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Gradient estimates for f-exponentially harmonic functions on complete Riemannian manifolds

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Abstract: For smooth metric measure spaces $(M, g, e^{-f}d\text{vol})$, the gradient estimates of positive solutions to the f-exponentially harmonic functions was considered by using the maximum principle. Then a Liouville type theorem was obtained when the Bakry-Emery Ricci tensor was nonnegtive and the sectional curvature was bounded by a negative constant. This generalizes a result in Ref. [Wu J, Ruan Q, Yang Y H. Gradient estimates for exponentially harmonic functions on complete Riemannian manifolds. Manuscripta Mathematica, 2014, 143(3-4): 483-489], which is covered in the case where f is a constant.

Key words: f-exponentially harmonic function; gradient estimate; Liouville type theorem

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完备黎曼流形上 f 指数调和型函数的梯度估计

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摘要:对于光滑的度量测度空间$(M, g, e^{-f}d\text{vol})$,通过使用极大值原理,考虑了 f 指数调和型函数的梯度估计.当 Bakry-Emery Ricci 张量非负并且截面曲率有负下界,可以得到刘维尔型定理.当 f 为常数时,即为文献[Wu J, Ruan Q, Yang Y H. Gradient estimate for exponentially harmonic functions on complete Riemannian manifolds. Manuscripta Mathematica, 2014, 143(3-4): 483-489]中的结果.

关键词:f 指数调和型函数;梯度估计;刘维尔型定理

0 Introduction

The notion of exponentially harmonic function was put forward by Eells and Lemaire[1]. For some useful properties of exponentially harmonic functions, see Ref. [2]. In Ref. [3], Hong obtained a Liouville type theorem for exponentially harmonic functions by assuming that the sectional curvature is nonnegtive.

Recently, Wu et al.[4] considered the same

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question under a weaker condition. Actually, they obtained a Liouville type theorem for positive exponentially harmonic functions. They proved the following:

Theorem 0.1 Let M be an m dimensional complete Riemannian manifold with nonnegative Ricci curvature and sectional curvature bounded below by $-K$, $K>0$, $p\in M$, $B_p(R)$ the geodesic ball at p with radius R. Then for a positive exponentially harmonic function on M, one has the following estimate on $B_p(R)$

$$|\nabla u|^2\leqslant\left(\frac{C_1(m)}{R^2}+\frac{C_2(m,K)}{R}\right)(\sup_{B_p(2R)}u)^2,$$

where C_1 and C_2 are constants.

In Ref. [5], Kotschwar et al. deal with the p-harmonic function in a general way.

In this paper, we study the f-exponentially harmonic function. Let M be a complete Riemannian manifold with a smooth metric measure spaces $(M, g, e^{-f}d\text{vol})$, where f is a smooth real valued function on M. Consider the following equation:

$$\operatorname{div}(\exp(e(u))\nabla u)-\exp(e(u))\nabla f\cdot\nabla u=0 \tag{1}$$

on M, where $e(u)=\frac{1}{2}|\nabla u|^2$. In fact, it is the Euler-Langrange equation of the following weighted exponentially functional

$$E_f(u)=\int_M\exp(e(u))e^{-f}d\text{vol}.$$

If u satisfies (1), we call the function u an f-exponentially harmonic function.

The Bakry-Emery Ricci tensor is defined by $\text{Ric}_f=\text{Ric}+\text{Hess } f$. Based on Ref. [4]'s argument we obtained the following Liouville type theorem:

Theorem 0.2 Let M be a complete Riemanian manifold with smooth metric measure $(M, g, e^{-f}d\text{vol})$ with $\text{Ric}_f\geqslant 0$ and sectional curvature is bounded below. If u is a bounded f-exponentially harmonic function defined on M, then u is a constant.

Actually, we will show the following gradient estimates for the f-exponentially harmonic functions.

Theorem 0.3 Let $(M, g, e^{-f}d\text{vol})$ be a complete smooth metric measure space with $\text{Ric}_f\geqslant 0$ and sectional curvature bounded from below by $-K$, $K>0$, $p\in M$, $B_p(R)$ the geodesic ball at p with radius R. Assume $R\geqslant 1$. Then for a positive f-exponentially harmonic function on M, one has the following estimate on $B_p(R)$:

$$|\nabla u|^2\leqslant\left(\frac{C_1(\alpha)}{R^2}+\frac{C_2(\alpha,K)}{R}\right)(\sup_{B_p(2R)}u)^2,$$

where $\alpha=\max_{q\in\{q:d(p,q)=1\}}\Delta_f r(q)$, C_1 and C_2 are constants.

1 Proof of Theorem 0.3

We will calculate in a local orthonormal frame field $\{e_1,e_2,\cdots,e_m\}$. Under this local orthonormal frame, the f-exponentially harmonic function equation can be written as

$$\exp(e(u))\Big(\sum_{i,j}(a_{ij}u_{ij}-f_iu_i)\Big)=0,$$

where $a_{ij}=\delta_{ij}+u_iu_j$, $\delta_{ij}=0$, $i\neq j$ and $\delta_{ij}=1$, $i=j$. It is easy to see that (a_{ij}) is a positive definite matrix.

We also use a C^2 cut-off function $\eta=\eta(t)$, $t\in[0,+\infty)$, which is defined as follows

$$\eta(t)=\begin{cases}1, & t\in[0,1];\\ >0, & t\in(1,2);\\ 0, & t\in[2,+\infty)\end{cases} \tag{2}$$

satisfying that as $t\in(1,2)$,

$$0\geqslant\frac{\eta'(t)}{\eta^{\frac{1}{2}}(t)}\geqslant -C \tag{3}$$

and

$$|\eta''(t)|\leqslant C \tag{4}$$

for some constant $C>0$.

Let $\rho(x)$ denote the geodesic distance between p and x and set

$$\phi(x)=\eta\left(\frac{\rho(x)}{R}\right) \tag{5}$$

Then we have

$$\frac{|\nabla\phi|^2}{\phi}=\frac{|\eta'|^2}{\eta R^2}\leqslant\frac{C^2}{R^2} \tag{6}$$

We sometimes use η and its derivatives to express their composition with $\frac{\rho(x)}{R}$, e.g. $\eta=\eta\left(\frac{\rho(x)}{R}\right)$.

We begin to prove Theorem 0.3. Consider the function

$$G=\frac{\phi\,|\nabla u|^2}{(\theta-u)^\beta},$$

where $\theta=2\sup_{B_p(2R)}u$ and β is a positive constant which will be determined later on. Since G vanishes on the boundary of $B_p(2R)$, we can assume G achieves its maximum at an interior point $x_0\in B_p(2R)$. Without loss of generality, we can assume that $G(x_0)>0$ and that x_0 is not in the cut-locus of p (a standard argument, see Ref. [6]). We set $F=\ln G$. Then by means of maximum principle, we have at x_0,

$$\nabla F=0 \tag{7}$$

and

$$(F_{ij})\leqslant 0 \tag{8}$$

Since (a_{ij}) is positive, we have

$$a_{ij}F_{ij}-F_if_i\leqslant 0 \tag{9}$$

A direct computation shows that

$$F_i=\frac{\phi_i}{\phi}+\frac{|\nabla u|_i^2}{|\nabla u|^2}+\frac{\beta u_i}{\theta-u}=0 \tag{10}$$

$$F_{ij}=\frac{\phi_{ij}}{\phi}-\frac{\phi_i\phi_j}{\phi^2}+\frac{|\nabla u|_{ij}^2}{|\nabla u|^2}+\frac{\beta u_iu_j}{(\theta-u)^2}+\frac{\beta u_{ij}}{\theta-u}-\frac{|\nabla u|_i^2\,|\nabla u|_j^2}{|\nabla u|^4} \tag{11}$$

As u is f-exponentially harmonic, we have $a_{ij}u_{ij}-f_iu_i=0$. So the above (9) together with (11) can be written as

$$\frac{a_{ij}\phi_{ij}-f_i\phi_i}{\phi}-\frac{a_{ij}\phi_i\phi_j}{\phi^2}+\frac{a_{ij}\,|\nabla u|_{ij}^2-|\nabla u|_i^2f_i}{|\nabla u|^2}+\frac{\beta a_{ij}u_iu_j}{(\theta-u)^2}-\frac{a_{ij}\,|\nabla u|_i^2\,|\nabla u|_j^2}{|\nabla u|^4}\leqslant 0. \tag{12}$$

We begin to estimate the first term of (12). By using (4), we get

$$\begin{aligned}\frac{a_{ij}\phi_{ij}-f_i\phi_i}{\phi}&=\frac{\eta''(1+u_iu_j\rho_i\rho_j)+R\eta'(\Delta_f\rho+u_iu_j\rho_{ij})}{R^2\phi}\geqslant\\&-\frac{C}{\eta R^2}\left(1+\frac{u_i^2\rho_j^2+u_j^2\rho_i^2}{2}\right)+\frac{\eta'}{\eta R}(\Delta_f\rho+u_iu_j\rho_{ij})=\\&-\frac{C}{\eta R^2}(1+|\nabla u|^2)+\frac{\eta'}{\eta R}(\Delta_f\rho+u_iu_j\rho_{ij})\end{aligned} \tag{13}$$

If $1\leqslant R<\rho(x_0)<2R$, by the Theorem 2.1 in Ref. [7], we have

$$\frac{\eta'}{\eta R}\Delta_f\rho\geqslant\frac{\eta'}{\eta R}\alpha\geqslant-\frac{C}{\eta R}\,|\alpha| \tag{14}$$

where $\alpha=\max_{q\in\{q:d(p,q)=1\}}\Delta_f r(q)$. Also, using the Hessian comparison theorem, we have

$$\begin{aligned}\frac{\eta'}{\eta R}u_iu_j\rho_{ij}&\geqslant\frac{\eta'}{\eta R}\sqrt{K}\coth(\sqrt{K}\rho)\,|\nabla u|^2\geqslant\\&\frac{\eta'}{\eta R}(\rho^{-1}+\sqrt{K})\,|\nabla u|^2\geqslant\\&-\frac{C}{\eta R}(\rho^{-1}+\sqrt{K})\,|\nabla u|^2\end{aligned} \tag{15}$$

Now we have the estimate of the first term

$$\begin{aligned}\frac{a_{ij}\phi_{ij}-f_i\phi_i}{\phi}&\geqslant-\frac{C}{\eta R^2}(1+|\nabla u|^2)-\\&\frac{|\alpha|\,C}{\eta R}-\frac{C}{\eta R}(\rho^{-1}+\sqrt{K})\,|\nabla u|^2\geqslant\\&-\frac{C}{\eta R^2}(1+|\nabla u|^2)-\frac{|\alpha|\,C}{\eta R}(1+\sqrt{K})-\\&\frac{C}{\eta R}(1+\sqrt{K})\,|\nabla u|^2\geqslant\\&-\frac{C}{\eta R^2}(1+|\nabla u|^2)-\\&\frac{C}{\eta R}(1+\sqrt{K})(|\alpha|+|\nabla u|^2)\geqslant\\&-\frac{C}{\eta R^2}(A+|\nabla u|^2)-\\&\frac{C}{\eta R}(1+\sqrt{K})(A+|\nabla u|^2)\geqslant\\&\frac{1}{\eta}(A+|\nabla u|^2)\left(-\frac{C}{R^2}-\frac{C}{R}-\frac{C\sqrt{K}}{R}\right)\end{aligned} \tag{16}$$

where $A=\max\{|\alpha|,1\}$.

If $0\leqslant\rho(x_0)\leqslant R$, then $\eta'=0$, since $\eta(t)=1$ for $t\in[0,1]$. The above estimate still holds according to (13).

The second term of (12) can be estimated easily by using (6),

$$\begin{aligned}-\frac{a_{ij}\phi_i\phi_j}{\phi^2}&=-\frac{|\nabla\phi|^2+u_iu_j\phi_i\phi_j}{\phi^2}\geqslant\\&-\frac{C^2}{\eta R^2}-\frac{|\nabla u|^2\,|\nabla\phi|^2}{\phi^2}\geqslant\\&-\frac{C^2}{\eta R^2}-\frac{C^2}{\eta R^2}\,|\nabla u|^2\geqslant\\&-\frac{C^2}{\eta R^2}(A+|\nabla u|^2)\end{aligned} \tag{17}$$

Next, we estimate the third term of (12). Computing directly, we have

$$\begin{aligned}&\frac{a_{ij}\,|\nabla u|_{ij}^2-|\nabla u|_i^2f_i}{|\nabla u|^2}=\\&\frac{2a_{ij}u_{si}u_{sj}+2a_{ij}u_su_{sij}-2u_{ij}u_jf_i}{|\nabla u|^2}=\end{aligned}$$

$$\frac{2\mid\nabla^2 u\mid^2+\frac{1}{2}\mid\nabla\mid\nabla u\mid^2\mid^2+2a_{ij}u_s u_{sij}-2u_{ij}u_j f_i}{\mid\nabla u\mid^2} \tag{18}$$

Since $a_{ij}u_{ij}-f_iu_i=0$, we have

$$u_{is}u_ju_{ij}+u_iu_{js}u_{ij}+a_{ij}u_{ijs}=f_{is}u_i+f_iu_{is}.$$

On the other hand, observe that

$$R_{ikjs}u_iu_ku_ju_s=R(\nabla u,\nabla u,\nabla u,\nabla u)=0.$$

By means of Ricci identity,

$$2a_{ij}u_su_{sij}=2a_{ij}(u_{ijs}+u_kR_{ikjs})u_s=$$
$$2a_{ij}u_{ijs}u_s+2\mathrm{Ric}(\nabla u,\nabla u).$$

Hence, one get

$$2a_{ij}u_su_{sij}-2u_{ij}u_jf_i=2\mathrm{Ric}(\nabla u,\nabla u)+$$
$$2u_s(f_{is}u_i+f_iu_{is}-u_{is}u_ju_{ij}-u_iu_{js}u_{ij})-2u_{ij}u_jf_i=$$
$$2\mathrm{Ric}_f(\nabla u,\nabla u)-\mid\nabla\mid\nabla u\mid^2\mid^2 \tag{19}$$

Also, it is easy to see that

$$2\mid\nabla^2u\mid^2\geqslant\frac{\mid\nabla\mid\nabla u\mid^2\mid^2}{2\mid\nabla u\mid^2} \tag{20}$$

For the proof, see Ref.[8].

Hence, (20) together with (18) and (19) gives the estimate of the third term,

$$\frac{a_{ij}\mid\nabla u\mid^2_{ij}-\mid\nabla u\mid^2_if_i}{\mid\nabla u\mid^2}=$$
$$\frac{2\mid\nabla^2u\mid^2-\frac{1}{2}\mid\nabla\mid\nabla u\mid^2\mid^2+2\mathrm{Ric}_f(\nabla u,\nabla u)}{\mid\nabla u\mid^2}\geqslant$$
$$\frac{\mid\nabla\mid\nabla u\mid^2\mid^2}{2\mid\nabla u\mid^4}-\frac{\mid\nabla\mid\nabla u\mid^2\mid^2}{2\mid\nabla u\mid^2}\geqslant$$
$$-\frac{\mid\nabla\mid\nabla u\mid^2\mid^2}{2\mid\nabla u\mid^4}(A+\mid\nabla u\mid^2) \tag{21}$$

The fourth term of (12) can be estimate as

$$\frac{\beta a_{ij}u_iu_j}{(\theta-u)^2}=\frac{\beta\mid\nabla u\mid^2}{(\theta-u)^2}(1+\mid\nabla u\mid^2)\geqslant$$
$$\frac{\beta\mid\nabla u\mid^2}{(\theta-u)^2}\left(1+\frac{1}{A}\mid\nabla u\mid^2\right)=$$
$$\frac{\beta\mid\nabla u\mid^2}{A(\theta-u)^2}(A+\mid\nabla u\mid^2) \tag{22}$$

The final term of (12) can be estimate as follows

$$\frac{a_{ij}\mid\nabla u\mid^2_i\mid\nabla u\mid^2_j}{\mid\nabla u\mid^4}\leqslant$$
$$\frac{\mid\nabla\mid\nabla u\mid^2\mid^2}{\mid\nabla u\mid^4}(1+\mid\nabla u\mid^2)\leqslant$$
$$\frac{\mid\nabla\mid\nabla u\mid^2\mid^2}{\mid\nabla u\mid^4}(A+\mid\nabla u\mid^2) \tag{23}$$

Substituting (16),(17),(21),(22) and (23) into (12), we obtain

$$\frac{1}{\eta}\left(-\frac{C}{R^2}-\frac{C}{R}-\frac{C\sqrt{K}}{R}\right)-\frac{C^2}{\eta R^2}-$$
$$\frac{3\mid\nabla\mid\nabla u\mid^2\mid^2}{2\mid\nabla u\mid^4}+\frac{\beta\mid\nabla u\mid^2}{A(\theta-u)^2}\leqslant 0 \tag{24}$$

By (10), one has

$$\frac{-3\mid\nabla\mid\nabla u\mid^2\mid^2}{2\mid\nabla u\mid^4}=-\frac{3}{2}\left(\frac{\phi_i}{\phi}+\frac{\beta u_i}{\theta-u}\right)^2\geqslant$$
$$-3\left(\frac{\phi_i^{\ 2}}{\phi^2}+\frac{\beta^2\mid\nabla u\mid^2}{(\theta-u)^2}\right)\geqslant$$
$$\frac{-3C^2}{\eta R^2}-3\frac{\beta^2\mid\nabla u\mid^2}{(\theta-u)^2} \tag{25}$$

Substituting it into (24), we have

$$\frac{1}{\eta}\left(-\frac{C}{R^2}-\frac{C}{R}-\frac{C\sqrt{K}}{R}\right)-\frac{4C^2}{\eta R^2}+$$
$$\frac{\beta\mid\nabla u\mid^2(1-3A\beta)}{A(\theta-u)^2}\leqslant 0.$$

Then, at the point x_0

$$\frac{\phi\mid\nabla u\mid^2}{(\theta-u)^\beta}\leqslant$$
$$\left(\frac{C}{R^2}+\frac{C}{R}+\frac{C\sqrt{K}}{R}+\frac{4C^2}{R^2}\right)\frac{A(\theta-u)^{2-\beta}}{\beta(1-3A\beta)}.$$

Let $\beta\in\left(0,\frac{1}{3A}\right)$. Since G attains its maximum at x_0 on $B_p(2R)$, so $G(x)\leqslant G(x_0)$ for $x\in B_p(R)$. Also, we have $\phi=1$ for $x\in B_p(R)$. We conclude that

$$\mid\nabla u\mid^2\leqslant\left(\frac{C}{R^2}+\frac{C}{R}+\frac{C\sqrt{K}}{R}+\frac{4C^2}{R^2}\right)\frac{\theta^2A}{\beta(1-3A\beta)},$$

for $x\in B_p(R)$.

Let $R\to+\infty$, we get $\mid\nabla u\mid=0$, when u is bounded on M. So we get the Liouville type theorem for f-exponentially harmonic function.

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