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# Optimal $t$-pebbling on paths and cycles

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**Abstract**: A pebbling move removes two pebbles from a vertex and places one pebble on one of its neighbours. For $t \geqslant 1$, the optimal $t$-pebbling number of a graph G, $f'_t(G)$, is the minimum number of pebbles necessary so that from some initial distribution of them it is possible to move $t$ pebbles to any target vertex by a sequence of pebbling moves. $f'(G) = f'_1(G)$ be the optimal pebbling number of G. Here the optimal $t$-pebbling numbers of the path $P_n$ and the cycle $C_5$ were given, respectively. In the final section, it was obtained that $f'_{9t}(P_2 \times P_3) = 20t$, $f'_{9t+1}(P_2 \times P_3) = 20t + 3$, and $20t + 2r + 1 \leqslant f'_{9t+r}(P_2 \times P_3) \leqslant 20t + 2r + 2$, for $2 \leqslant r \leqslant 8$, the last equality holds for $r = 5, 6, 7, 8$.

**Key words**: optimal $t$-pebbling number; path; cycle; Cartesian product

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# 路与圈的优化 $t$-pebbling 数

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**摘要**:图上的一个 pebbling 移动,是从图的一个顶点同时移除 2 个 pebbles,并且在其某个邻点上放置 1 个 pebble. 图的优化 $t$-pebbling 数,记为 $f'_t(G)$,是指图 G 中所需要的 pebbled 的最小数目,使得存在该 $f'_t(G)$ 个 pebbles 在图上的一种分布,可以在经过一系列 pebbling 移动后,$t$ 个 pebbles 可以移动到任意一个给定的目标顶点上. $f'(G) = f'_1(G)$称为图 G 的优化 pebbling 数. 这里给出了路 $P_n$ 和圈 $C_5$ 的优化 $t$-pebbling 数,证明了 $f'_{9t}(P_2 \times P_3) = 20t$; $f'_{9t+1}(P_2 \times P_3) = 20t + 3$;当 $2 \leqslant r \leqslant 8$ 时,$20t + 2r + 1 \leqslant f'_{9t+r}(P_2 \times P_3) \leqslant 20t + 2r + 2$,其中,当 $5 \leqslant r \leqslant 8$ 时,最后一个不等式取到等号.

**关键词**:优化 $t$-pebbling 数;路;圈;笛卡尔乘积

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## 0 Introduction

Throughout this paper, $G=(V,E)$ denotes a simple, connected graph with $n$ vertices. Let $P_n=u_1u_2\cdots u_n$ and $C_n=u_1u_2\cdots u_nu_1$ be the path and the cycle with $n$ vertices, respectively. A function $D: V\to N\cup\{0\}$ is called a distribution on the vertices of $G$. Let $D(v)$ be the number of pebbles on the vertex $v\in V(G)$ ($D_i=D(u_i)$), $|D|$ be the total number of pebbles on $V(G)$ under $D$. A pebbling move consists of the removal of two pebbles from a vertex and placing one pebble at an adjacent vertex. The optimal $t$-pebbling number of $G$, denoted by $f'_t(G)$, is the least $p$ such that, for some distribution of $p$ pebbles on the vertices of $G$, a pebble can be moved to any vertex by a sequence of pebbling moves. Moreover, $f'(G)=f'_1(G)$ is called the optimal pebbling number of $G$.

Let $W(D,v)$ be the maximal number of pebbles on $v$ by some (possibly empty) pebbling moves on $G$ from the original distribution $D$. Then we call $v$ $t$-reachable under $D$ for some positive integer $t$ if $W(D,v)\geqslant t$. A distribution $D$ is called $t$-fold solvable if every vertex is $t$-reachable under $D$.

The known results about the optimal $t$-pebbling number of paths, cycles and the product of paths are given as follows.

**Theorem 0.1**[1,3] $f'(P_n)=f'(C_n)=\lceil 2n/3\rceil$.

**Theorem 0.2**[1] $f'_2(P_n)=n+1$.

**Theorem 0.3**[4] $f'_{3t+r}(P_2)=4t+2r$ if $r<2$, $f'_{3t+2}(P_2)=4t+3$.

The Cartesian product $G\times H$ is defined to be the graph with vertex set $V(G\times H)$ and edge set: the union of $\{((a,v),(b,v))\mid(a,b)\in E(G), v\in E(H)\}$ and $\{((u,x),(u,y))\mid u\in V(G),(x,y)\in E(H)\}$.

**Theorem 0.4**[1,4] $f'(P_m\times P_2)=f'(C_m\times P_2)=m$ for $m\geqslant 2$, except that $f'(P_2\times P_2)=3$ and $f'(P_5\times P_2)=6$. If $t=6q+r$,

$$f'_t(K_2\times K_3)=\begin{cases}12q & \text{if } r=0,\\ 12q+2r+1 & \text{otherwise.}\end{cases}$$

In this paper, we will give the optimal $t$-pebbling numbers of the path $P_n$ and the cycle $C_5$. First, we give some lemmas.

**Lemma 0.5**[4] $f'_{s+t}(G)\leqslant f'_s(G)+f'_t(G)$.

For a given distribution $D$ on $V(G)$, assume $d(v)=2$ and $D(v)\geqslant 3$. A smoothing move from $v$ changes $D$ by removing two pebbles from $v$, and add one pebble on each neighbour of $v$.

**Lemma 0.6**[1] Let $D$ be a distribution on a graph $G$ with distinct vertices $u$ and $v$, where $d(v)=2$, $D(v)\geqslant 3$, and $u$ is $t$-reachable under $D$, then $u$ is $t$-reachable under the distribution $D'$ obtained by making a smoothing move from $v$.

For more background and related topics of this article, we refer to [1-8].

## 1 The optimal $t$-pebbling number of path

First, we give an upper bound of $f'_t(P_n)$.

**Theorem 1.1** $f'_{3t}(P_n)\leqslant t(n+2)$, $f'_{3t+1}(P_n)\leqslant t(n+2)+\lfloor n/2\rfloor+1$, $f'_{3t+2}(P_n)\leqslant t(n+2)+(n+1)$ for $t\geqslant 1$.

**Proof** From Lemma 0.5, we have that $f'_{3t}(P_n)\leqslant tf'_3(P_n)$, $f'_{3t+1}(P_n)\leqslant(t-1)f'_3(P_n)+f'_4(P_n)$ and $f'_{3t+2}(P_n)\leqslant tf'_3(P_n)+f'_2(P_n)$, for $t\geqslant 1$.

Clearly, it is sufficient to show that $f'_3(P_n)\leqslant n+2$, $f'_4(P_n)\leqslant n+\lfloor n/2\rfloor+3$.

Let $D_1=D_n=2$, and $D_i=1$ for $2\leqslant i\leqslant n-1$. Then it is a 3-fold solvable distribution on $P_n$ with $|D|=n+2$. Thus $f'_3(P_n)\leqslant n+2$.

For $t=4$, we use induction on $n$ to prove that there exists a 4-fold solvable distribution $D$ on $P_n$ with $D_n\geqslant 2$ and $|D|=n+\lfloor n/2\rfloor+3$.

If $n=1$, $D_1=4$; $n=2$, $D_1=D_2=3$. So it holds for $n=1,2$.

Assume that it holds for $n-2$, let $D'$ be a 4-fold solvable distribution on $P_{n-2}$, so that $D'_{n-2}\geqslant 2$ and $|D'|=n-2+\lfloor(n-2)/2\rfloor+3$.

Let $D$ be a distribution on $P_n$ such that $D_i=D'_i$ for $i<n-2$, $D_{n-2}=D'_{n-2}-2$, $D_{n-1}=3$, $D_n=2$. It is clear that $D$ is a 4-fold solvable distribution on $P_n$, $D_n\geqslant 2$, and

$|D|=|D'|+3=n-2+\lfloor(n-2)/2\rfloor+3+3=$

$$n+\lfloor n/2 \rfloor+3,$$

this completes the proof. □

Lemma 0.6 shows that a smoothing move on any vertex $v$ with degree 2 keeps a $t$-reachable vertex $u\neq v$ still be $t$-reachable, but it does not hold for $v$ itself. The following lemma shows that $v$ may still be $t'$-reachable for some $t'\leqslant t$.

**Lemma 1.2** Let $D$ be a distribution on $P_3$ with $W(D,u_i)\geqslant 3t+r$ for $1\leqslant i\leqslant 3$, $D_2\geqslant t+r+2-\lfloor r/2 \rfloor$. If we make a smoothing move on $u_2$, then $u_2$ is at least $(3t+r)$-reachable.

**Proof** We only prove the cases $r=0,1$. The case $r=2$ can be proved similarly.

Let $a=D_1$, $b=D_2$, $c=D_3$. The distribution, after a smoothing move on $u_2$, is denoted by $D'$. Without loss of generality, we assume that $a\geqslant c$.

If $a>c+2$, then we remove two pebbles from $u_1$ and add them onto $u_3$ to get $D^*$. Clearly, we have that $W(D^*,u_i)\geqslant 3t+r$, and $W(D^{*\prime},u_2)=W(D',u_2)$ ($D^{*\prime}$ is the distribution after a smoothing move on $u_2$ from $D^*$).

So we only need to deal with $c\leqslant a\leqslant c+2$.

**Case 1** $a=c$.

If $a$ is odd, then

$$W(D',u_2)\geqslant D_2-2+2\lfloor (a+1)/2 \rfloor = D_2+2\lfloor a/2 \rfloor = W(D,u_2)\geqslant 3t+r.$$

If $a$ is even, then $W(D,u_3)=\lfloor (a/2+b)/2 \rfloor+a\geqslant 3t+r$. Note that $\min\{a+b\}$ can be achieved while $b$ is at its minimum $\min b=t+r+2$, then $a\geqslant 2t$. Thus

$$W(D',u_2)=\lfloor a/2 \rfloor+\lfloor a/2 \rfloor+b-2 = a+b-2\geqslant 3t+r.$$

**Case 2** $a=c+1$. $W(D,u_3)=\lfloor (a/2+b)/2 \rfloor+a-1\geqslant 3t+r$. Then $\min\{a+b\}$ can be achieved while $b=\min b=t+r+2$, then $a\geqslant 2t$.

$$W(D',u_2)=\lfloor (a+1)/2 \rfloor+\lfloor a/2 \rfloor+b-2 = a+b-2\geqslant 3t+r.$$

**Case 3** $a=c+2$. If $a$ is odd, then we are done. If $a$ is even,

$$W(D,u_3)=\lfloor (a/2+b)/2 \rfloor+a-2\geqslant 3t+r.$$

Then $\min\{a+b\}$ can be achieved if $b=\min b=t+r+2$, then $a\geqslant 2t+1$. Moreover,

$$W(D',u_2)=\lfloor (a-2)/2 \rfloor+\lfloor a/2 \rfloor+b-2 = a+b-3\geqslant 3t+r.$$ □

**Corollary 1.3** There exists a $(3t+r)$-fold solvable distribution with $f'_{3t+r}(P_n)$ pebbles on $P_n$ so that $D_1\geqslant 2t+\lfloor r/2 \rfloor$, $D_n\geqslant 2t+\lfloor r/2 \rfloor$, $D_i\leqslant t+r+1-\lfloor r/2 \rfloor$ for $1<i<n$.

**Proof** We only prove the cases $r=0,1$. The case $r=2$ can be similarly proved.

By Lemma 1.2, we can make a smoothing move on vertex $u_i$ if $D_i\geqslant t+r+2$ for $1<i<n$, and the smoothing moves must be finished. Hence we can make sure that $D_i\leqslant t+r+1$. Then if we can move at least $t+r+1$ pebbles from $P_n\backslash u_n$ to $u_n$, then there are at least $t+r+1$ pebbles that can be moved from $P_n\backslash\{u_{n-1},u_n\}$ to $u_{n-1}$, and so on. Then the number of pebbles on $P_n$ is at least

$$2t+2r+2+(n-2)(t+r+1)+2t-1 = nt+nr+n+2t-1,$$

which is incompatible with Theorem 1.1. Thus there are at most $t+r$ pebbles that can be moved from $P_n\backslash u_n$ to $u_n$, but $W(D,u_n)\geqslant 3t+r$, so $D_n\geqslant 2t$, and similarly $D_1\geqslant 2t$. □

**Lemma 1.4** $f'_{3t+r}(P_n)\leqslant f'_{3t+r}(P_{n-2})+2t+r$ for $t\geqslant 1$.

**Proof** Let $D'$ be a $(3t+r)$-fold solvable distribution with $f'_{3t+r}(P_{n-2})$ pebbles on $P_{n-2}$ so that $D'_{n-2}\geqslant 2t+\lfloor r/2 \rfloor$.

If $2t+\lfloor r/2 \rfloor\geqslant t+r$, then let $D_{n-2}=D'_{n-2}-t-r$, $D_{n-1}=t+2r$, $D_n=2t$, and $D_i=D'_i$ for $i<n-2$.

If $2t+\lfloor r/2 \rfloor<t+r$, we must have that $t=1$, $r=2$. Then let $D_i=D'_i$ for $i<n-2$,

$$D_{n-2}=D'_{n-2}-2,\ D_{n-1}=3,\ D_n=3.$$

It is easy to see that the new distribution $D$ is $(3t+r)$-fold solvable on $P_n$ with $|D|\leqslant|D'|+2t+r$. Therefore this lemma holds. □

**Lemma 1.5** $f'_{3t+r}(P_n)\geqslant f'_{3t+r}(P_{n-2})+2t+r$ for $t\geqslant 1$ and $r=0,1$.

**Proof** Assume that $D$ is a $(3t+r)$-fold solvable distribution with $f'_{3t+r}(P_n)$ pebbles on $P_n$ which was provided by Corollary 1.3. Then we let $D'$ be a new distribution such that $D'_i=D_i$ for $i<n-2$ and $D'_{n-2}=D_{n-2}+D_{n-1}+D_n-2t-r$.

First we note that

$$D_{n-1}+D_n\geqslant 2t+r \tag{1}$$

Second we show that

$$W(D', u_{n-2}) \geqslant W(D, u_{n-2}) \quad (2)$$

It is sufficient to show that

$$D_{n-1} + D_n - 2t - r \geqslant \lfloor (D_{n-1} + \lfloor D_n/2 \rfloor)/2 \rfloor \quad (3)$$

Let $a = D_n$, $b = D_{n-1}$. From Corollary 1.3 and its proof, it follows that $a \geqslant 2t$ and there are at most $t+r$ pebbles that can be moved from $P_n \setminus \{u_{n-1}, u_n\}$ to $u_{n-1}$, and $W(D, u_{n-1}) \geqslant 3t+r$, $W(D, u_n) \geqslant 3t+r$, we have

$$\begin{cases} b + \lfloor a/2 \rfloor + t + r \geqslant 3t + r \\ \quad \Rightarrow b + a/2 \geqslant 2t, \\ \lfloor (b + t + r)/2 \rfloor + a \geqslant 3t + r \\ \quad \Rightarrow b/2 + a \geqslant 5t/2 + r/2. \end{cases}$$

Let $g = a + b - 2t - r - \lfloor (b + \lfloor a/2 \rfloor)/2 \rfloor$ (similarly let $g_i = a_i + b_i - 2t - r - \lfloor (b_i + \lfloor a_i/2 \rfloor)/2 \rfloor$). Then $g = 3a/4 + b/2 - 2t - r + \delta$ for some $\delta \in \{0, 1/4, 1/2, 3/4\}$ and min $g$ can be reached in a small neighbourhood of

$$\{(a,b) \mid b + a/2 = 2t,\ b/2 + a = 5t/2 + r/2\} = (2t + 2r/3, t - r/3).$$

If $a = 2t$, $b = t + r$, then $g \geqslant 0$.

If $a > 2t$, then min $g$ can be reached along the line $b + a/2 = 2t$. Note that if $a_1 = a_2 + 2$, $b_1 = b_2 - 1$, then $g_1 > g_2$. So we only need to consider $a = 2t+1$, $b = t$ and $a = 2t+2$, $b = t-1$. In both cases, $g \geqslant 0$.

Therefore, $D'$ is $(3t+r)$-fold solvable on $P_{n-2}$. □

Similarly, we have the following lemma.

**Lemma 1.6** $f'_{3t+2}(P_n) \geqslant f'_{3t+2}(P_{n-2}) + 2t + 2$ for $t \geqslant 1$.

**Proof** Assume that $D$ is a $(3t+2)$-fold solvable distribution with $f'_{3t+2}(P_n)$ pebbles on $P_n$.

Let $D'_i = D_i$ for $i < n-2$ and $D'_{n-2} = D_{n-2} + D_{n-1} + D_n - 2t - 2$.

First we show that $D_{n-1} + D_n \geqslant 2t + 2$, we know that $D_n \geqslant 2t+1$, if $D_n \geqslant 2t+2$, then we are done; if $D_n = 2t+1$, then, similar to the proof in Corollary 1.3, at most $t+1$ pebbles that can be moved from $P_n \setminus \{u_{n-1}, u_n\}$ to $u_{n-1}$. But $W(D, u_{n-1}) \geqslant 3t+2$, so $D_{n-1} \geqslant t+1$.

Second we show that $W(D', u_{n-2}) \geqslant W(D, u_{n-2})$, we only need to show

$$D_{n-1} + D_n - 2t - 2 \geqslant \lfloor (D_{n-1} + \lfloor D_n/2 \rfloor)/2 \rfloor.$$

Let $a = D_n$, $b = D_{n-1}$, from Corollary 1.3, assume that $a \geqslant 2t+1$. For at most $t+1$ pebbles that can be moved from $P_n \setminus \{u_{n-1}, u_n\}$ to $u_{n-1}$, and $W(D, u_{n-1}) \geqslant 3t+2$, $W(D, u_n) \geqslant 3t+2$, they imply that

$$\begin{cases} b + \lfloor a/2 \rfloor + t + 1 \geqslant 3t + 2 \\ \quad \Rightarrow b + a/2 \geqslant 2t + 1, \\ \lfloor (b + t + 1)/2 \rfloor + a \geqslant 3t + 2 \\ \quad \Rightarrow b/2 + a \geqslant 5t/2 + 3/2. \end{cases}$$

Let $g = a + b - 2t - 2 - \lfloor (b + \lfloor a/2 \rfloor)/2 \rfloor$ (similarly let $g_i = a_i + b_i - 2t - 2 - \lfloor (b_i + \lfloor a_i/2 \rfloor)/2 \rfloor$). Then we should prove that $g \geqslant 0$.

Let $g = 3a/4 + b/2 - 2t - 2 + \delta$ for some $\delta \in \{0, 1/4, 1/2, 3/4\}$. It is not hard to see that min $g$ can be reached in a small neighbourhood of

$$\{(a,b) \mid b + a/2 = 2t+1,\ b/2 + a = 5t/2 + 3/2\} = (2t + 4/3, t + 1/3).$$

If $a = 2t+1$, $b = t+1$, then $g \geqslant 0$.

If $a > 2t+1$, then min $g$ can be reached along the line $b + a/2 = 2t+1$. Note that if $a_1 = a_2 + 2$, $b_1 = b_2 - 1$, then $g_1 > g_2$. So we only need to consider $a = 2t+2$, $b = t$ and $a = 2t+3$, $b = t$. In both cases, $g \geqslant 0$.

So $D'$ is $(3t+2)$-fold solvable on $P_{n-2}$, and we are done. □

By Lemmas 1.4 ~ 1.6, we can get the following theorem immediately.

**Theorem 1.7** $f'_{3t+r}(P_n) = f'_{3t+r}(P_{n-2}) + 2t + r$, for $t \geqslant 1$.

From Theorems 1.7 and 0.3, also note that $f'_{3t+r}(P_1) = 3t + r$, we can get the following theorem.

**Theorem 1.8** $f'_{3t}(P_n) = t(n+2)$,

$$f'_{3t+1}(P_n) = t(n+2) + \lfloor n/2 \rfloor + 1,$$

$$f'_{3t+2}(P_n) = t(n+2) + (n+1),$$

for $t \geqslant 1$.

## 2 The optimal $t$-pebbling number of cycle

The optimal $t$-pebbling numbers of $C_3$ and $C_4$ were obtained in Ref. [4]. In this section, we will give the optimal $t$-pebbling number of $C_5$.

A distribution $D$ is smooth if it has at most two pebbles on every vertex of degree 2.

**Theorem 2.1** $f_2'(C_n)=n$ for $n>3$, $f_3'(C_n)=n+2$.

**Proof** If $f_2'(C_n)\leqslant n-1$, then it is not hard to see that $f'(C_n\times K_2)\leqslant n-1$, a contradiction to Theorem 6.6 in Ref.[1], so $f_2'(C_n)\geqslant n$.

Let $D$ be a distribution on $C_n$ such that $D_i=2$ if $i$ is odd and $D_i=0$ if $i$ is even, except for $D_n=1$ if $n$ is odd. Then $D$ is 2-fold solvable and $|D|=n$, so $f_2'(C_n)\leqslant n$.

Let $D$ be a distribution on $C_n$ so that $D_1=D_2=2$ and $D_i=1$ otherwise. Then $D$ is 3-fold solvable and $|D|=n+2$, so $f_3'(C_n)\leqslant n+2$.

Let $D'$ be a distribution on $C_n$ with $f_3'(C_n)$ pebbles such that it has at most two pebbles on every vertex. If all the vertices of $C_n$ are occupied and there exists one vertex $u_i$ with $D_i\geqslant 3$, then from the upper bound $n+2$ it follows that $D_j=1$ for every $j\neq i$, and $D_i=3$. It is easy to see that it is not a 3-fold solvable distribution. If $D_i=0$ for some $i$, then from Lemma 4.4 in Ref.[1] it follows that at most two pebbles can be moved to $u_i$, a contradiction. Thus $D_i\geqslant 1$ for $1\leqslant i\leqslant n$, but no distribution with $n+1$ pebbles can be 3-fold solvable and hence $f_3'(C_n)\geqslant n+2$. □

**Definition 2.2** Assume $u$ is the target vertex, a pebbling move from $v$ to $w$ is greedy if $d(w,u)<d(v,u)$.

**Theorem 2.3** $f'_t(C_{2n})\geqslant \frac{2^{n+1}nt}{3\cdot 2^n-3}$, the equality holds if and only if $(3\cdot 2^n-3)\mid t$; $f_t'(C_{2n+1})\geqslant\frac{2^{n-1}(2n+1)t}{3\cdot 2^{n-1}-1}$, the equality holds if and only if $(3\cdot 2^n-2)\mid t$.

**Proof** We only prove that for even cycle, the case for the odd cycle can be similarly proved.

Let $D$ be a $t$-fold solvable distribution with $f_t'(C_{2n})$ pebbles on $C_{2n}=u_1u_2\cdots u_{2n}u_1$. For simplicity, let $a_i=D_i$ for $1\leqslant i\leqslant 2n$ ($a_{2n+i}=a_i$), $\tilde{a}_i$ be the number of pebbles on $u_i$ after some pebbling moves.

Then we have

$$t\leqslant W(D,u_i)\leqslant a_i+(a_{i+1}/2+a_{i-1}/2)+(a_{i+2}/4+a_{i-2}/4)+\cdots+(a_{i+n-1}/2^{n-1}+a_{i-n+1}/2^{n-1})+a_{i+n}/2^n.$$

Adding these $2n$ inequalities, we can get the inequality.

If $t=(3\cdot 2^n-3)m$ for some integer $m$, then we put $2^nm$ pebbles on each vertex, which is a $t$-fold solvable distribution, so the equality holds.

Conversely, if the equality holds, then for $1\leqslant i\leqslant 2n$,

$$W(D,u_i)=a_i+(a_{i+1}/2+a_{i-1}/2)+(a_{i+2}/4+a_{i-2}/4)+\cdots+(a_{i+n-1}/2^{n-1}+a_{i-n+1}/2^{n-1})+a_{i+n}/2^n=t.$$

This means:

① The pebbling moves must be greedy.

② In the sequence of pebbling moves, we can not lose any one pebble. In other words, if $t$ pebbles have been moved to $u_i$, then the number of pebbles left on any other vertex must be 0.

First we prove that $a_i$ is a constant for all $1\leqslant i\leqslant 2n$.

Let $d=\min\{a_j\mid 1\leqslant j\leqslant 2n\}$. Without loss of generality, we assume that $a_1=d$. Let $\alpha=a_2+a_3/2+\cdots+a_n/2^{n-2}$ and $\beta=a_{2n}+a_{2n-1}/2+\cdots+a_{n+2}/2^{n-2}$. Then from $W(D,u_{2n})=W(D,u_2)$, $W(D,u_{2n})=W(D,u_1)$, we can get

$$\alpha+\beta/4+a_{n+1}/2^{n-1}+d/2=\beta+\alpha/4+a_{n+1}/2^{n-1}+d/2,$$

$$d+\beta/2+\alpha/2+a_{n+1}/2^n=\beta+\alpha/4+a_{n+1}/2^{n-1}+d/2.$$

So $\alpha=\beta=2d-a_{n+1}/2^{n-2}\leqslant 2d-\frac{1}{2^{n-2}}d$. From

$$\alpha=a_2+a_3/2+\cdots+a_n/2^{n-2},$$

it follows that

$$\alpha\geqslant d+\left(1-\frac{1}{2^{n-2}}\right)d=2d-\frac{1}{2^{n-2}}d,$$

where equality holds if and only if $a_2=a_3=\cdots=a_n=d$. Again from

$$2d-\frac{1}{2^{n-2}}d\leqslant\alpha=2d-a_{n+1}/2^{n-2}\leqslant 2d-\frac{1}{2^{n-2}}d$$

it follows that $a_{n+1}=d$. In the same way, we know that $a_{n+2}=a_{n+3}=\cdots=a_{2n-1}=a_{2n}=d$.

Now we prove that $2^n | d$.

If $a_i = d$ is odd and $u_{i+n+1}$ is the target vertex, there is at least one pebble on $u_i$ which can not be moved, so $2 | n$.

If $a_i = d = 4k+2$ and $u_{i+n+1}$ is the target vertex, after all the pebbles are removed from $u_i$, one of $\tilde{a}_{i+1}$, $\tilde{a}_{i-1}$ must be odd. So $4 | d$.

If $2^j | d$ for some $j < n$, but then $2^{j+1} \nmid d$, namely, there is some integer $k$ such that $a_i = 2^{j+1}k + 2^j$. Let the target vertex be $u_{i+n+1}$, then when we move all pebbles off $u_s$ for $i-j < s < i+j$, then one of $\tilde{a}_{i+j}$, $\tilde{a}_{i-j}$ must be odd, a contradiction. So $2^{j+1} | d$.

From the above argument, it follows that $2^n | d$. Assume that $d = 2^n m$ for some integer $m$, then $t = (3 \cdot 2^n - 3)m$, and hence $(3 \cdot 2^n - 3) | t$. □

Now, we give the optimal $t$-pebbling number of $C_5$.

From Theorem 2.3, we have the following corollary.

**Corollary 2.4** $f'_t(C_5) \geqslant 2t+1$ if $10 \nmid t$, $f'_t(C_5) = 2t$ if $10 | t$.

Let $C_5 = u_1 u_2 u_3 u_4 u_5$, $a_i = D(u_i)$. First we give the optimal $t$-pebbling numbers of $C_5$ for $2 \leqslant t \leqslant 11$, which were obtained by the direct calculation.

**Lemma 2.5** For $2 \leqslant t \leqslant 11$, the optimal $t$-pebbling number of $C_5$ is given in Tab. 1.

**Theorem 2.6** The optimal $t$-pebbling number of $C_5$ is

$$f'_t(C_5) = \begin{cases} 4, & \text{if } t = 1; \\ 2t, & \text{if } 10 \mid t; \\ 2t+1, & \text{otherwise.} \end{cases}$$

**Proof** For $t \leqslant 10$, it follows from Tab. 1.

For $t > 10$, assume that $t = 10n + r$, where $0 \leqslant r \leqslant 9$.

If $r = 0$, it follows from Corollary 2.4.

If $r = 1$, then from Lemma 0.5 and Corollary 2.4, we have

$$2t+1 \leqslant f'_t(C_5) \leqslant (n-1)f'_{10}(C_5) + f'_{11}(C_5) = 20(n-1) + 23 = 2t+1.$$

So $f'_t(C_5) = 2t+1$ if $t = 10n+1$, where $n \geqslant 1$.

If $r \neq 1$, then from Lemma 0.5 and Corollary 2.4,

$$2t+1 \leqslant f'_t(C_5) \leqslant nf'_{10}(C_5) + f'_r(C_5) = 20n + 2r + 1 = 2t+1.$$

So $f'_t(C_5) = 2t+1$ if $t = 10n + r$, where $r \neq 0, 1$. □

## 3 Optimal pebbling on product of paths

In this section, we give the optimal $t$-pebbling number of $P_2 \times P_3$. Let $D = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}$ be a $t$-fold solvable distribution on

$$P_2 \times P_3 = \begin{pmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \end{pmatrix}.$$

First we give a lower bound of $f'_t(P_2 \times P_3)$.

**Lemma 3.1** $f'_t(P_2 \times P_3) \geqslant \lceil 20t/9 \rceil$, equality holds if $t \equiv 0 \pmod 9$.

**Proof** Let $D = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}$ be a distribution on $P_2 \times P_3$ with $f'_t(P_2 \times P_3)$ pebbles. Then we can get $W(1) \doteq a_1 + a_2/2 + a_3/4 + a_4/2 + a_5/4 + a_6/8 \geqslant t$. For the other 5 vertices, we can get similar inequalities according to pebbling moves.

Since $W(1) + W(3) + W(4) + W(6) \geqslant 4t$,

$$\frac{15}{8}(a_1 + a_3 + a_4 + a_6) + \frac{3}{2}(a_2 + a_5) \geqslant 4t \quad (4)$$

Since $W(2) + W(5) \geqslant 2t$,

$$\frac{3}{4}(a_1 + a_3 + a_4 + a_6) + \frac{3}{2}(a_2 + a_5) \geqslant 2t \quad (5)$$

$(4) \times 2 + (5)$, we can get

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \geqslant 20t/9,$$

**Tab. 1** $f'_t(C_5)$ **for** $2 \leqslant t \leqslant 11$

| $t$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| $f'_t(C_5)$ | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 20 | 23 |
| $a_1, a_2, a_3$ | 2,0,2 | 2,2,1 | 1,3,1 | 2,2,2 | 3,3,2 | 3,3,3 | 3,4,3 | 4,4,4 | 4,4,4 | 5,4,5 |
| $a_4, a_5$ | 0,1 | 1,1 | 2,2 | 2,3 | 3,2 | 3,3 | 3,4 | 4,3 | 4,4 | 4,5 |

**Tab. 2** $f'_t(P_2\times P_3)$ **for** $1\leqslant t\leqslant 9$

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $\lceil 20t/9\rceil$ | 3 | 5 | 7 | 9 | 12 | 14 | 16 | 18 | 20 |
| $f'_t(P_2\times P_3)$ | 3 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| $a_1,a_2,a_3$ | 0,2,0 | 1,2,1 | 1,2,1 | 2,2,2 | 2,2,2 | 2,3,2 | 3,2,3 | 3,2,3 | 4,2,4 |
| $a_4,a_5,a_6$ | 0,1,0 | 0,2,0 | 1,2,1 | 1,2,1 | 2,2,2 | 2,3,2 | 3,2,3 | 4,2,4 | 4,2,4 |

so $f'_t(P_2\times P_3)\geqslant 20t/9$.

If $t\equiv 0 \pmod 9$, then assume $t=9m$ for some integer $m$. Let $a_1=a_3=a_4=a_6=4m$, $a_2=a_4=2m$, which is a $t$-fold solvable distribution with $20m$ pebbles, so the equality holds. □

From direct computation, we can get

**Lemma 3.2** $f'_t(P_2\times P_3)=2t+2$ for $2\leqslant t\leqslant 9$.

**Proof** We use Tab. 2, where the last row is a $t$-fold solvable distribution $D$ with $f'_t(P_2\times P_3)$ pebbles on $P_2\times P_3$. □

**Theorem 3.3** $f'_{9t}(P_2\times P_3)=20t$,

$$f'_{9t+1}(P_2\times P_3)=20t+3,$$

and

$$20t+2r+1\leqslant f'_{9t+r}(P_2\times P_3)\leqslant 20t+2r+2,\ \text{for } 2\leqslant r\leqslant 8,$$

the last equality holds for $r=5,6,7,8$.

**Proof** From Lemma 3.1, we have

$$f'_{9t+r}(P_2\times P_3)\geqslant 20t+\lceil 20r/9\rceil.$$

By Lemma 0.5, we know that

$$f'_{9t+r}(P_2\times P_3)\leqslant tf'_9(P_2\times P_3)+f'_r(P_2\times P_3)=20t+f'_r(P_2\times P_3).$$

By Tab. 2, we are done. □

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