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The Lagrangian surfaces with constant curvature in Q_2

LI Kang[1], HU Sen[1,2]

(1. *School of Mathematical Sciences, University of Science and Technology of China, Hefei* 230026, *China*;
2. *Wu Wen-Tsun Key Laboratory of Mathematics, USTC, Chinese Academy of Sciences, Hefei* 230026, *China*)

Abstract: A class of H-minimal Lagrangian surfaces with constant curvature in Q_2 was described, and an example was given of minimal Lagrangian S^2 with Gaussian curvature $K=2$.

Key words: Lagrangian surfaces; constant curvature; H-minimal

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Q_2 中常曲率拉格朗日曲面

李 康[1],胡 森[1,2]

(1.中国科学技术大学数学科学学院,安徽合肥 230026;
2.中国科学技术大学中国科学院吴文俊数学重点实验室,安徽合肥 230026)

摘要:描述了复流形 Q_2 中一类常曲率 H 极小拉格朗日曲面,并且给出 Q_2 中一个高斯曲率 $K=2$ 的极小拉格朗日球面.

关键词:拉格朗日曲面;常曲率;H 极小

0 Introduction

Let (N, J, ω) be a Kähler manifold with $\dim_{\mathbb{C}} N=n$, where J is the complex structure and ω is the Kähler form. An immersion $f: M\to N$ from a q-dimensional manifold M into N is called totally real if $f^*\omega=0$. In particular, a totally real immersion f is called Lagrangian if $q=n$.

A vector field V along a Lagrangian immersion $f: M\to N$ is called a Hamiltonian variation if the 1-form $\alpha_V := \omega(V, \cdot)|_M$ is exact on M. A smooth family $\{f_t\}$ of immersions from M into N is called a Hamiltonian deformation if its derivative is Hamiltonian, and a Lagrangian immersion $f: M\to N$ is called Hamiltonian minimal or H-minimal if it satisfies

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{vol}\ f_t(M) = 0$$

for all Hamiltonian deformation. The Euler-Lagrange equation of H-minimal Lagrangian submanifolds is $\delta\alpha_H = 0$, where H is the mean curvature vector field of f and δ is the codifferential operator on M with respect to the induced metric. In particular, minimal Lagrangian

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Biography: LI Kang, male, born in 1986, PhD. Research field: differential geometry. E-mail: likang@mail.ustc.edu.cn

Corresponding author: HU Sen, PhD/ Prof. E-mail: shu@ustc.edu.cn

submanifolds are trivially H-minimal.

Many examples of H-minimal Lagrangian submanifolds in complex space form have been constructed in the past years. Castro et al.[1] classified S^1-invariant H-minimal Lagrangian submanifolds in $\mathbb{C}^2$. Schoen et al.[2] studied the minimal Lagrangian cones in $\mathbb{C}^2$. Ma et al.[3] gave a family of Hamiltonian stationary Lagrangian tori in CP^2. Mironov et al.[4] constructed a family of conformally flat H-minimal Lagrangian tori in CP^3. Jiao et al.[5] completely determined all the totally real conformal minimal two-spheres with constant curvature in Q_2.

However, there are less results about the H-minimal Lagrangian submanifolds which are not in the complex space form. In this paper, we describe a class of H-minimal Lagrangian surfaces with constant curvature in Q_2, and give a example of minimal Lagrangian S^2 with Gaussian curvature $K=2$.

1 Preliminary

In this section, we give the basic formulae of surfaces in a Kähler surface, for a more general case, see Ref. [6]. Throughout this paper, we use the following conventions for index ranges:

$$1 \leqslant A, B, \cdots \leqslant 4;\ 1 \leqslant i, j, \cdots \leqslant 2;$$
$$3 \leqslant \alpha, \beta, \cdots \leqslant 4.$$

Let M be a smooth surface. Locally, we choose an orthonormal frame $\{e_1, e_2\}$ of M, and its dual $\{\theta_1, \theta_2\}$. The first Cartan structure equation of M is given by

$$d\theta_i = -\theta_{ij} \wedge \theta_j,\ \theta_{ij} + \theta_{ji} = 0 \tag{1}$$

where θ_{ij} are connection forms with respect to the coframe θ_i. Let N be a Kähler surface. Locally, we choose a unitary frame field $\{\varepsilon_1, \varepsilon_2\}$ of (1,0)-type on N, and denote its dual by $\{\varphi_1, \varphi_2\}$. The first structure equation is given by

$$d\varphi_i = -\varphi_{ij} \wedge \varphi_j,\ \varphi_{ij} + \bar{\varphi}_{ji} = 0 \tag{2}$$

where φ_{ij} are the connection forms with respect to the coframe φ_i.

Let $f: M \to N$ be an isometric immersion. Set

$$f^* \varphi_i = f^i_j \theta_j \tag{3}$$

Taking the exterior differentiation of Eq. (3), we obtain

$$(df^i_j - f^i_k \theta_{kj} + \varphi_{ik} f^k_j) \wedge \theta_j = 0 \tag{4}$$

Set

$$Df^i_j = df^i_j - f^i_k \theta_{kj} + \varphi_{ik} f^k_j = f^i_{jk} \theta_k \tag{5}$$

the covariant derivative of f^i_j, then $f^i_{jk} = f^i_{kj}$ by Eq. (4). The tensor field $\prod^{\mathbb{C}} = \sum_{ijk} f^i_{jk} \theta_j \otimes \theta_k \otimes \varepsilon_i$ is called the complex second fundamental form of f, and the vector field $H^{\mathbb{C}} = \sum_{ij} f^i_{jj} \varepsilon_i$ is called the complex mean curvature vector field of f.

Proposition 1.1[6] Let $f: M \to N$ be an isometric immersion from a surface M into a Kähler surface N, H the mean curvature vector of f, and ω the Kähler form of N, then

$$\left.\begin{aligned} \alpha_H &:= \omega(H, \cdot)_M = h_j \theta_j, \\ h_j &= \frac{\mathrm{i}}{2}(f^l_{kk} \bar{f}^l_j - \bar{f}^l_{kk} f^l_j) \end{aligned}\right\} \tag{6}$$

Therefore, the codifferential of α_H is given by

$$\delta \alpha_H = -\sum_j h_{jj} \tag{7}$$

where $h_{jk}\theta_k = dh_j - h_k \theta_{kj}$.

2 The Lagrangian surfaces in the complex quadric Q_2

Let Q_2 denote the hyperquadric in CP^3, which is identified with $G(2,4)$, the Grassmann manifold of oriented two planes in $\mathbb{R}^4$: $[v + \mathrm{i}w] \leftrightarrow [v \wedge w]$, where $[v + \mathrm{i}w]$ denotes the point in Q_2 given by the homogeneous vector $v + iw$ in $\mathbb{C}^4$ and $[v \wedge w]$ denotes the oriented two-plane in $\mathbb{R}^4$ spanned by the ordered pair $v, w \in \mathbb{R}^4$.

As a homogeneous space $Q_2 = SO(4)/SO(2) \times SO(2)$. Let $\{e_A\}$ be a basis of $\mathbb{R}^4$, then

$$de_A = \omega_{AB} e_B,\ d\omega_{AB} = \omega_{AC} \wedge \omega_{CB} \tag{8}$$

where ω_{AB} are the Maurer-Cartan forms of $SO(4)$ satisfying $\omega_{AB} + \omega_{BA} = 0$. Let $f: M \to Q_2$ be an isometric immersion from a surface M, and locally $f = [e_1 + \mathrm{i}e_2] = [e_1 \wedge e_2]$. Set

$$\omega_3 = \omega_{13} + \mathrm{i}\omega_{23},\ \omega_4 = \omega_{14} + \mathrm{i}\omega_{24} \tag{9}$$

then the metric on Q_2 coming from the Fubini-Study metric on CP^3 is given by

$$ds^2_{FS} = \frac{1}{2}(\omega_3\bar{\omega}_3 + \omega_4\bar{\omega}_4) = \varphi_1\bar{\varphi}_1 + \varphi_2\bar{\varphi}_2 \quad (10)$$

where $\varphi_1 = \frac{1}{\sqrt{2}}\omega_3$, $\varphi_2 = \frac{1}{\sqrt{2}}\omega_4$. And the Kähler form of Q_2 is

$$\omega = \frac{i}{4}(\omega_3 \wedge \bar{\omega}_3 + \omega_4 \wedge \bar{\omega}_4) = \frac{i}{2}(\varphi_1 \wedge \bar{\varphi}_1 + \varphi_2 \wedge \bar{\varphi}_2) \quad (11)$$

Locally, we choose an orthonormal coframe θ_1, θ_2 on M, then

$$f^* ds^2_{FS} = \theta_1\theta_1 + \theta_2\theta_2 \quad (12)$$

The connection form $\theta_{12} = -\theta_{21}$ is fixed by the structure equation of M,

$$d\theta_1 = -\theta_{12} \wedge \theta_2,\ d\theta_2 = -\theta_{21} \wedge \theta_1$$

Set

$$\omega_{AB} = a_{AB}\theta_1 + b_{AB}\theta_2 \quad (13)$$

then

$$\omega_3 = a_3\theta_1 + b_3\theta_2,\ \omega_4 = a_4\theta_1 + b_4\theta_2 \quad (14)$$

where

$$a_3 = a_{13} + ia_{23},\ b_3 = b_{13} + ib_{23},$$
$$a_4 = a_{14} + ia_{24},\ b_4 = b_{14} + ib_{24}.$$

Let

$$C = \frac{1}{\sqrt{2}}\begin{pmatrix} a_3 & a_4 \\ b_3 & b_4 \end{pmatrix},$$

then we have the following proposition.

Proposition 2. 1 Let $f: M \to Q_2$ be an immersion from a surface M, then f is isometric and Lagrangian if and only if the matrix C is Hermitian, i. e. $CC^{\dagger} = I$.

Proof By Eqs. (12) and (14), f is isometric iff

$$a_3\bar{a}_3 + a_4\bar{a}_4 = 2,\ b_3\bar{b}_3 + b_4\bar{b}_4 = 2,$$
$$a_3\bar{b}_3 + a_4\bar{b}_4 + b_3\bar{a}_3 + b_4\bar{a}_4 = 0,$$

and $f^*\omega = 0$ iff

$$a_3\bar{b}_3 + a_4\bar{b}_4 - b_3\bar{a}_3 - b_4\bar{a}_4 = 0.$$

Consequently, $CC^{\dagger} = I$. □

From structure equations of Q_2:

$$d\varphi_1 = -\varphi_{11} \wedge \varphi_1 - \varphi_{12} \wedge \varphi_2,$$
$$d\varphi_2 = -\varphi_{21} \wedge \varphi_1 - \varphi_{22} \wedge \varphi_2,$$

we, by Eqs. (9) and (10), get the connection forms of Q_2:

$$\varphi_{11} = \varphi_{22} = i\omega_{12},\ \varphi_{21} = -\varphi_{12} = \omega_{34} \quad (15)$$

3 The construction of Lagrangian surfaces with constant curvature in Q_2

In this section, we will describe a family H-minimal surfaces with constant curvature in Q_2, and give some examples of Lagrangian two-spheres with constant curvature in Q_2. Giving a Lagrangian isometric immersion $f: M \to Q_2$ from a surface M, locally choose a isothermal coordinates (x, y) on M such that

$$\theta_1 = e^u dx,\ \theta_2 = e^u dy \quad (16)$$

where $u(x, y)$ is a differentiable function on M. Taking exterior differentiation of Eq. (16), we get

$$\theta_{12} = e^{-u}\left(\frac{\partial u}{\partial y}\theta_1 - \frac{\partial u}{\partial x}\theta_2\right) \quad (17)$$

Let $U = (a_{AB})$, $V = (b_{AB})$, and suppose $a_{2\alpha} = \lambda a_{1\alpha}$, $b_{2\alpha} = \lambda b_{1\alpha}$, i. e. U, V have the following forms

$$U = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & \lambda a_{13} & \lambda a_{14} \\ a_{31} & \lambda a_{31} & 0 & a_{34} \\ a_{41} & \lambda a_{41} & a_{43} & 0 \end{pmatrix} \quad (18)$$

$$V = \begin{pmatrix} 0 & b_{12} & b_{13} & b_{14} \\ b_{21} & 0 & \lambda b_{13} & \lambda b_{14} \\ b_{31} & \lambda b_{31} & 0 & b_{34} \\ b_{41} & \lambda b_{41} & b_{43} & 0 \end{pmatrix} \quad (19)$$

where $a_{AB} = -a_{BA}$, $b_{AB} = -b_{BA}$, and λ is a smooth function. Let $\varphi_i = f^i_j\theta_j$, then by Eqs. (18) and (19) we have

$$f^1_1 = \frac{1}{\sqrt{2}}(1 + i\lambda)a_{13},\ f^1_2 = \frac{1}{\sqrt{2}}(1 + i\lambda)b_{13} \quad (20)$$

$$f^2_1 = \frac{1}{\sqrt{2}}(1 + i\lambda)a_{14},\ f^2_2 = \frac{1}{\sqrt{2}}(1 + i\lambda)b_{14} \quad (21)$$

By Eqs. (5), (15) and (17) we get

$$f^1_{11} = \frac{1}{\sqrt{2}}\left(e^{-u}\frac{\partial a_3}{\partial x} + e^{-u}b_3\frac{\partial u}{\partial y} + ia_{12}a_3 - a_{34}a_4\right) \quad (22)$$

$$f^1_{22} = \frac{1}{\sqrt{2}}\left(e^{-u}\frac{\partial b_3}{\partial y} + e^{-u}a_3\frac{\partial u}{\partial x} + ib_{12}b_3 - b_{34}b_4\right) \quad (23)$$

$$f^2_{11} = \frac{1}{\sqrt{2}}\left(e^{-u}\frac{\partial a_4}{\partial x} + e^{-u}b_4\frac{\partial u}{\partial y} + ia_{12}a_4 + a_{34}a_3\right) \quad (24)$$

$$f_{22}^2=\frac{1}{\sqrt{2}}(\mathrm{e}^{-u}\frac{\partial b_4}{\partial y}+\mathrm{e}^{-u}a_4\frac{\partial u}{\partial x}+\mathrm{i}b_{12}b_4+b_{34}b_3) \quad (25)$$

So, by Eq. (6) and Proposition 2.1,

$$h_1=-\frac{1}{1+\lambda^2}\frac{\partial\lambda}{\partial x}-a_{12},\ h_2=-\frac{1}{1+\lambda^2}\frac{\partial\lambda}{\partial y}-b_{12} \quad (26)$$

The structure equation $\mathrm{d}\omega_{AB}=\omega_{AC}\wedge\omega_{CB}$ of $SO(4)$ give

$$\mathrm{e}^{-u}\Big(\frac{\partial b_{AB}}{\partial x}-\frac{a_{AB}}{\partial y}-a_{AB}\frac{\partial u}{\partial y}+b_{AB}\frac{\partial u}{\partial x}\Big)=a_{AC}b_{CB}-b_{AC}a_{CB} \quad (27)$$

which, by Eqs. (18) and (19), are

$$\mathrm{e}^{-u}\Big(\frac{\partial b_{34}}{\partial x}-\frac{a_{34}}{\partial y}-a_{34}\frac{\partial u}{\partial y}+b_{34}\frac{\partial u}{\partial x}\Big)=(1+\lambda^2)(-a_{13}b_{14}+b_{13}a_{14}) \quad (28)$$

$$\mathrm{e}^{-u}\Big(\frac{\partial b_{13}}{\partial x}-\frac{a_{13}}{\partial y}-a_{13}\frac{\partial u}{\partial y}+b_{13}\frac{\partial u}{\partial x}\Big)=-a_{14}b_{34}+b_{14}a_{34}+\lambda(a_{12}b_{13}-b_{12}a_{13}) \quad (29)$$

$$\mathrm{e}^{-u}\Big(\frac{\partial b_{14}}{\partial x}-\frac{a_{14}}{\partial y}-a_{14}\frac{\partial u}{\partial y}+b_{14}\frac{\partial u}{\partial x}\Big)=a_{13}b_{34}-b_{13}a_{34}+\lambda(a_{12}b_{14}-b_{12}a_{14}) \quad (30)$$

$$\mathrm{e}^{-u}\Big(\frac{\partial b_{23}}{\partial x}-\frac{a_{23}}{\partial y}-a_{23}\frac{\partial u}{\partial y}+b_{23}\frac{\partial u}{\partial x}\Big)=-a_{12}b_{13}+b_{12}a_{13}+\lambda(-a_{14}b_{34}+b_{14}a_{34}) \quad (31)$$

$$\mathrm{e}^{-u}\Big(\frac{\partial b_{24}}{\partial x}-\frac{a_{24}}{\partial y}-a_{24}\frac{\partial u}{\partial y}+b_{24}\frac{\partial u}{\partial x}\Big)=-a_{12}b_{14}+b_{12}a_{14}+\lambda(a_{13}b_{34}-b_{13}a_{34}) \quad (32)$$

$$\mathrm{e}^{-u}\Big(\frac{\partial b_{12}}{\partial x}-\frac{a_{12}}{\partial y}-a_{12}\frac{\partial u}{\partial y}+b_{12}\frac{\partial u}{\partial x}\Big)=0 \quad (33)$$

By Proposition 2.1, Eqs. (31) and (32) give

$$a_{12}=-\mathrm{e}^{-u}\frac{1}{1+\lambda^2}\frac{\partial\lambda}{\partial x},\ b_{12}=-\mathrm{e}^{-u}\frac{1}{1+\lambda^2}\frac{\partial\lambda}{\partial y} \quad (34)$$

which imply, by Eqs. (7) and (26), that $\delta\alpha_H=0$ if and only if

$$(\mathrm{e}^{-u}-1)\Big(\frac{\partial^2\lambda}{\partial x^2}+\frac{\partial^2\lambda}{\partial y^2}\Big)-\frac{2\lambda(\mathrm{e}^{-u}-1)}{1+\lambda^2}\Big[\Big(\frac{\partial\lambda}{\partial x}\Big)^2+\Big(\frac{\partial\lambda}{\partial y}\Big)^2\Big]-\Big(\frac{\partial\lambda}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial\lambda}{\partial y}\frac{\partial u}{\partial y}\Big)=0 \quad (35)$$

Considering Proposition 2.1, we let

$$a_{13}=\sqrt{\frac{2}{1+\lambda^2}}\cos\theta,\ a_{14}=\sqrt{\frac{2}{1+\lambda^2}}\sin\theta \quad (36)$$

$$b_{13}=-\sqrt{\frac{2}{1+\lambda^2}}\sin\theta,\ b_{14}=\sqrt{\frac{2}{1+\lambda^2}}\cos\theta \quad (37)$$

where $\theta(x,y)$ is a smooth function on M. By Eqs. (28)～(30), we get

$$\mathrm{e}^{-u}\Big(\frac{\partial b_{34}}{\partial x}-\frac{a_{34}}{\partial y}-a_{34}\frac{\partial u}{\partial y}+b_{34}\frac{\partial u}{\partial x}\Big)=-2 \quad (38)$$

$$-\mathrm{e}^{-u}\Big(\frac{\partial\theta}{\partial x}+\frac{\partial u}{\partial y}\Big)=a_{34} \quad (39)$$

$$-\mathrm{e}^{-u}\Big(\frac{\partial\theta}{\partial y}-\frac{\partial u}{\partial x}\Big)=b_{34} \quad (40)$$

Theorem 3.1 Given an isometric immersion $f: M\to Q_2$ from a surface M with the induced metric $\mathrm{d}s^2=\mathrm{e}^{2u}(\mathrm{d}x^2+\mathrm{d}y^2)$, where (x, y) is an isothermal coordinates and u a smooth function on M. Suppose the coefficients of the pullback of the Maurer-Cartan form are Eqs. (18) and (19), then f is H-minimal Lagrangian if and only if the Eq. (35) holds and the function $u(x,y)$ satisfies

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-2\mathrm{e}^{2u} \quad (41)$$

Proof It suffices to insert Eqs. (39) and (40) into Eq. (38). □

By the surface uniformization theorem, we get

Corollary 3.2 Suppose as in Theorem 3.1, if M is closed, then M is the two-sphere S^2 with constant curvature 2.

So it is known

$$u=-\ln(1+\frac{1}{2}(x^2+y^2)) \quad (42)$$

and Eq. (35) becomes

$$\frac{\partial^2\lambda}{\partial x^2}+\frac{\partial^2\lambda}{\partial y^2}-\frac{2\lambda}{1+\lambda^2}\Big[\Big(\frac{\partial\lambda}{\partial x}\Big)^2+\Big(\frac{\partial\lambda}{\partial y}\Big)^2\Big]+\frac{4}{(x^2+y^2)(2+x^2+y^2)}\Big(x\frac{\partial\lambda}{\partial x}+y\frac{\partial\lambda}{\partial y}\Big)=0 \quad (43)$$

Theorem 3.3 Suppose as in Theorem 3.1 and M is closed, if λ is constant, then $f: S^2\to Q_2$ is a Lagrangian immersion with the Gaussian curvature $K=2$ which is totally geodesic, and the coefficients of the pullback of the Maurer-Cartan form are

$$U=\begin{pmatrix} 0 & 0 & \sqrt{\frac{2}{1+\lambda^2}}\cos\theta & \sqrt{\frac{2}{1+\lambda^2}}\sin\theta \\ 0 & 0 & \lambda\sqrt{\frac{2}{1+\lambda^2}}\cos\theta & \lambda\sqrt{\frac{2}{1+\lambda^2}}\sin\theta \\ -\sqrt{\frac{2}{1+\lambda^2}}\cos\theta & -\lambda\sqrt{\frac{2}{1+\lambda^2}}\cos\theta & 0 & -e^{-u}\left(\frac{\partial\theta}{\partial x}+\frac{\partial u}{\partial y}\right) \\ -\sqrt{\frac{2}{1+\lambda^2}}\sin\theta & -\lambda\sqrt{\frac{2}{1+\lambda^2}}\sin\theta & e^{-u}\left(\frac{\partial\theta}{\partial x}+\frac{\partial u}{\partial y}\right) & 0 \end{pmatrix} \tag{44}$$

$$V=\begin{pmatrix} 0 & 0 & -\sqrt{\frac{2}{1+\lambda^2}}\sin\theta & \sqrt{\frac{2}{1+\lambda^2}}\cos\theta \\ 0 & 0 & -\lambda\sqrt{\frac{2}{1+\lambda^2}}\sin\theta & \lambda\sqrt{\frac{2}{1+\lambda^2}}\cos\theta \\ \sqrt{\frac{2}{1+\lambda^2}}\sin\theta & \lambda\sqrt{\frac{2}{1+\lambda^2}}\sin\theta & 0 & -e^{-u}\left(\frac{\partial\theta}{\partial y}-\frac{\partial u}{\partial x}\right) \\ -\sqrt{\frac{2}{1+\lambda^2}}\cos\theta & -\lambda\sqrt{\frac{2}{1+\lambda^2}}\cos\theta & e^{-u}\left(\frac{\partial\theta}{\partial y}-\frac{\partial u}{\partial x}\right) & 0 \end{pmatrix} \tag{45}$$

where $\theta(x,y)$ is a function on S^2 and $u(x,y)$ is as Eq. (42).

Proof It only needs to verify that f is totally geodesic. Taking Eqs. (44) and (45) into Eqs. (22)～(25), we know $f_{11}^1=f_{22}^1=f_{11}^2=f_{22}^2=0$. Following the same procedure yields

$$f_{12}^1=f_{21}^1=f_{12}^2=f_{21}^2=0. \qquad \square$$

Example 3.4 Let $\theta=\lambda=0$ in the above, then we get

$$U=\begin{pmatrix} 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 & -e^{-u}\frac{\partial u}{\partial y} \\ 0 & 0 & e^{-u}\frac{\partial u}{\partial y} & 0 \end{pmatrix},$$

$$V=\begin{pmatrix} 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-u}\frac{\partial u}{\partial x} \\ -\sqrt{2} & 0 & -e^{-u}\frac{\partial u}{\partial x} & 0 \end{pmatrix}.$$

We note in this case that the coefficients of the pullback of the Maurer-Cartan form of $SO(4)$ are the same as in Ref. [4], and by that result, up to a rigid motion, in local coordinates, f is given by

$$f=[x^2+y^2-1,2x,2y,\mathrm{i}(x^2+y^2+1)].$$

We guess Eq. (43) has no non-constant solution.

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References

[1] Castro I, Urbano F. Examples of unstable Hamiltonian-minimal Lagrangian tori in C^2 [J]. Compositio Math, 1998, 111: 1-14.

[2] Schoen R, Wolfson J. Minimizing volume among Lagrangian submanifolds [C]// Proc Sympos Pure Math: Vol 65. Providence, RI: Amer Math Soc, 1999: 181-199.

[3] Ma H, Schmies M. Examples of Hamiltonian stationary Lagrangian tori in CP^2 [J]. Geom Dedicata, 2006, 118: 173-183.

[4] Mironov A E, Zuo D. On a family of conformally flat Hamiltonian minimal Lagrangian tori in CP^3 [J]. Int Math Res Not, 2008, 2008: 10.1093/imrn/rnn078.

[5] Jiao X X, Wang J. Totally real minimal surfaces in the complex hyperquadrics[J]. Diff Geo Appl, 2013, 31: 540-555.

[6] Hu S, Chen Q, Xu X W. Construction of Lagrangian submanifolds in CP^n [J]. Pacific Journal of Math, 2012, 258: 31-49.

[7] Chern S S, Wolfson J. Minimal surfaces by moving frames[J]. Amer J Math, 1983, 105(1): 59-83.

[8] Yang K. Complete and Compact Minimal Surface[M]. Dordrecht: Kluwer Academic, 1989.