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Numerical simulation between long and short waves by multisymplectic method

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Abstract: The multisymplectic structure-preserving scheme for the Schrödinger-KdV equation was investigated. First the canonical formulation of the equation was discussed. Then, it was discretized by the multisymplectic integrator, such as a midpoint integrator. Numerical results were presented to illustrate the validity of the new scheme.

Key words: Schrödinger-KdV equation; long and short waves; multisymplectic method

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长短波方程多辛数值模拟

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摘要: 主要研究了 Schrödinger-KdV 方程的保多辛结构的数值格式. 首先讨论了它的正则方程组, 然后对此方程组用多辛格式, 例如中点格式离散. 数值实验验证了格式的有效性.

关键词: Schrödinger-KdV 方程; 长短波; 多辛

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0 Introduction

The interaction of nonlinear wave between long and short waves in dispersive media can be modeled by the coupled Schrödinger-Korteweg-de-Vries (SKdV henceforth):

$$\left.\begin{aligned} &iS_t + S_{xx} - SL = 0, \\ &i^2 = -1,\ x \in \mathbb{R},\ t > 0 \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} &L_t + \beta L_{xxx} + \frac{1}{2}\alpha(L^2)_x - (|S|^2)_x = 0, \\ &x \in \mathbb{R},\ t > 0 \end{aligned}\right\} \tag{2}$$

where S and L denote the complex amplitude of the short wave and the long wave, respectively, α and β are the nonlinear and dispersive parameters. The model is used in situations where the phase velocity of the long wave is almost equal to the group velocity of the short wave. We perfect (1)～(2) by prescribing the initial-boundary conditions

$$\lim_{|x|\to\infty} S(x,t) = 0,\ \lim_{|x|\to\infty} L(x,t) = 0 \tag{3}$$

$$S(x,0) = S_0(x),\ L(x,0) = L_0(x) \tag{4}$$

By straightforward computation, we have the following proposition.

Proposition 1 The SKdV system (1)～(4) has at least the following four conserved quantities or invariants.

The first one is wave energy

$$\mathscr{D}(t) = \int_{\mathbb{R}} |S|^2 dx = \int_R |S_0(x)|^2 dx = \mathscr{D}(0) \tag{5}$$

The second one is the number of particles

$$\mathscr{P}(t) = \int_{\mathbb{R}} L dx = \mathscr{P}(0) \tag{6}$$

The third one is the Hamiltonian energy

$$\mathscr{H}(t) = \int_{\mathbb{R}} \left(|S_x|^2 + L|S|^2 + \frac{\beta}{2} L_x^2 - \frac{\alpha}{6} L^3 \right) dx = \mathscr{H}(0) \tag{7}$$

The last one is the wave momentum

$$\mathscr{M}(t) = \int_{\mathbb{R}} [(S\overline{S}_x - \overline{S}S_x) + L^2] dx = \mathscr{M}(0) \tag{8}$$

For detailed proof of the proposition, see Ref. [22].

The SKdV equation has been studied theoretically by some authors[1,2,5,10,19,22]. There were some numerical investigation about the SKdV-like equation[4,9]. However, fewer numerical methods have been proposed for SKdV equation (1)～(2).

Multisymplectic integrator has been a hot topic over the last decades[3,8,11-13,15-18,21]. Hong et al. studies the multisymplecticity for a Runge-Kutta method for Hamiltonian systems thoroughly[12-15]. Wang et al. consider the theories and application of multisymplectic integrators and extended the idea to general local structure-preserving integrators[6-7,11]. Now multisymplectic integrator was widely used in numerical simulations for Hamiltonian PDEs, such as Dirac equation[13], various Schrödinger equations[18], KdV equation[3,21], SRLW equation[16]. The SKdV equation (1)～(2) is a multisymplectic Hamiltonian system. We will discuss its multisymplectic integrator in the paper.

The paper is outlined as follows: In Section 1, the multisymplectic structure of SKdV equation (1)～(2) is presented. In Section 2, a multisymplectic integrator is constructed. Some numerical results are presented in Section 3.

1 Multisymplectic structure of the SKdV

In this section, we investigate the multisymplectic structure of the SKdV equation and other local conservation laws.

Let the short wave complex function $S=p+iq$. Introducing the Legendre transformation $p_x=\varphi$, $q_x=\psi$, $L_x=\gamma$, $\eta_x=L$, and let $z=(p,q,\varphi,\psi,\eta,L,\gamma,w)^{\mathrm{T}}$, we get the multisymplectic formation of SKdV equation (1)～(2)

$$M\partial_t z + K\partial_x z = \nabla_z S(z) \tag{9}$$

where M, K are skew-symmetric matrices of the SKdV (1)～(2) with

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$K=\begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \end{bmatrix},$$

and with the Hamiltonian function

$$S(z)=-\frac{1}{2}L(p^2+q^2)+\frac{1}{2}(\varphi^2+\psi^2)+\frac{1}{12}\alpha L^3-\frac{1}{2}Lw+\frac{1}{4}\beta\gamma^2.$$

Correspondingly, we have the following three local conservation laws.

① The multisymplectic conservation law is

$$\partial_t\omega+\partial_x\kappa=0 \tag{10}$$

where the symplectic density is

$$\omega=\mathrm{d}z\wedge\mathrm{d}z=2\mathrm{d}p\wedge\mathrm{d}q+\frac{1}{2}\mathrm{d}\eta\wedge\mathrm{d}L,$$

and the symplectic flux is

$$\kappa(z)=2\mathrm{d}\varphi\wedge\mathrm{d}p+2\mathrm{d}\psi\wedge\mathrm{d}q+\mathrm{d}\eta\wedge\mathrm{d}w+\beta\mathrm{d}\gamma\wedge\mathrm{d}L.$$

② The local energy conservation law is

$$\partial_t E(z)+\partial_x F(z)=0 \tag{11}$$

where the energy density is

$$E(z)=L(p^2+q^2)-\frac{1}{6}\alpha L^3+(\varphi^2+\psi^2)+\frac{1}{2}\beta\gamma^2,$$

and the energy flux is

$$F(z)=(w\eta_t-\beta L_t\gamma)-2(\varphi p_t+\psi q_t).$$

③ The local momentum conservation law is

$$\partial_t I(z)+\partial_x G(z)=0 \tag{12}$$

where the momentum density is

$$I(z)=q\varphi-p\psi+\frac{1}{2}L^2=\mathscr{I}(S\overline{S}_x)+\frac{1}{2}L^2,$$

the momentum flux is

$$G(z)=L(p^2+q^2)-(\varphi^2+\psi^2)-\frac{1}{6}\alpha L^3+Lw-\frac{1}{2}\beta\gamma^2+(pq_t-qp_t)-\frac{1}{2}L\eta_t.$$

The local conservation laws are point-by-point, which means that changes in density along the time direction are just offset by changes of flux in the space direction. With suitable boundary conditions, such as periodic or homogenous boundary conditions, integrating the local conservation laws (10), (11) and (12), respectively, over the whole considered domain, the global symplecticity, energy and momentum conservation laws were obtained

$$\left.\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\omega(x,t)\mathrm{d}x=0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}E(x,t)\mathrm{d}x=0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}I(x,t)\mathrm{d}x=0 \end{aligned}\right\} \tag{13}$$

It should be noted that the latter two global conservation laws of (13) are just the conservation laws (7) and (8), respectively. This provides a new way to obtain global conservation laws.

2 Multisymplectic structure-preserving integrator

As mentioned in the previous section, preserving the original geometry structure as much as possible has become a basic principle for construct numerical integrators. The Hamiltonian form of the SKdV satisfies the three local conservation laws. However, in general, it is impossible to require a numerical method to preserve all the three local conservation laws. We will devise a multisymplectic numerical integrator which preserves the multisymplectic conservation law (10).

Application of the midpoint scheme to (9), results in the multisymplectic preserving scheme[13]

$$M\frac{z_{j+\frac{1}{2}}^{n+1}-z_{j+\frac{1}{2}}^{n}}{\tau}+K\frac{z_{j+1}^{n+\frac{1}{2}}-z_{j}^{n+\frac{1}{2}}}{h}=\nabla_z S(z_{j+\frac{1}{2}}^{n+\frac{1}{2}}) \tag{14}$$

where

$$z_{j+\frac{1}{2}}^{n+\frac{1}{2}}=\frac{1}{2}(z_j^{n+\frac{1}{2}}+z_{j+1}^{n+\frac{1}{2}})=\frac{1}{4}(z_j^n+z_{j+1}^n+z_j^{n+1}+z_{j+1}^{n+1}).$$

The component formulation is

$$\frac{1}{\tau}(q_{j+\frac{1}{2}}^{n+1}-q_{j+\frac{1}{2}}^{n})-\frac{1}{h}(\varphi_{j+1}^{n+\frac{1}{2}}-\varphi_{j}^{n+\frac{1}{2}})=-L_{j+\frac{1}{2}}^{n+\frac{1}{2}}p_{j+\frac{1}{2}}^{n+\frac{1}{2}} \tag{15a}$$

$$\frac{1}{\tau}(p^{n+1}_{j+\frac{1}{2}} - p^{n}_{j+\frac{1}{2}}) + \frac{1}{h}(\psi^{n+\frac{1}{2}}_{j+1} - \psi^{n+\frac{1}{2}}_{j}) = L^{n+\frac{1}{2}}_{j+\frac{1}{2}} q^{n+\frac{1}{2}}_{j+\frac{1}{2}} \tag{15b}$$

$$\frac{1}{h}(p^{n+\frac{1}{2}}_{j+1} - p^{n+\frac{1}{2}}_{j}) = \varphi^{n+\frac{1}{2}}_{j+\frac{1}{2}} \tag{15c}$$

$$\frac{1}{h}(q^{n+\frac{1}{2}}_{j+1} - q^{n+\frac{1}{2}}_{j}) = \psi^{n+\frac{1}{2}}_{j+\frac{1}{2}} \tag{15d}$$

$$\frac{1}{2\tau}(L^{n+1}_{j+\frac{1}{2}} - L^{n}_{j+\frac{1}{2}}) = -\frac{1}{h}(w^{n+\frac{1}{2}}_{j+1} - w^{n+\frac{1}{2}}_{j}) \tag{15e}$$

$$\frac{1}{2\tau}(\eta^{n+1}_{j+\frac{1}{2}} - \eta^{n}_{j+\frac{1}{2}}) + \frac{\beta}{h}(\gamma^{n+\frac{1}{2}}_{j+1} - \gamma^{n+\frac{1}{2}}_{j}) = -\frac{\alpha}{2}(L^{n+\frac{1}{2}}_{j+\frac{1}{2}})^2 + |S^{n+\frac{1}{2}}_{j+\frac{1}{2}}|^2 + w^{n+\frac{1}{2}}_{j+\frac{1}{2}} \tag{15f}$$

$$\frac{1}{h}(L^{n+\frac{1}{2}}_{j+1} - L^{n+\frac{1}{2}}_{j}) = \gamma^{n+\frac{1}{2}}_{j+\frac{1}{2}} \tag{15g}$$

$$\frac{1}{h}(\eta^{n+\frac{1}{2}}_{j+1} - \eta^{n+\frac{1}{2}}_{j}) = L^{n+\frac{1}{2}}_{j+\frac{1}{2}} \tag{15h}$$

Eliminating the canonical momentum, one obtains a more convenient and more practical multisymplectic integrator

$$\frac{i}{\tau}\delta_t(S^{n}_{j+\frac{1}{2}} + S^{n}_{j-\frac{1}{2}}) + \frac{2}{h^2}\delta_x^2 S^{n+\frac{1}{2}}_{j} = L^{n+\frac{1}{2}}_{j+\frac{1}{2}} S^{n+\frac{1}{2}}_{j+\frac{1}{2}} + L^{n+\frac{1}{2}}_{j-\frac{1}{2}} S^{n+\frac{1}{2}}_{j-\frac{1}{2}} \tag{16}$$

$$\begin{aligned}
&\frac{1}{\tau}[(L^{n+\frac{1}{2}}_{j+\frac{1}{2}} + 2L^{n+\frac{1}{2}}_{j-\frac{1}{2}} + L^{n+\frac{1}{2}}_{j-\frac{3}{2}}) - (L^{n-\frac{1}{2}}_{j+\frac{1}{2}} + 2L^{n-\frac{1}{2}}_{j-\frac{1}{2}} + L^{n-\frac{1}{2}}_{j-\frac{3}{2}})] + \\
&\frac{2\beta}{h^3}[(L^{n+\frac{1}{2}}_{j+1} - 3L^{n+\frac{1}{2}}_{j} + 3L^{n+\frac{1}{2}}_{j-1} - L^{n+\frac{1}{2}}_{j-2}) + (L^{n-\frac{1}{2}}_{j+1} - 3L^{n-\frac{1}{2}}_{j} + 3L^{n-\frac{1}{2}}_{j-1} - L^{n-\frac{1}{2}}_{j-2})] = \\
&-\frac{\alpha}{2h}[(L^{n+\frac{1}{2}}_{j+\frac{1}{2}})^2 - (L^{n+\frac{1}{2}}_{j-\frac{3}{2}})^2 + (L^{n-\frac{1}{2}}_{j+\frac{1}{2}})^2 - (L^{n-\frac{1}{2}}_{j-\frac{3}{2}})^2] + \\
&\frac{1}{h}[|S^{n+\frac{1}{2}}_{j+\frac{1}{2}}|^2 - |S^{n+\frac{1}{2}}_{j-\frac{3}{2}}|^2 + |S^{n-\frac{1}{2}}_{j+\frac{1}{2}}|^2 - |S^{n-\frac{1}{2}}_{j-\frac{3}{2}}|^2]
\end{aligned} \tag{17}$$

3 Numerical experiments

In this part, we investigate the proposed multisymplectic integrator (16)～(17) numerically, including the convergence rate and the conservation properties. For this purpose, we present the exact solution of the SKdV system (1)～(2)[20]:

$$\left.\begin{aligned}
&S(x,t) = A\operatorname{sech}^2 p(x-\lambda t)\exp\left(\frac{\lambda}{2}i(x-Vt)\right), \\
&L(x,t) = -6p^2\operatorname{sech}^2 p(x-\lambda t)
\end{aligned}\right\} \tag{18}$$

where $p^2=\frac{\lambda}{4\beta}$, $A^2=18p^4(\alpha+2\beta)$ and $V=\frac{\lambda}{2}-\frac{2}{\beta}$.

Three parameters are taken from (18): $\beta=1$, $\alpha=1/2$ and $\lambda=1$. To satisfy the boundary condition (3), we take the spatial interval $[a,b]$ large enough such that the error due to boundary truncation is negligible. The spatial-temporal domain is $[-200, 200]\times[0, 6]$. We use multisymplectic scheme (16)～(17) to simulate the problem with the mesh division $h=0.1$, $\tau=0.03$. We present the profiles of wave functions over the spatial interval $[-16, 24]$ instead of the whole interval $[-200, 200]$ so that the pictures can be clearer. The profiles of the numerical and exact solutions at $t=6$ are plotted in Figs. 1 and 2(a). The numerical error in the maximum norm of the solution is presented in Fig. 2(b). In the figure, we use different time levels $N=[100, 200,$

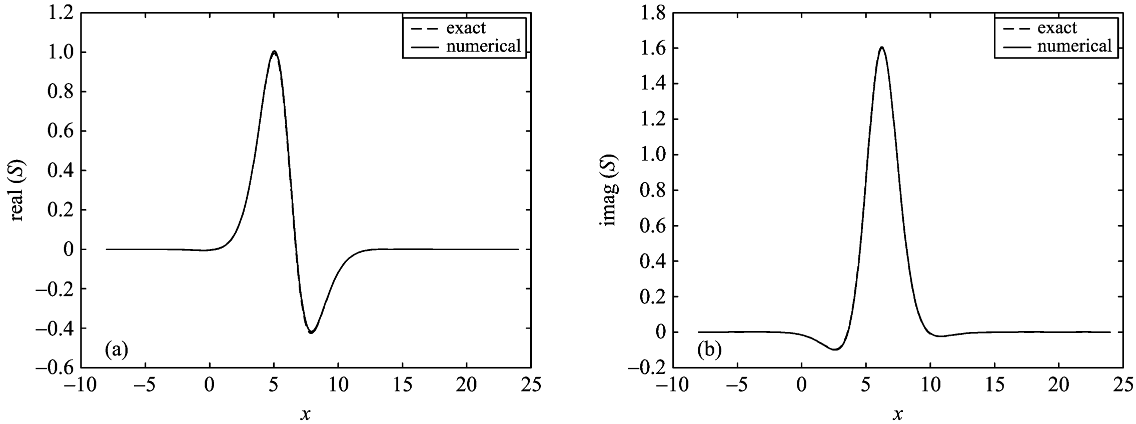

Fig. 1 The real and imaginary parts of the wave function. (a) real; (b) imaginary

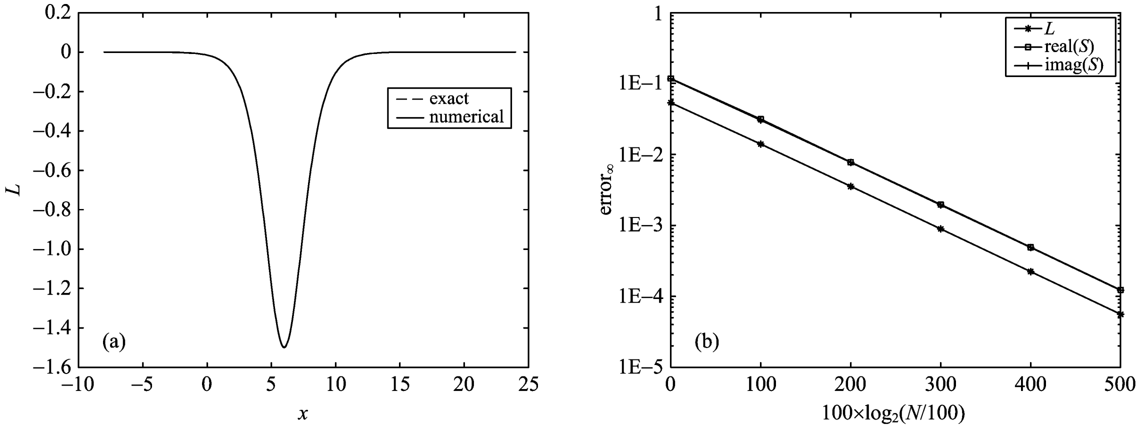

Fig. 2　The wave function L (a) and the error v. s. N (b)

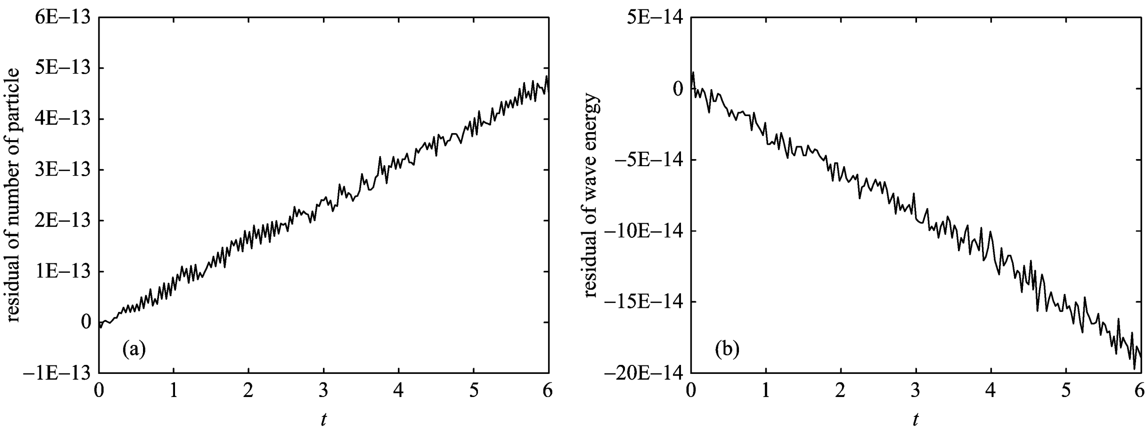

Fig. 3　The residuals of the number of particles (a) and wave energy (b)

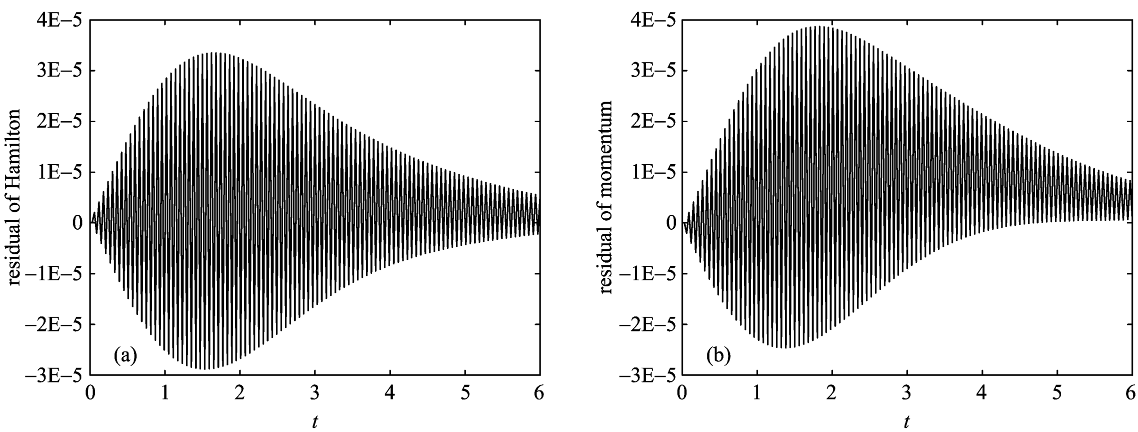

Fig. 4　The residuals of Hamilton (a) and momentum (b)

400, 800, 1 600, 3 200] with fixed h. Figs. 3 and 4 report the residual of conservation quantities, including the number of the particles, the wave energy, the Hamiltonian energy and the momentum. From these figures, it is observed that the scheme can simulate the original problem very well and is of second convergence rate. Moreover, the scheme can preserve the number of particles and wave energy exactly, and the residuals of Hamiltonian energy and momentum are very small, up to 1×10^{-5} all along.

4 Conclusion

In this paper, we developed a multisymplectic method for the coupled Schrödinger-KdV equation to describe the motion between long and short waves. The new method can perfectly simulate the motion of the waves during a long time. Furthermore, the invariants of the continuous problem can be well preserved. Their residuals can be controlled in a small range despite of not exactly. Establishing the numerical theory for the multisymplectic integrator of SKdV equation is our future work. Constructing efficient multisymplectic method for SKdV equation will be considered.

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