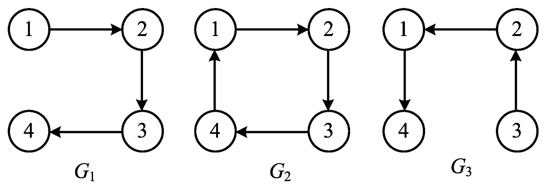

Fig. 1 Interconnection topologies for four agents

1.2 Related lemmas

Lemma 1.1[21] For Laplacian $\boldsymbol{L}$ associated with digraph G, then there exists a non-singular matrix

$$\boldsymbol{U} = \begin{pmatrix} 1 & * & \cdots & * \\ 1 & * & \cdots & * \\ \vdots & \vdots & \cdots & \vdots \\ 1 & * & \cdots & * \end{pmatrix} R^{n\times n} \tag{4}$$

such that

$$\boldsymbol{U}^{-1}\boldsymbol{L}\boldsymbol{U} = \begin{pmatrix} 0 & \boldsymbol{\alpha}^{\mathrm{T}} \\ 0_{n-1} & \bar{\boldsymbol{L}} \end{pmatrix} = \boldsymbol{\Lambda} \in R^{n\times n},\ \boldsymbol{\alpha} \in R^{n-1} \tag{5}$$

Lemma 1.2[22] For any $\boldsymbol{a}, \boldsymbol{b} \in R^n$, $K > 0$ and real positive definite matrix $\boldsymbol{\Psi}$, we have

$$2\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} \leqslant K\boldsymbol{a}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{a} + \frac{1}{K}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{\Psi}\boldsymbol{b} \tag{6}$$

Lemma 1.3[23] Suppose that a symmetric matrix is partitioned as

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{S}_1 & \boldsymbol{S}_2 \\ \boldsymbol{S}_2^{\mathrm{T}} & \boldsymbol{S}_3 \end{pmatrix} \tag{7}$$

where S_1 and S_3 are square. $\boldsymbol{S} < 0$ if and only if $\boldsymbol{S}_1 < 0$, $\boldsymbol{S}_3 - \boldsymbol{S}_2^{\mathrm{T}}\boldsymbol{S}_1^{-1}\boldsymbol{S}_2 < 0$ or $\boldsymbol{S}_3 < 0$, $\boldsymbol{S}_1 - \boldsymbol{S}_2\boldsymbol{S}_3^{-1}\boldsymbol{S}_2^{\mathrm{T}} < 0$.

1.3 Model description and problem formulation

In this paper, the continuous-time model of n agents is described as follows:

$$\ddot{x}_i = u_i \in R^m,\ i = 1,2,\cdots,n \tag{8}$$

where R^m denote the space of a real m-dimensional vector.

This information states can also be written by

$$\left.\begin{array}{l} \dot{x}_i = v_i \\ \dot{v}_i = u_i \quad i = 1,2,\cdots,n \end{array}\right\} \tag{9}$$

where $x_i \in R^m$, $v_i \in R^m$, $u_i \in R^m$ are the state, information state derivative and control input, respectively.

We say the consensus is globally asymptotically achieved if the states of the agent satisfy:

$$\| x_i - x_j \| \to 0,\ \| v_i - v_j \| \to 0,\quad t \to \infty,\ i, j \in V \tag{10}$$

for any $x_i(0) \in R^m$, $v_i(0) \in R^m$. Without loss of generality in the following analysis, let $m = 1$ just for notational simplicity.

In order to solve the consensus problem, a second-order consensus protocol is proposed as follows[13]:

$$u_i = -\sum_{j\in N_i} a_{ij}[(x_i - x_j) + \gamma(v_i - v_j)] \tag{11}$$

where $i = 1,2,\cdots,n$, $\gamma > 0$ are constants. However, in practice, there may be interconnection delays, and not all agents can instantly get the information from others. Thus, the feedback $u_i(t)$ should be constructed based on $x_j(t-\tau(t))$ and $v_j(t-\tau(t))$ for some $j \in N_i$ and time-varying delay $\tau(t) > 0$, a continuously differentiable function satisfying:

$$0 < \tau(t) < d_1,\ \dot{\tau} \leqslant d_2 < 1 \tag{12}$$

Therefore, for each agent, we use the following two local control schemes:

(Ⅰ) Fixed topology with time-varying delay.

$$u_i = -\sum_{j\in N_i} a_{ij}[(x_i(t-\tau(t)) - x_j(t-\tau(t))) + \gamma(v_i(t-\tau(t)) - v_j(t-\tau(t)))] \tag{13}$$

(Ⅱ) Switched topology with time-varying delay.

$$u_i = -\sum_{j\in N_i} a_{ij}(\sigma)[(x_i(t-\tau(t)) - x_j(t-\tau(t))) + \gamma(v_i(t-\tau(t)) - v_j(t-\tau(t)))] \tag{14}$$

Take $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$, $\boldsymbol{v} = (v_1, v_2, \cdots, v_n)^{\mathrm{T}}$. The closed-loop system (9) with Eq. (13) can be written in a matrix form

$$\left.\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{v} \\ \dot{\boldsymbol{v}} &= -\boldsymbol{L}\boldsymbol{x}(t-\tau(t)) - \gamma\boldsymbol{L}\boldsymbol{v}(t-\tau(t))\end{aligned}\right\} \quad (15)$$

and system(9) with Eq. (14) can be written as

$$\left.\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{v} \\ \dot{\boldsymbol{v}} &= -\boldsymbol{L}_k\boldsymbol{x}(t-\tau(t)) - \gamma\boldsymbol{L}_k\boldsymbol{v}(t-\tau(t))\end{aligned}\right\} \quad (16)$$

where $k = \sigma(t)$, $\boldsymbol{L}_k = \boldsymbol{L}(G_k)$ is the Laplacian of graph G_k is defined as above.

In the following, we will demonstrate the convergence of the dynamics systems (15) and (16), that is, $x_i \to x_j$, $v_i \to v_j$ as $t \to \infty$.

2 Main result

In this section, we will solve the consensus problem for second-order consensus multi-agent system (8). Two cases for the communication topology are considered : ① directed networks with fixed topology and time-varying delays, where the directed graph is assumed to have a spanning tree; ② directed networks with dynamically changing topologies and time-varying delays, where each directed graph is assumed to have a spanning tree.

This section will give two subsections to discuss the consensus problem for system (8). In the first subsection, we consider the communication topology as fixed directed network with time-varying delays, and can prove that protocol (13) makes all agents reach a consensus state. In the second subsection, we consider the communication topology as directed networks with dynamically changing topologies and time-varying delays, and will prove protocol (14) guarantees that the states of system (8) converge to a same value.

2.1 Fixed coupling topology with time-varying delays

Theorem 2.1 Consider a directed network of agents with both fixed topology and time-varying delay t, which satisfies Eq. (12). Suppose the communication topology G has a spanning tree. Given protocol (13), globally asymptotical consensus of Eq. (8) can be achieved if there exist some symmetric matrices $\boldsymbol{P} > 0$, $\boldsymbol{Q} > 0$, $\boldsymbol{M} > 0$ satisfying:

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{PB} & 0 \\ \boldsymbol{B}^{\mathrm{T}}\boldsymbol{P} & -\dfrac{1}{d_1}\boldsymbol{M} & 0 \\ 0 & 0 & \boldsymbol{A}_{22} \end{bmatrix} < 0 \quad (17)$$

and

$$\boldsymbol{A}_{11} = \boldsymbol{P}(\boldsymbol{C}+\boldsymbol{B}) + (\boldsymbol{C}+\boldsymbol{B})^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{Q} + 2d_1\boldsymbol{C}^{\mathrm{T}}\boldsymbol{MC} \quad (18a)$$

$$\boldsymbol{A}_{22} = -(1-d_2)\boldsymbol{Q} + 2d_1\boldsymbol{B}^{\mathrm{T}}\boldsymbol{MB} \quad (18b)$$

Proof Three steps will be given to prove the theorem.

Step 1 According to Lemma 1.1, with a coordinate transformation.

$$\tilde{\boldsymbol{x}} = \boldsymbol{U}^{-1}\boldsymbol{x},\ \tilde{\boldsymbol{v}} = \boldsymbol{U}^{-1}\boldsymbol{v} \quad (19)$$

system (15) becomes:

$$\left.\begin{aligned}\dot{\tilde{\boldsymbol{x}}}_1 &= \tilde{v}_1 \\ \dot{\tilde{\boldsymbol{v}}}_1 &= -\alpha^{\mathrm{T}}\tilde{\boldsymbol{x}}_2(t-\tau(t)) - \gamma\boldsymbol{\alpha}^{\mathrm{T}}\tilde{\boldsymbol{v}}_2(t-\tau(t))\end{aligned}\right\} (20)$$

and

$$\left.\begin{aligned}\dot{\tilde{\boldsymbol{x}}}_2 &= \tilde{\boldsymbol{v}}_2 \\ \dot{\tilde{\boldsymbol{v}}}_2 &= -\bar{\boldsymbol{L}}\tilde{\boldsymbol{x}}_2(t-\tau(t)) - \gamma\bar{\boldsymbol{L}}\tilde{\boldsymbol{v}}_2(t-\tau(t))\end{aligned}\right\} \quad (21)$$

For subsystem (21), let

$$\boldsymbol{\varepsilon} = \begin{pmatrix}\tilde{\boldsymbol{x}}_2 \\ \tilde{\boldsymbol{v}}_2\end{pmatrix} \in R^{n-1}$$

Then we have a compact form:

$$\dot{\boldsymbol{\varepsilon}}(t) = \boldsymbol{C}\boldsymbol{\varepsilon}(t) + \boldsymbol{B}\boldsymbol{\varepsilon}(t-\tau) \quad (22)$$

where,

$$\boldsymbol{C} = \begin{pmatrix} 0 & \boldsymbol{I}_{n-1} \\ 0 & 0 \end{pmatrix},\ \boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ -\bar{\boldsymbol{L}} & -\gamma\bar{\boldsymbol{L}} \end{pmatrix} \quad (23)$$

Step 2 For system (22), take a Lyapunov function:

$$\boldsymbol{V} = \boldsymbol{V}_1 + \boldsymbol{V}_2 + \boldsymbol{V}_3 \quad (24)$$

where,

$$\boldsymbol{V}_1 = \boldsymbol{\varepsilon}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{\varepsilon}(t),\ \boldsymbol{V}_2 = \int_{t-\tau}^{t}\boldsymbol{\varepsilon}^{\mathrm{T}}(s)\boldsymbol{Q}\,\boldsymbol{\varepsilon}(s)\mathrm{d}s \quad (25)$$

$$\boldsymbol{V}_3 = \int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{\boldsymbol{\varepsilon}}^{\mathrm{T}}(s)\boldsymbol{M}\,\dot{\boldsymbol{\varepsilon}}(s)\mathrm{d}s\mathrm{d}\theta \quad (26)$$

and $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{M} \in R^{(n-1)\times(n-1)}$. Consider the time derivative of $\boldsymbol{V}_i (i = 1,2,3)$ along with the solution of Eq. (22) respectively, yields:

$$\dot{\boldsymbol{V}}_1 = \boldsymbol{\varepsilon}^{\mathrm{T}}(t)(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{PC})\boldsymbol{\varepsilon}(t) + 2\boldsymbol{\varepsilon}^{\mathrm{T}}(t)\boldsymbol{PB}\boldsymbol{\varepsilon}(t-\tau) \quad (27)$$

$$\dot{V}_2 = \boldsymbol{\varepsilon}^{\mathrm{T}}(t)\boldsymbol{Q}\,\boldsymbol{\varepsilon}(t) - (1-\dot{\tau})\boldsymbol{\varepsilon}^{\mathrm{T}}(t-\tau)\boldsymbol{Q}\,\boldsymbol{\varepsilon}(t-\tau) \tag{28}$$

$$\dot{V}_3 = \tau\dot{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t)\boldsymbol{M}\dot{\boldsymbol{\varepsilon}}(t) - \int_{t-\tau}^{t}\dot{\boldsymbol{\varepsilon}}^{\mathrm{T}}(s)\boldsymbol{M}\dot{\boldsymbol{\varepsilon}}\mathrm{d}s \tag{29}$$

According to Lemma 1. 2, let $\boldsymbol{a}^{\mathrm{T}} = -(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\varepsilon}^{\mathrm{T}}(t))^{\mathrm{T}}$, $\boldsymbol{b}=\dot{\boldsymbol{\varepsilon}}(s)$, $\boldsymbol{\Psi}=\boldsymbol{M}$, $K=1$ and from Leibniz Newton formula we can obtain that:

$$\begin{aligned}2\boldsymbol{\varepsilon}^{\mathrm{T}}(t)\boldsymbol{PB\varepsilon}(t-\tau) &= \boldsymbol{\varepsilon}^{\mathrm{T}}(t)(\boldsymbol{PB}+\boldsymbol{B}^{\mathrm{T}}P)\boldsymbol{\varepsilon}(t) \\ &- \int_{t-\tau}^{t}2\boldsymbol{\varepsilon}^{\mathrm{T}}(t)\boldsymbol{PB}\dot{\boldsymbol{\varepsilon}}(s)\mathrm{d}s \leqslant \\ &\boldsymbol{\varepsilon}^{\mathrm{T}}(t)(\boldsymbol{PB}+\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P})\boldsymbol{\varepsilon}(t) + \\ &\tau\boldsymbol{\varepsilon}^{\mathrm{T}}(t)\boldsymbol{PBM}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P\varepsilon}(t) + \\ &\int_{t-\tau}^{t}\dot{\boldsymbol{\varepsilon}}^{\mathrm{T}}(s)\boldsymbol{M}\dot{\boldsymbol{\varepsilon}}(s)\mathrm{d}s\end{aligned} \tag{30}$$

Consequently,

$$\begin{aligned}\dot{V}_1 \leqslant\ & \boldsymbol{\varepsilon}^{\mathrm{T}}(t)[(\boldsymbol{C}+\boldsymbol{B})^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}(\boldsymbol{C}+\boldsymbol{B}) + \\ &\tau\boldsymbol{PBM}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}]\boldsymbol{\varepsilon}(t) + \\ &\int_{t-\tau}^{t}\dot{\boldsymbol{\varepsilon}}^{\mathrm{T}}(s)\boldsymbol{M}\dot{\boldsymbol{\varepsilon}}(s)\mathrm{d}s\end{aligned} \tag{31}$$

Similarly, we can get that

$$\begin{aligned}\dot{V}_3 =\ & \tau[\boldsymbol{\varepsilon}^{\mathrm{T}}(t)\boldsymbol{C}^{\mathrm{T}}\boldsymbol{MC\varepsilon}(t) + \boldsymbol{\varepsilon}^{\mathrm{T}}(t-\tau)\boldsymbol{B}^{\mathrm{T}}\boldsymbol{MB\varepsilon}(t-\tau) + \\ &2\boldsymbol{\varepsilon}^{\mathrm{T}}(t)\boldsymbol{C}^{\mathrm{T}}\boldsymbol{MB\varepsilon}(t-\tau)] - \int_{t-\tau}^{t}\dot{\boldsymbol{\varepsilon}}^{\mathrm{T}}(s)\boldsymbol{M}\dot{\boldsymbol{\varepsilon}}(s)\mathrm{d}s \leqslant \\ &2\tau[\boldsymbol{\varepsilon}^{\mathrm{T}}(t)\boldsymbol{C}^{\mathrm{T}}\boldsymbol{MC\varepsilon}(t) + \boldsymbol{\varepsilon}^{\mathrm{T}}(t-\tau)\boldsymbol{B}^{\mathrm{T}}\boldsymbol{MB\varepsilon}(t-\tau)] - \\ &\int_{t-\tau}^{t}\dot{\boldsymbol{\varepsilon}}^{\mathrm{T}}(s)\boldsymbol{M}\dot{\boldsymbol{\varepsilon}}(s)\mathrm{d}s\end{aligned} \tag{32}$$

From Eqs. (28), (31) and (32), we can obtain that

$$\begin{aligned}\dot{V} \leqslant\ & \boldsymbol{\varepsilon}^{\mathrm{T}}(t)[(\boldsymbol{C}+\boldsymbol{B})^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}(\boldsymbol{C}+\boldsymbol{B}) + d_1\boldsymbol{PBM}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P} + \\ &\boldsymbol{Q} + 2d_1\boldsymbol{C}^{\mathrm{T}}\boldsymbol{MC}]\boldsymbol{\varepsilon}(t) + \\ &\boldsymbol{\varepsilon}^{\mathrm{T}}(t-\tau)[-(1-d_2)\boldsymbol{Q} + 2d_1\boldsymbol{B}^{\mathrm{T}}\boldsymbol{MB}]\boldsymbol{\varepsilon}(t-\tau)\end{aligned} \tag{33}$$

Then, by Lemma 1. 3, the sufficient condition for $\dot{V} < 0$ is that matrix inequality (17) holds.

Step 3 From the discussion above, that Eq. (17) holds means that $\tilde{\boldsymbol{x}}_2 \to 0_{n-1}$, $\tilde{\boldsymbol{v}}_2 \to 0_{n-1}$ as $t \to \infty$. For subsystem (20), let $\tilde{\boldsymbol{x}}_1(0)$, $\tilde{\boldsymbol{v}}_1(0)$ be the initial values of $\tilde{\boldsymbol{x}}_1(t)$, $\tilde{\boldsymbol{v}}_1(t)$, we have

$$\left.\begin{aligned}\tilde{\boldsymbol{x}}_1 &= \tilde{\boldsymbol{v}}_1(0)t + \tilde{\boldsymbol{x}}_1(0) \\ \tilde{\boldsymbol{v}}_1 &= \tilde{\boldsymbol{x}}_1(0)\end{aligned}\right\} \tag{34}$$

From transformation (19), we can get

$$\boldsymbol{x} - \boldsymbol{U}\tilde{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{U}\begin{bmatrix}\tilde{\boldsymbol{v}}_1(0)t + \tilde{\boldsymbol{x}}_1(0) \\ 0_{n-1}\end{bmatrix} \to 0_n \tag{35}$$

$$\boldsymbol{v} - \boldsymbol{U}\tilde{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{U}\begin{bmatrix}\tilde{\boldsymbol{v}}_1(0) \\ 0_{n-1}\end{bmatrix} \to 0_n \tag{36}$$

So we derive the explicit formulation of consensus:

$$\boldsymbol{x} = 1_n(\tilde{\boldsymbol{v}}_1(0)t + \tilde{\boldsymbol{x}}_1(0)),\ \boldsymbol{v} = 1_n\tilde{\boldsymbol{v}}_1(0) \tag{37}$$

Therefore, $x_i - x_j \to 0_n$, $v_i - v_j \to 0_n$ for all i, $j \in V$ as $t \to \infty$, This completes the proof.

Remark 1 Note that, if we choose $\frac{1_n}{n}$ as the first column vectors of $\boldsymbol{U}$, protocol (13) solves the average consensus problems.

Remark 2 Note that in Theorems 2. 1, the communication topologies are both assumed to have a spanning tree. This is the lowest condition.

2. 2 Switched coupling topology with time-varying delays

Theorem 2. 2 Consider a directed network of agents withboth switching topology and time-varying delay τ, which satisfies Eq. (12). Suppose that the communication topology G_k is kept to have a spanning tree. Given protocol (14), globally asymptotical consensus of Eq. (8) can be achieved if there exist some symmetric matrices $\bar{\boldsymbol{P}} > 0$, $\bar{\boldsymbol{Q}} > 0$, $\bar{\boldsymbol{M}} > 0$ satisfying:

$$\begin{bmatrix}\boldsymbol{B}_{11} & \bar{\boldsymbol{P}}\boldsymbol{B}_k & 0 \\ \boldsymbol{B}_k^{\mathrm{T}}\bar{\boldsymbol{P}} & -\frac{1}{d_1}\bar{\boldsymbol{M}} & 0 \\ 0 & 0 & \boldsymbol{B}_{22}\end{bmatrix} < 0 \tag{38}$$

and

$$\boldsymbol{B}_{11} = \bar{\boldsymbol{P}}(\boldsymbol{C}+\boldsymbol{B}_k) + (\boldsymbol{C}+\boldsymbol{B}_k)^{\mathrm{T}}\bar{\boldsymbol{P}} + \bar{\boldsymbol{Q}} + 2d_1\boldsymbol{C}^{\mathrm{T}}\bar{\boldsymbol{M}}\boldsymbol{C} \tag{39a}$$

$$\boldsymbol{B}_{22} = -(1-d_2)\bar{\boldsymbol{Q}} + 2d_1\boldsymbol{B}_k^{\mathrm{T}}\bar{\boldsymbol{M}}\boldsymbol{B}_k \tag{39b}$$

$\boldsymbol{C}$ is defined in Theorem 2. 1.

Proof Similar to the proof procedure of Theorem 2. 1.

Step 1 According to Lemma 1. 1, we give the following coordinate transformation:

$$\dot{\bar{\boldsymbol{x}}} = \bar{\boldsymbol{U}}^{-1}\boldsymbol{x},\ \bar{\boldsymbol{v}} = \tilde{\boldsymbol{U}}^{-1}\boldsymbol{v} \tag{40}$$

combining Eg. (16) with Eg. (40), it can obtain

$$\left.\begin{aligned}\dot{\bar{\boldsymbol{x}}}_1 &= \bar{\boldsymbol{v}}_1 \\ \dot{\bar{\boldsymbol{v}}}_1 &= -\bar{\boldsymbol{\alpha}}^{\mathrm{T}}\bar{\boldsymbol{x}}_2(t-\tau)) - \gamma\bar{\boldsymbol{\alpha}}^{\mathrm{T}}\bar{\boldsymbol{v}}_2(t-\tau(t)))\end{aligned}\right\} \tag{41}$$

and

$$\left.\begin{aligned}\dot{\bar{\boldsymbol{x}}}_2 &= \bar{\boldsymbol{v}}_2 \\ \dot{\bar{\boldsymbol{v}}}_2 &= -\bar{\boldsymbol{L}}_k\bar{\boldsymbol{x}}_2(t-\tau)) - \gamma\bar{\boldsymbol{L}}_k\bar{\boldsymbol{v}}_2(t-\tau(t)))\end{aligned}\right\} \tag{42}$$

For the subsystem (42), let

$$\bar{\boldsymbol{\varepsilon}} = \begin{pmatrix} \bar{\boldsymbol{x}}_2 \\ \bar{\boldsymbol{v}}_2 \end{pmatrix} \in R^{n-1}.$$

Then we have a compact form:

$$\dot{\bar{\boldsymbol{\varepsilon}}}(t) = \boldsymbol{C}\bar{\boldsymbol{\varepsilon}}(t) + \boldsymbol{B}_k\bar{\boldsymbol{\varepsilon}}(t-\tau(t)) \tag{43}$$

where

$$\boldsymbol{B}_k = \begin{pmatrix} 0 & 0 \\ -\bar{\boldsymbol{L}}_k & -\gamma\bar{\boldsymbol{L}}_k \end{pmatrix} \tag{44}$$

Step 2 In the next part, we mainly consider system (43). Consider the following candidate Lyapunov function:

$$\bar{\boldsymbol{V}} = \bar{\boldsymbol{V}}_1 + \bar{\boldsymbol{V}}_2 + \bar{\boldsymbol{V}}_3 \tag{45}$$

where

$$\bar{\boldsymbol{V}}_1 = \bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t)\bar{\boldsymbol{P}}\bar{\boldsymbol{\varepsilon}}(t),\ \bar{\boldsymbol{V}}_2 = \int_{t-\tau(t)}^{t} \bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(s)\bar{\boldsymbol{Q}}\bar{\boldsymbol{\varepsilon}}(s)\mathrm{d}s \tag{46}$$

$$\bar{\boldsymbol{V}}_3 = \int_{-\tau(t)}^{0}\int_{t+\theta}^{t} \dot{\bar{\boldsymbol{\varepsilon}}}^{\mathrm{T}}(s)\bar{\boldsymbol{M}}\dot{\bar{\boldsymbol{\varepsilon}}}(s)\mathrm{d}s\mathrm{d}\theta \tag{47}$$

and $\bar{\boldsymbol{P}}, \bar{\boldsymbol{Q}}, \bar{\boldsymbol{M}} \in R^{(n-1)\times(n-1)}$.

Consider the time derivative of $\bar{\boldsymbol{V}}_i (i = 1,2,3)$ along with system(43) respectively, we can have

$$\dot{\bar{\boldsymbol{V}}}_1 = \bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t)(\boldsymbol{C}^{\mathrm{T}}\bar{\boldsymbol{P}} + \bar{\boldsymbol{P}}\boldsymbol{C})\bar{\boldsymbol{\varepsilon}}(t) + 2\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t)\bar{\boldsymbol{P}}\boldsymbol{B}_k\bar{\boldsymbol{\varepsilon}}(t-\tau(t)) \tag{48}$$

$$\dot{\bar{\boldsymbol{V}}}_2 = \bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t)\bar{\boldsymbol{Q}}\bar{\boldsymbol{\varepsilon}}(t) - (1-\dot{\tau}(t))\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t-\tau(t))\bar{\boldsymbol{Q}}\bar{\boldsymbol{\varepsilon}}(t-\tau(t)) \tag{49}$$

$$\dot{\bar{\boldsymbol{V}}}_3 = \dot{\bar{\boldsymbol{\varepsilon}}}^{\mathrm{T}}(t)\bar{\boldsymbol{M}}\dot{\bar{\boldsymbol{\varepsilon}}}(t) - \int_{t-\tau(t)}^{t} \bar{\boldsymbol{M}}\dot{\bar{\boldsymbol{\varepsilon}}}(s)\mathrm{d}s \tag{50}$$

According to Lemma 1. 2, let $a^{\mathrm{T}} = -(\boldsymbol{B}_k^{T}\bar{\boldsymbol{P}}\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t))^{\mathrm{T}}$, $\boldsymbol{b} = \dot{\bar{\boldsymbol{\varepsilon}}}(s)$, $\Psi = \bar{\boldsymbol{M}}$, $K=1$ and from Leibniz Newton formula we can obtain that

$$2\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t)\bar{\boldsymbol{P}}\boldsymbol{B}_k\bar{\boldsymbol{\varepsilon}}(t-\tau(t)) = \bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t)(\bar{\boldsymbol{P}}\boldsymbol{B}_k + \boldsymbol{B}_k^{\mathrm{T}}\bar{\boldsymbol{P}})\bar{\boldsymbol{\varepsilon}}(t) - \int_{t-\tau(t)}^{t} 2\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t)\bar{\boldsymbol{P}}\boldsymbol{B}_k\dot{\bar{\boldsymbol{\varepsilon}}}(s)\mathrm{d}s \leqslant \bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t)(\bar{\boldsymbol{P}}\boldsymbol{B}_k + \boldsymbol{B}_k^{\mathrm{T}}\bar{\boldsymbol{P}})\bar{\boldsymbol{\varepsilon}}(t) + \tau(t)\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t)\bar{\boldsymbol{P}}\boldsymbol{B}_k\bar{\boldsymbol{M}}^{-1}\boldsymbol{B}_k^{\mathrm{T}}\bar{\boldsymbol{P}}\bar{\boldsymbol{\varepsilon}}(t) + \int_{t-\tau(t)}^{t} \dot{\bar{\boldsymbol{\varepsilon}}}^{\mathrm{T}}(s)\bar{\boldsymbol{M}}\dot{\bar{\boldsymbol{\varepsilon}}}(s)\mathrm{d}s \tag{51}$$

Hence, the derivative of $\bar{V}_1$ can be written as

$$\dot{\bar{\boldsymbol{V}}}_1 \leqslant \bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t)[(\boldsymbol{C}+\boldsymbol{B}_k)^{\mathrm{T}}\bar{\boldsymbol{P}} + \bar{\boldsymbol{P}}(\boldsymbol{C}+\boldsymbol{B}_k) + \tau(t)\bar{\boldsymbol{P}}\boldsymbol{B}_k\bar{\boldsymbol{M}}^{-1}\boldsymbol{B}_k^{\mathrm{T}}\bar{\boldsymbol{P}}]\bar{\boldsymbol{\varepsilon}}(t) + \int_{t-\tau(t)}^{t} \dot{\bar{\boldsymbol{\varepsilon}}}^{\mathrm{T}}(s)\bar{\boldsymbol{M}}\dot{\bar{\boldsymbol{\varepsilon}}}(s)\mathrm{d}s \tag{52}$$

Similarly, the derivative of $\bar{\boldsymbol{V}}_3$ can be written as:

$$\dot{\bar{\boldsymbol{V}}}_3 = \tau(t)[\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t)\boldsymbol{C}^{\mathrm{T}}\bar{\boldsymbol{M}}\boldsymbol{C}\bar{\boldsymbol{\varepsilon}}(t) + \bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t-\tau(t))\boldsymbol{B}_k^{\mathrm{T}}\bar{\boldsymbol{M}}\boldsymbol{B}_k\bar{\boldsymbol{\varepsilon}}(t-\tau(t)) + 2\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t)\boldsymbol{C}^{\mathrm{T}}\bar{\boldsymbol{M}}\boldsymbol{B}_k\bar{\boldsymbol{\varepsilon}}(t-\tau(t))] - \int_{t-\tau(t)}^{t} \dot{\bar{\boldsymbol{\varepsilon}}}^{\mathrm{T}}(s)\bar{\boldsymbol{M}}\dot{\bar{\boldsymbol{\varepsilon}}}(s)\mathrm{d}s \leqslant 2\tau(t)[\bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t)\boldsymbol{C}^{\mathrm{T}}\bar{\boldsymbol{M}}\boldsymbol{C}\bar{\boldsymbol{\varepsilon}}(t) + \bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t-\tau(t))\boldsymbol{B}_k^{\mathrm{T}}\bar{\boldsymbol{M}}\boldsymbol{B}_k\bar{\boldsymbol{\varepsilon}}(t-\tau(t))] - \int_{t-\tau(t)}^{t} \dot{\bar{\boldsymbol{\varepsilon}}}(s)\bar{\boldsymbol{M}}\dot{\bar{\boldsymbol{\varepsilon}}}(s)\mathrm{d}s \tag{53}$$

Based on Eqs. (49), (52) and (53), we have

$$\dot{\bar{\boldsymbol{V}}} \leqslant \bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}[(\boldsymbol{C}+\boldsymbol{B}_k)^{\mathrm{T}}\bar{\boldsymbol{P}} + \bar{P}(C+\boldsymbol{B}_k) + d_1\bar{\boldsymbol{P}}\boldsymbol{B}_k\bar{\boldsymbol{M}}^{-1}\boldsymbol{B}_k^{\mathrm{T}}\bar{\boldsymbol{P}} + \bar{\boldsymbol{Q}} + 2d_1\boldsymbol{C}^{\mathrm{T}}\bar{\boldsymbol{M}}\boldsymbol{C}]\bar{\boldsymbol{\varepsilon}}(t) + \bar{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t-\tau(t))[-(1-d_2)\bar{\boldsymbol{Q}} + 2d_1\boldsymbol{B}_k^{\mathrm{T}}\bar{\boldsymbol{M}}\boldsymbol{B}]\bar{\boldsymbol{\varepsilon}}(t-\tau(t)) \tag{54}$$

According to Lemma 1. 3, the sufficient condition for $\dot{\bar{\boldsymbol{V}}} < 0$ is that matrix inequality (38) holds.

Step 3 From the discussion above, that Eq. (38) holds means that $\bar{\boldsymbol{x}}_2 \to 0_{n-1}$, $\bar{\boldsymbol{v}}_2 \to 0_{n-1}$ as $t \to \infty$. For subsystem (41), let $\bar{\boldsymbol{x}}_1(0)$, $\bar{\boldsymbol{v}}_1(0)$ be the initial values of $\bar{\boldsymbol{x}}_1(t)$, $\bar{\boldsymbol{v}}_1(t)$, we have

$$\left.\begin{aligned} \bar{\boldsymbol{x}}_1 &= \bar{\boldsymbol{v}}_1(0)t + \bar{\boldsymbol{x}}_1(0) \\ \bar{\boldsymbol{v}}_1 &= \bar{\boldsymbol{v}}_1(0) \end{aligned}\right\} \tag{55}$$

From the transformation (40), we can get

$$\boldsymbol{x} - \bar{\boldsymbol{U}}\bar{\boldsymbol{x}} = \boldsymbol{x} - \bar{\boldsymbol{U}}\begin{bmatrix} \bar{\boldsymbol{v}}_1(0)t + \bar{\boldsymbol{x}}_1(0) \\ 0_{n-1} \end{bmatrix} \to 0_n \tag{56}$$

$$\boldsymbol{v} - \bar{\boldsymbol{U}}\bar{\boldsymbol{v}} = \boldsymbol{v} - \bar{\boldsymbol{U}}\begin{bmatrix} \bar{\boldsymbol{v}}_1(0) \\ 0_{n-1} \end{bmatrix} \to 0_n \tag{57}$$

so we derive the explicit formulation of consensus:

$$\boldsymbol{x} = 1_n(\bar{\boldsymbol{v}}_1(0)t + \bar{\boldsymbol{x}}_1(0)),\ \boldsymbol{v} = 1_n\bar{\boldsymbol{v}}_1(0) \tag{58}$$

Therefore, $x_i - x_j \to 0_n$, $v_i - v_j \to 0_n$ for all i, $j \in \boldsymbol{V}$ as $t \to \infty$, This completes the proof.

Remark 3 Since $\boldsymbol{L}_k$ is time-varying, the matrix inequality (38) should be satisfied for all the possible graphs.

Remark 4 Consensus problem for second-order agents under switching topologies and with communication delay are also investigated[24], and the consensus protocol is given as:

$$\boldsymbol{u}_i = -k_r\sum_{j\in N_i}\boldsymbol{a}_{ij}(\sigma)(\boldsymbol{v}_i(t-\tau(t)) - \boldsymbol{v}_j(t-\tau(t))) - \sum_{j\in N_i}\boldsymbol{a}_{ij}(\sigma)(\boldsymbol{x}_i(t-\tau(t)) - \boldsymbol{x}_j(t-\tau(t)))] \tag{59}$$

where $\tau(t)$ are the time-varying communication delays satisfying condition (12), $k_r > 0$ is the relative damping gain, the communication topology G [24] is assumed to be strongly connected and balanced. The consensus protocol (14) is similar to algorithm (59). However, in this paper, the directed graph is only assumed to have a directed spanning tree, which is a weaker condition for communication topology.

3 Simulation results

In this section, two numerical examples are provided to illustrate our theoretical results derived in the previous section. The directed network graph is shown in Fig. 1. There are four agents for each directed graph (G_1, G_2 and G_3), and we can see that each of them has a spanning tree. We choose $a_{ij} = 1$ if $(i, j) \in \varepsilon$ and $a_{ij} = 0$ otherwise.

Example 3. 1 Note that in Ref. [19], when the communication topology G is fixed and has a spanning tree, the following consensus protocol

$$U_i = -b_i v_i(t) - \sum_{j \in N_i} a_{ij}[x_i(t - d_{ij}(t)) - x_j(t - d_{ij}(t))] \quad (60)$$

is proposed for system (8), where $d_{ij}(t)$ is time-varying communication delay. In order to show the performance of our control laws proposed in this paper, we give a comparison between our controllers and the previous controllers[19], i. e., we will compare the performance of controllers (13) with that of controller (60).

In this example, the digraph network G_1 in Fig. 1 is considered. In order to show the relation between graph theory and multi-agent system under digraph network G_1, we first give the mathematical model for system (8) with the consensus protocol (13).

$$\dot{x} = I_4 v \quad (61a)$$

$$\dot{v} = (L_1 \quad \gamma L_1)\begin{pmatrix} x(t-\tau(t)) \\ v(t-\tau(t)) \end{pmatrix} \quad (61b)$$

where $x = (x_1, x_2, x_3, x_4)^T$, $v = (v_1, v_2, v_3, v_4)^T$, $x(t-\tau(t)) = (x_1(t-\tau(t)), x_2(t-\tau(t)), x_3(t-\tau(t)), x_4(t-\tau(t)))^T$, $v(t-\tau(t)) = (v_1(t-\tau(t)), v_2(t-\tau(t)), v_3(t-\tau(t)), v_4(t-\tau(t)))^T$, I_4 is an 4×4 dimensional identity matrix.

For both kinds of controllers, we choose the time-varying communication delays to be $\tau(t) = d_{ij}(t) = 0.1\|\cos(3t)\|$. To have a fair comparison, the control efforts are limited to 5. 5. Under these circumstence, the initial values and parameters are chosen as follows: $\gamma = 1.5$, $b_i = 0.7$ $(i = 1,2,3,4)$. The simulation results are shown in Fig. 2～Fig. 5. Figs. 2～3 show response state trajectories (position and velocity state) for each agent by using consensus protocols (13) and (60), respectively. Figs. 4～5 show the control inputs (13) and (60), respectively. It is easy to see that the control efforts are limited to 5. 5 for both control algorithms. From Figs. 2～3 we can see that the consensus algorithm (13) can offer faster convergence performance than controller (60).

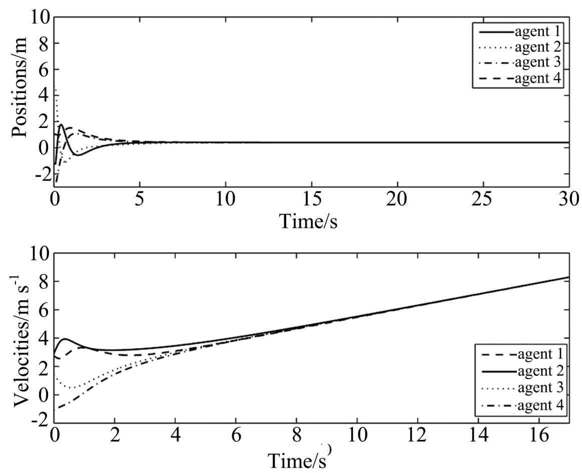

Fig. 2 Response state trajectories by using protocol (13) corresponding to fixed topology G_1

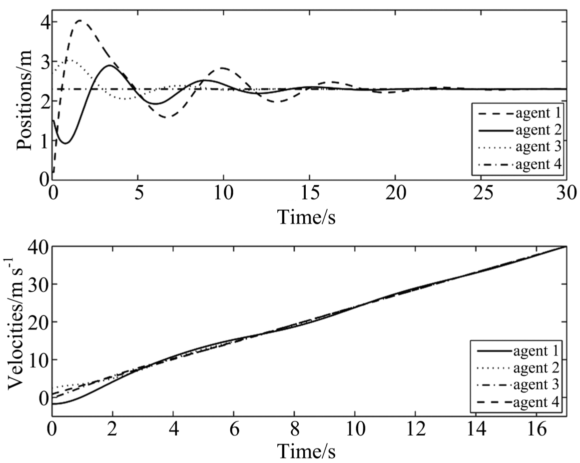

Fig. 3 Response state trajectories by using protocol (60) corresponding to fixed topology G_1

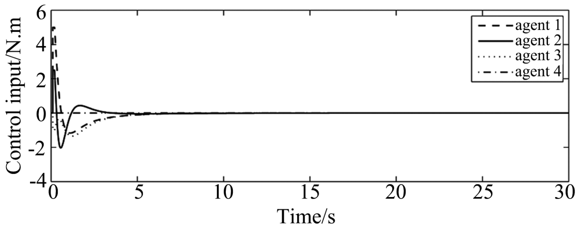

Fig. 4 Response curves of control inputs (13)

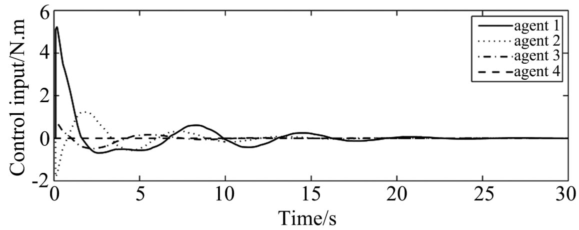

Fig. 5 Response curves of control inputs (60)

Example 3.2 In this example, the consensus protocol (14) is considered. The directed graphs G_1, G_2 and G_3 represent the information exchange among agents, as shown in Fig. 1. We assumed the switch sequence of the directed network is $G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow G_2 \rightarrow G_1$, and communication topologies are switched every five seconds. The parameter of consensus protocol (14) is chosen as $\gamma = 1.5$ and the time-varying delay is taken as $\tau(t) = 3\cos(6t)$. The simulation results are shown in Figs. 6~7. It can be seen that all agents will reach consensus while the interconnection is dynamically changing.

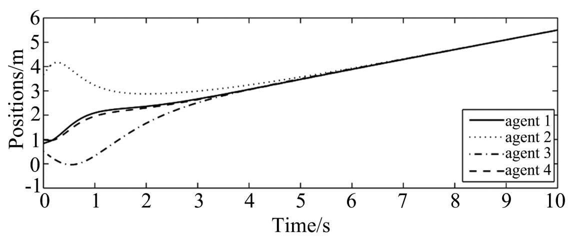

Fig. 6 Response trajectories by using protocol (14) corresponding to switching topologies G_1, G_2, G_3.

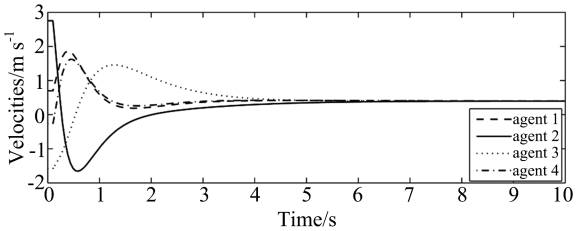

Fig. 7 Response trajectories by using protocol (14) corresponding to switching topologies G_1, G_2, G_3.

4 Conclusion

In this paper, the consensus problems of second-order multi-agent systems with time-varying delays have been investigated. The consensus stability has been guaranteed under both fixed and switched interconnection topologies. In terms of linear matrix inequalities (LMIs), Lyapunov-Krasovskii functional method has been employed in the stability analysis. Numerical examples have been given to illustrate the theoretical result.

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