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A C^1 bivariate rational cubic interpolating spline

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Abstract: A bivariate rational bicubic interpolating spline (BRIS) with biquadratic denominator and six shape parameters was constructed using both function values and partial derivatives of the function as the interpolation data in a rectangular domain. The C^1 continuous condition of BRIS was discussed. Some properties of BRIS such as symmetry were given. BRIS was proved to be bounded and its error was estimated. In the end, a numerical example was given to illustrate the effect of the shape parameters on the shape of BRIS surface.

Key words: bivariate rational interpolating spline; shape parameter; bounded property; error estimate; symmetry

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一种 C^1 连续的二元有理三次插值样条

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摘要:利用矩形域上的函数值及偏导数值作为插值数据,构造了一种带6个形状参数、分母为双二次的二元有理三次插值样条,讨论了其 C^1 连续条件,给出了诸如对称性的一些性质,证明了其有界性并作了误差分析. 最后给出了一个数值例子说明了形状参数对于曲面构造的有效性.

关键词:二元有理三次插值样条;形状参数;有界性;误差估计;对称性

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0 Introduction

The construction method of curves and surfaces and their mathematical description is a key issue in CAGD. There are many ways to deal with this problem[1-13], such as the polynomial spline method, Bézier spline method and NURBS method. These methods are applied widely in the shape design of industrial products. Some spline methods are interpolating cases, where local shape can not be modified for the interpolating surfaces while interpolating data is unchanged. Some spline methods such as NURBS and Bézier methods are no-interpolating cases, that is to say, the constructed curve and surface do not fit the given data and the given points are the control points. Therefore, when we construct the interpolating functions required for CAGD, we should consider not only the expressions of interpolating functions are simple and explicit, but also the parameters of constructed curves and surfaces can be modified without changing the given data.

Recently, there has been many works[14-21] about univariate rational spline interpolation with parameters. Some univariate rational spline interpolations are generalized to bivariate cases, whose expressions with parameters are simple and explicit. Some interesting results are presented. In Refs. [22-28], the authors constructed several bivariate spline interpolations over rectangular mesh, and derived some properties such as the sufficient conditions of down-constraint and up-constraint for the shape control, the matrix expression, bounded property, stability, convexity control, the preserving positivity.

In this paper, motivated by Ref.[25], we will construct a kind of bivariate rational bicubic interpolating spline with biquadratic denominator (BRIS) and six shape parameters in a rectangle domain. The definition of BRIS presented and the effect of the parameter choice on the curve shape discussed in Section 1. The sufficient condition that BRIS is C^1 continuous in Section 2. Some properties of BRIS such as the symmetry is discussed in Section 3. The bounded property is proved and the error of BRIS is analysed in Section 4. A numerical example is given to illustrate the effect of the shape parameters on the shape of BRIS surface, and comparison of our error with that from Ref.[25] is given in Section 5.

1 Rational interpolating spline

Let D be the rectangular domain $[a,b;c,d]$, $\{(x_i,y_i,f_{ij},f_{ij}^x,f_{ij}^y),i=1,2,\cdots,n+1;j=1,2,\cdots,m+1\}$ be a given set of data points, where $a=x_1<x_2<\cdots<x_{n+1}=b;c=y_1<y_2<\cdots<y_{m+1}=d$. Set

$$f_{ij}=f(x_i,y_j),f_{ij}^x=\left.\frac{\partial f(x,y)}{\partial x}\right|_{(x_i,y_j)},$$

$$f_{ij}^y=\left.\frac{\partial f(x,y)}{\partial y}\right|_{(x_i,y_j)},$$

$$h_i=x_{i+1}-x_i,l_j=y_{j+1}-y_j.$$

Denote $D_{ij}=[x_i,x_{i+1};y_j,y_{j+1}]$. For any point $(x,y)\in D$, let

$$\theta=(x-x_i)/h_i,\eta=(y-y_j)/l_j.$$

For each y_j, $j=1,2,\cdots,m+1$, we construct the x-direction interpolation curve as follows

$$P_{ij}^*(x)=\frac{p_{ij}^*(x)}{q_{ij}^*(x)},i=1,2,\cdots,n \quad (1)$$

where

$$p_{ij}^*(x)=\alpha_{ij}^*f_{ij}(1-\theta)^3+V_{ij}^*\theta(1-\theta)^2+W_{ij}^*\theta^2(1-\theta)+f_{i+1,j}\gamma_{ij}^*\theta^3,$$

$$q_{ij}^*(x)=\alpha_{ij}^*(1-\theta)^2+\beta_{ij}^*\theta(1-\theta)+\gamma_{ij}^*\theta^2$$

and

$$V_{ij}^*=(\alpha_{ij}^*+\beta_{ij}^*)f_{ij}+h_i\alpha_{ij}^*f_{ij}^x,$$

$$W_{ij}^*=(\beta_{ij}^*+\gamma_{ij}^*)f_{i+1,j}-h_i\gamma_{ij}^*f_{i+1,j}^x$$

with $\alpha_{ij}^*,\beta_{ij}^*,\gamma_{ij}^*$ positive. One can prove that

$$P_{ij}^*(x_r)=f_{rj},\ P_{ij}^{*\prime}(x_r)=f_{rj}^x,r=i,i+1.$$

The interpolation Eq.(1) is called the rational cubic interpolating spline based on function values and derivatives. It is clear that the interpolation is local in the interval $[x_i,x_{i+1}]$ and depends on the shape parameters $\alpha_{ij}^*,\beta_{ij}^*,\gamma_{ij}^*$.

It is interesting to note that the interpolating spline Eq.(1) becomes a standard cubic Hermite spline with the values of the shape parameters

$\alpha_{ij}^{*}=1$, $\beta_{ij}^{*}=2$, $\gamma_{ij}^{*}=1$. We now illustrate the mathematical and graphical effect of the shape parameters $\alpha_{ij}^{*}, \beta_{ij}^{*}, \gamma_{ij}^{*}$ on the shape of a curve. The three free parameters can be exploited properly to modify the shape of curve according to the designer's choice. We rewrite Eq.(1) as follows:

$$P_{ij}^{*}(x)=\frac{\alpha_{ij}^{*}(1-\theta)^{2}(f_{ij}+f_{ij}^{x}h_{i}\theta)+\beta_{ij}^{*}\theta(1-\theta)(f_{ij}(1-\theta)+f_{i+1,j}\theta)+\gamma_{ij}^{*}\theta^{2}(f_{i+1,j}-h_{i}(1-\theta)f_{i+1,j}^{x})}{\alpha_{ij}^{*}(1-\theta)^{2}+\beta_{ij}^{*}\theta(1-\theta)+\gamma_{ij}^{*}\theta^{2}} \tag{2}$$

By the computation for Eq.(2) we have the following formulas:

$$\lim_{\alpha_{ij}^{*}\to\infty} P_{ij}^{*}(x)=f_{ij}+h_{i}f_{ij}^{x}\theta \tag{3}$$

$$\lim_{\beta_{ij}^{*}\to\infty} P_{ij}^{*}(x)=f_{ij}(1-\theta)+f_{i+1,j}\theta \tag{4}$$

$$\lim_{\gamma_{ij}^{*}\to\infty} P_{ij}^{*}(x)=f_{i+1,j}-h_{i}f_{i+1,j}^{x}(1-\theta) \tag{5}$$

We can see from Eqs.(3)~(5) that the increase in the shape parameter α_{ij}^{*}, β_{ij}^{*} or γ_{ij}^{*} reduces the rational interpolation spline Eq.(2) to three different straight lines. According to data points in Tab. 1, by choosing different shape parameters, we can observe the corresponding shape changes of the interpolating curves in Fig.1. The curve in Fig. 1(a) is a cubic Hermite interpolating spline. Each piecewise curve in Figs.1(b) and 1(c) incline to a line segment. Each piecewise curve in Fig.1(d) tilts heavily to the left-hand side.

Tab.1 Data points

i	x_i	f_i	f_i^x
1	0	2	5.6
2	0.25	0.6	−2
3	0.5	0.1	0.06
4	1	0.13	1.74
5	1.5	1	−1
6	2	0.5	1.2
7	2.5	1.1	−1.7
8	3	0.25	−0.05
9	4	0.2	0

We now use the x-direction interpolation $P_{ij}^{*}(x)$, $i=1,2,\cdots,n-1$; $m=1,2,\cdots,m+1$ to construct the BRIS in D. Denote $D_{ij}=[x_i,x_{i+1};y_j,y_{j+1}]$, $i=1,2,\cdots,n$, $j=1,2,\cdots,m$ and

$$f_{i,j+s}^{*}(x)=\sum_{r=0}^{1}\rho_r(\theta,\bullet_{i,j+s}^{*})f_{i+r,j+s}^{y},\quad s=0,1 \tag{6}$$

where

$$\rho_0(\theta,\bullet_{i,j+s}^{*})=\frac{(1-\theta)^2(\alpha_{i,j+s}^{*}+\beta_{i,j+s}^{*}\theta)}{\alpha_{i,j+s}^{*}(1-\theta)^2+\beta_{i,j+s}^{*}\theta(1-\theta)+\gamma_{i,j+s}^{*}\theta^2},$$

$$\rho_1(\theta,\bullet_{i,j+s}^{*})=\frac{(\gamma_{i,j+s}^{*}+\beta_{i,j+s}^{*}(1-\theta))\theta^2}{\alpha_{i,j+s}^{*}(1-\theta)^2+\beta_{i,j+s}^{*}\theta(1-\theta)+\gamma_{i,j+s}^{*}\theta^2},$$

It is clear that

$$f_{i,j+s}^{*}(x_{i+r})=f_{i+r,j+s}^{y},\ f_{i+r,j+s}^{*\prime}(x_r)=0,\ r,s=0,1.$$

For each pair (i,j), $i=1,2,\cdots,n$; $j=1,2,\cdots,m$, we define a bivariate interpolating spline $P_{ij}(x,y)$ in D_{ij} as follows

$$P_{ij}(x,y)=\frac{p_{ij}(x,y)}{q_{ij}(x,y)} \tag{7}$$

where

$$p_{ij}(x,y)=\alpha_{ij}P_{ij}^{*}(x)(1-\eta)^3+V_{ij}\eta(1-\eta)^2+W_{ij}\eta^2(1-\eta)+\gamma_{ij}\eta^3P_{i,j+1}^{*}(x),$$

$$q_{ij}(x,y)=\alpha_{ij}(1-\eta)^2+\beta_{ij}\eta(1-\eta)+\gamma_{ij}\eta^2$$

and

$$V_{ij}=(\alpha_{ij}+\beta_{ij})P_{ij}^{*}(x)+l_j\alpha_{ij}f_{ij}^{*}(x),$$

$$W_{ij}=(\beta_{ij}+\gamma_{ij})P_{i,j+1}^{*}(x)-l_j\gamma_{ij}f_{i,j+1}^{*}(x)$$

with $\alpha_{ij}, \beta_{ij}, \gamma_{ij}$ positive. $P_{ij}(x,y)$ is called a BRIS with a bi-quadratic denominator. We can easily prove that

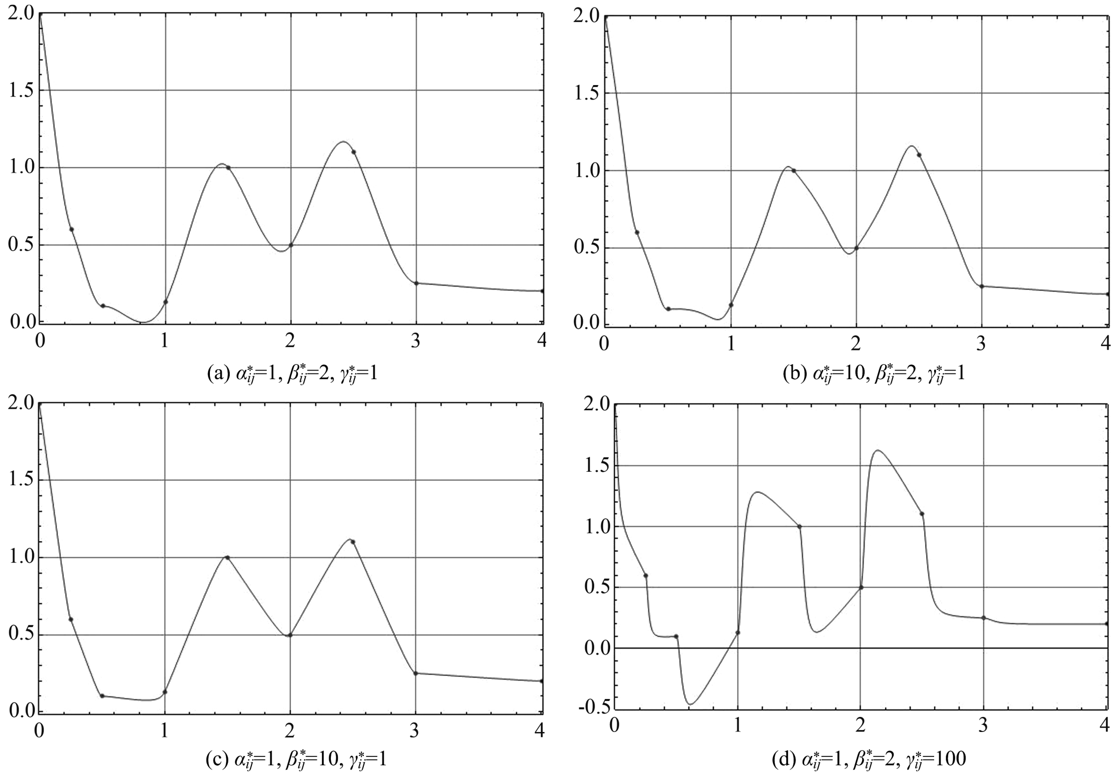
$$\frac{\partial P_{ij}(x,y_s)}{\partial}=f_{is}^{*}(x),\ s=j,j+1 \tag{8}$$

Meanwhile, it is clear that $P_{ij}(x,y)$ satisfies the interpolating conditions

$$P_{ij}(x_r,y_s)=f(x_r,y_s),$$

$$\frac{\partial P_{ij}(x_r,y_s)}{\partial x}=f_{rs}^{x},$$

$$\frac{\partial P_{ij}(x_r,y_s)}{\partial y}=f_{rs}^{y},\ r=i,i+1,\ s=j,j+1.$$



(a) $\alpha_{ij}^*=1, \beta_{ij}^*=2, \gamma_{ij}^*=1$ (b) $\alpha_{ij}^*=10, \beta_{ij}^*=2, \gamma_{ij}^*=1$
(c) $\alpha_{ij}^*=1, \beta_{ij}^*=10, \gamma_{ij}^*=1$ (d) $\alpha_{ij}^*=1, \beta_{ij}^*=2, \gamma_{ij}^*=100$

Fig.1 The rational interpolating spline(1)

The piecewise function Eq.(7) on D_{ij} is unique.

We remark that if we choose $\alpha_{ij}^* = \alpha = 1, \beta_{ij}^* = \beta = 2, \gamma_{ij}^* = \gamma = 1$, $P_{ij}(x, y)$ becomes a bicubic Hermite spline.

2 Smoothing conditions

For each y_j, $j = 1, 2, \cdots, m+1$, from the definition of $P_{ij}(x, y)$, we can see that it is C^1 continuous along the x-direction. Similarly, we can see from Eq.(8) that $P_{ij}(x, y)$ have a continuous first-order partial derivative $\partial P_{ij}(x, y)/\partial$ in D. So it is sufficient for $P_{ij}(x, y) \in C^1$ in D if $\partial P_{ij}(x_i+, y)/\partial x = \partial P_{i-1,j}(x_i-, y)/\partial x$ holds. We then have the following theorem.

Theorem 2.1 BRIS $P_{ij}(x, y)$ is C^1 in the whole interpolating region D if the parameters $\alpha_{i-1,j} = \alpha_{ij}, \beta_{i-1,j} = \beta_{ij}, \gamma_{i-1,j} = \gamma_{ij}$ for each j, $j = 1, 2, \cdots, m$ and all $i = 1, 2, \cdots, n$.

Proof Based on above discussion, we only need to prove that

$$\frac{\partial P_{ij}(x_i+, y)}{\partial x} = \frac{\partial P_{i-1,j}(x_i-, y)}{\partial x} \tag{9}$$

We first compute that

$$\frac{\partial P_{ij}(x, y)}{\partial x} = \frac{1}{q_{ij}(y)}\Big[(1-\eta)^3 \alpha_{ij} \frac{\mathrm{d}P_{ij}^*(x)}{\mathrm{d}x} + \eta(1-\eta)^2 \frac{\mathrm{d}V_{ij}}{\mathrm{d}x} + \eta^2(1-\eta)\frac{\mathrm{d}W_{ij}}{\mathrm{d}x} + \gamma^3 \beta_{ij} \frac{\mathrm{d}P_{i,j+1}^*(x)}{\mathrm{d}x}\Big] \tag{10}$$

Since

$$P_{is}^{*\prime}(x_i+) = f_{is}^x, s = j, j+1 \tag{11}$$

hence

$$\left.\begin{aligned} V'_{ij}|_{x_i^+} &= (\alpha_{ij} + \beta_{ij}) f_{ij}^x, \\ W'_{ij}|_{x_i^+} &= (\beta_{ij} + \gamma_{ij}) f_{i,j+1}^x \end{aligned}\right\} \tag{12}$$

Substituting Eqs.(11) and (12) into (10), we obtain that

$$\frac{\partial P_{ij}(x_i+, y)}{\partial x} = \frac{1}{\alpha_{ij}(1-\eta)^2 + \beta_{ij}\eta(1-\eta) + \gamma_{ij}\eta^2}\Big[(1-\eta)^3 \alpha_{ij} f_{ij}^x + \eta(1-\eta)^2(\alpha_{ij} + \beta_{ij}) f_{ij}^x + \eta^2(1-\eta)(\beta_{ij} + \gamma_{ij}) f_{i,j+1}^x + \eta^3 \gamma_{ij} f_{i,j+1}^x\Big] \tag{13}$$

Similarly, since

$$P_{i-1,s}^{*\prime}(x_i-) = f_{is}^x, s = j, j+1,$$

we can also compute that

$$\frac{\partial P_{i-1,j}(x_i -, y)}{\partial x} = \frac{1}{\alpha_{i-1,j}(1-\eta)^2 + \beta_{i-1,j}\eta(1-\eta) + \gamma_{i-1,j}\eta^2}[(1-\eta)^3\alpha_{i-1,j} f^x_{ij} + \eta(1-\eta)^2(\alpha_{i-1,j} + \beta_{i-1,j}) f^x_{ij} + \eta^2(1-\eta)((\beta_{i-1,j} + \gamma_{i-1,j}) f^x_{i,j+1} + \eta^3\gamma_{i-1,j} f^x_{i,j+1}]. \tag{14}$$

By comparing Eq.(13) with Eq.(14), we can yield $\alpha_{i-1,j} = \alpha_{ij}, \beta_{i-1,j} = \beta_{ij}, \gamma_{i-1,j} = \gamma_{ij}$. Therefore Eq.(9) holds.

3 Some properties of BRIS

We rewrite $P^*_{ij}(x)$ as follows:

$$P^*_{ij}(x) = \sum_{r=0}^{1}[\omega_{0r}(\theta, \bullet^*_{ij}) f_{i+r,j} + \omega_{1r}(\theta, \bullet^*_{ij}) h_i f^x_{i+r,j}] \tag{15}$$

where

$$\omega_{00}(\theta, \bullet^*_{ij}) = \frac{(1-\theta)^2(\alpha^*_{ij} + \beta^*_{ij}\theta)}{\alpha^*_{ij}(1-\theta)^2 + \beta^*_{ij}\theta(1-\theta) + \gamma^*_{ij}\theta^2} \tag{16}$$

$$\omega_{01}(\theta, \bullet^*_{ij}) = \frac{(\gamma^*_{ij} + \beta^*_{ij}(1-\theta))\theta^2}{\alpha^*_{ij}(1-\theta)^2 + \beta^*_{ij}\theta(1-\theta) + \gamma^*_{ij}\theta^2} \tag{17}$$

$$\omega_{10}(\theta, \bullet^*_{ij}) = \frac{\alpha^*_{ij}(1-\theta)^2\theta}{\alpha^*_{ij}(1-\theta)^2 + \beta^*_{ij}\theta(1-\theta) + \gamma^*_{ij}\theta^2} \tag{18}$$

$$\omega_{11}(\theta, \bullet^*_{ij}) = -\frac{\gamma^*_{ij}(1-\theta)\theta^2}{\alpha^*_{ij}(1-\theta)^2 + \beta^*_{ij}\theta(1-\theta) + \gamma^*_{ij}\theta^2} \tag{19}$$

where $\omega_{rs}(\theta, \bullet^*_{ij}), r, s = 0, 1$, are called the basis functions of $P^*_{ij}(x)$. We see that $\omega_{0r}(\theta, \bullet^*_{ij}) = \rho_r(\theta, \bullet^*_{ij}), r = 0, 1$ in Eq.(6). It is easy to compute that

$$\omega_{00}(\theta, \bullet^*_{ij}) + \omega_{01}(\theta, \bullet^*_{ij}) = 1 \tag{20}$$

Similarly, we rewrite $P_{ij}(x, y)$ as follows:

$$P_{ij}(x, y) = \sum_{s=0}^{1}[\omega_{0s}(\eta, \bullet_{ij}) P^*_{i,j+s}(x) + \omega_{1s}(\eta, \bullet_{ij}) l_j f^*_{i,j+s}(x)] \tag{21}$$

Using Eq.(15), Eq.(21) has the following form:

$$P_{ij}(x, y) = \sum_{s=0}^{1}\sum_{r=0}^{1}[\omega_{0s}(\eta, \bullet_{ij})\omega_{0r}(\theta, \bullet^*_{i,j+s}) f_{i+r,j+s} + \omega_{0s}(\eta, \bullet_{ij})\omega_{1r}(\theta, \bullet_{i,j+1}) h_i f^x_{i+r,j+s} + \omega_{1s}(\eta, \bullet_{ij})\omega_{0r}(\theta, \bullet^*_{i,j+s}) l_j f^y_{i+r,j+s}] \tag{22}$$

If $\alpha^*_{ij} = \alpha^*_{i,j+1}, \beta^*_{ij} = \beta^*_{i,j+1}, \gamma^*_{ij} = \gamma^*_{i,j+1}$, Eq.(22) has the following matrix form

$$\begin{aligned} P_{ij}(x, y) = &[\omega_{00}(\theta, \bullet^*_{ij}) \quad \omega_{01}(\theta, \bullet^*_{ij})]\begin{bmatrix} f_{ij} & f_{i,j+1} \\ f_{i+1,j} & f_{i+1,j+1} \end{bmatrix}\begin{bmatrix} \omega_{00}(\eta, \bullet_{ij}) \\ \omega_{01}(\eta, \bullet_{ij}) \end{bmatrix} + \\ &h_i[\omega_{10}(\theta, \bullet^*_{ij}) \quad \omega_{11}(\theta, \bullet^*_{ij})]\begin{bmatrix} f^x_{ij} & f^x_{i,j+1} \\ f^x_{i+1,j} & f^x_{i+1,j+1} \end{bmatrix}\begin{bmatrix} \omega_{00}(\eta, \bullet_{ij}) \\ \omega_{01}(\eta, \bullet_{ij}) \end{bmatrix} + \\ &l_j[\omega_{00}(\theta, \bullet^*_{ij}) \quad \omega_{01}(\theta, \bullet^*_{ij})]\begin{bmatrix} f^y_{ij} & f^y_{i,j+1} \\ f^y_{i+1,j} & f^y_{i+1,j+1} \end{bmatrix}\begin{bmatrix} \omega_{10}(\eta, \bullet_{ij}) \\ \omega_{11}(\eta, \bullet_{ij}) \end{bmatrix} \end{aligned} \tag{23}$$

From Eq.(23), we have the following property.

Theorem 3.1 (symmetry) If $\alpha^*_{ij} = \alpha^*_{i,j+1}, \beta^*_{ij} = \beta^*_{i,j+1}, \gamma^*_{ij} = \gamma^*_{i,j+1}$, BRIS $P_{ij}(x, y)$ are the same whatever direction it begins with first.

Denote

$$a_{rs}(\theta, \eta) = \omega_{0r}(\theta, \bullet^*_{i,j+s})\omega_{0s}(\eta, \bullet_{ij}),$$
$$b_{rs}(\theta, \eta) = \omega_{1r}(\theta, \bullet_{i,j+s})\omega_{0s}(\eta, \bullet_{ij}),$$
$$c_{rs}(\theta, \eta) = \omega_{0r}(\theta, \bullet^*_{i,j+s})\omega_{1s}(\eta, \bullet_{ij}),$$

where $a_{rs}(\theta, \eta), b_{rs}(\theta, \eta), c_{rs}(\theta, \eta)$ are called the basis of BRIS. Then Eq.(22) can be rewrite as follows:

$$P_{ij}(x, y) = \sum_{s=0}^{1}\sum_{r=0}^{1}[a_{rs}(\theta, \eta) f_{i+r,j+s} + b_{rs}(\theta, \eta) h_i f^x_{i+r,j+s} + c_{rs}(\theta, \eta) l_j f^y_{i+r,j+s}] \tag{24}$$

From Eq.(20), $a_{rs}(\theta, \eta), r, s = 0, 1$ satisfy that

$$\sum_{r=0}^{1}\sum_{s=0}^{1} a_{rs}(\theta, \eta) = 1 \tag{25}$$

which leads to the following unit property.

Property 3.1 If $f(x,y)=1,(x,y)\in D$, $P_{ij}(x,y)$ is its BRIS in D_{ij}, no mater what positive number the parameters $\alpha_{ij}^*,\beta_{ij}^*,\gamma_{ij}^*$ are, the unit property holds, namely

$$\iint_{D_{ij}} P_{ij}(x,y)\mathrm{d}x\mathrm{d}y=h_i l_j.$$

From Eqs.(16)~(19), for any positive α_{ij}^*, $\beta_{ij}^*,\gamma_{ij}^*$, we can compute that

$$\left.\begin{array}{l} 0<\omega_{00}(\theta,\bullet_{ij}^*)<1, \\ 0<\omega_{01}(\theta,\bullet_{ij}^*)<1, \\ 0<\omega_{10}(\theta,\bullet_{ij}^*)<\theta, \\ \theta-1<\omega_{11}(\theta,\bullet_{ij}^*)<0 \end{array}\right\} \tag{26}$$

Define

$$\omega_{rs}^*=\int_0^1\omega_{rs}(\theta,\bullet_{ij}^*)\mathrm{d}\theta,\ a_{rs}^*=\iint_{\widetilde{D}} a_{rs}(\theta,\eta),$$

$$b_{rs}^*=\int_{\widetilde{D}} b_{rs}(\theta,\eta),\ c_{rs}^*=\iint_{\widetilde{D}} c_{rs}(\theta,\eta),$$

where $\widetilde{D}=[0,1;0,1]$. $a_{rs}^*,b_{rs}^*,c_{rs}^*$ are called the integral weights coefficients of BRIS. From Eq.(26), we obtain that

$$0<\omega_{00}^*<1, 0<\omega_{01}^*<1,$$

$$0<\omega_{10}^*<\frac{1}{2},-\frac{1}{2}<\omega_{11}^*<0.$$

Thus it is easy to get the following property:

Property 3.2 For the BRIS $P_{ij}(x,y)$ defined by Eq.(7) on the rectangle net D, the weight coefficients satisfy

$$0<a_{rs}^*<1, r,s=0,1,$$

$$0<b_{0s}^*<\frac{1}{2},-\frac{1}{2}<b_{1s}^*<0, s=0,1,$$

$$0<c_{r0}^*<\frac{1}{2},-\frac{1}{2}<c_{r1}<0, r=0,1.$$

We notice that when $\alpha_{ij}^*=1,\beta_{ij}^*=2,\gamma_{ij}^*=1$, $\alpha_{ij}=1,\beta_{ij}=2,\gamma_{ij}=1$, BRIS becomes the bivariate Hermite interpolating spline (BHIS) on the rectangular net D. In this case, we can derive that

$$\sum_{r=0}^1\sum_{s=0}^1 b_{rs}=\theta(1-\theta)(1-2\theta),$$

$$\sum_{r=0}^1\sum_{s=0}^1 c_{rs}=\eta(1-\eta)(1-2\eta).$$

By the definition of b_{rs}^* and c_{rs}^*, it can be shown that

$$\sum_{r=0}^1\sum_{s=0}^1 b_{rs}^*=0,$$

$$\sum_{r=0}^1\sum_{s=0}^1 c_{rs}^*=0.$$

Therefore the following property holds.

Property 3.3 For BHIS $P_{ij}(x,y)$ on D, the integral weight coefficients satisfy the following equation

$$\sum_{r=0}^1\sum_{s=0}^1(a_{rs}^*+b_{rs}^*+c_{rs}^*)=1.$$

4 Bounded property and error analysis

BRIS is a local interpolation in each sub-rectangle, which depends on the function values and the partial derivative values of the function being interpolated at four vertexes of the sub-rectangle. There are nine parameters $\alpha_{i,j+s}^*,\beta_{i,j+s}^*$, $\gamma_{i,j+s}^*, s=0,1,\alpha_{ij},\beta_{ij}$ and γ_{ij} in each interpolating region, and when the parameters vary, the interpolating surface changes accordingly. The first theorem in this section shows that the values of the interpolating function in each interpolating region must be in an interval related to the interpolating data. In the next theorem we analyze the error of BRIS.

Similarly to Eq. (25), the following inequations hold.

$$0<\omega_{00}(\eta,\bullet_{ij})<1,\ 0<\omega_{01}(\eta,\bullet_{ij})<1,$$

$$0<\omega_{10}(\eta,\bullet_{ij})<\eta,\ \eta-1<\omega_{11}(\eta,\bullet_{ij})<0.$$

Combined with the definition of $b_{rs}(\theta,\eta)$ and $c_{rs}(\theta,\eta)$, it implies that

$$\left.\begin{array}{l} 0<b_{0s}(\theta,\eta)<\theta\leqslant 1, s=0,1, \\ 0<|b_{1s}|(\theta,\eta)<1-\theta\leqslant 1, s=0,1, \\ 0<c_{r0}(\theta,\eta)<\eta\leqslant 1,\ r=0,1, \\ 0<|c_{r1}(\theta,\eta)|<1-\eta\leqslant 1,\ r=0,1 \end{array}\right\} \tag{27}$$

Denote

$$Q_1=\max\{|f_{i+r,j+s}|, r,s=0,1\},$$

$$Q_2=\max\{|f_{i+r,j+s}^x|, r,s=0,1\},$$

$$Q_3=\max\{|f_{i+r,j+s}^y|, r,s=0,1\}$$

and $M=Q_1+4(h_iQ_2+l_jQ_3)$.

Theorem 4.1 For the given interpolating data $\{f_{i+r,j+s},f_{i+r,j+s}^x,f_{i+r,j+s}^y, r,s=0,1\}$, $P_{ij}(x,y)$ is BRIS defined by Eq.(7), then

$$|P_{ij}(x,y)|\leqslant M,(x,y)\in D_{ij}.$$

Proof From Eqs.(24),(25) and (27), we yield that

$$|P_{ij}(x,y)| \leqslant \sum_{s=0}^{1}\sum_{r=0}^{1}[|a_{rs}(\theta,\eta)| \cdot |f_{i+r,j+s}| + |b_{rs}(\theta,\eta)| h_i |f^x_{i+r,j+s}| + |c_{rs}(\theta,\eta)| l_j |f^y_{i+r,j+s}|] \leqslant$$

$$\sum_{s=0}^{1}\sum_{r=0}^{1}[|a_{rs}(\theta,\eta)| Q_1 + h_i |b_{rs}(\theta,\eta)| Q_2 + l_j |c_{rs}(\theta,\eta)| Q_3] =$$

$$Q_1\sum_{s=0}^{1}\sum_{r=0}^{1}|a_{rs}(\theta,\eta)| + h_i Q_2 \sum_{s=0}^{1}\sum_{r=0}^{1}|b_{rs}(\theta,\eta)| + l_j Q_3 \sum_{s=0}^{1}\sum_{r=0}^{1}|c_{rs}(\theta,\eta)| \leqslant$$

$$Q_1 + 4(h_i Q_2 + l_j Q_3).$$

The proof is completed.

According the bounded property Theorem 4.1, we can deduce the following error estimation.

Theorem 4.2 Suppose $f(x, y)$ has a continuous first-order partial derivative, then the following error formula of BRIS $P_{ij}(x,y)$ holds

$$|f(x,y) - P_{ij}(x,y)| \leqslant 5\left[h_i\left\|\frac{\partial f}{\partial x}\right\| + l_j\left\|\frac{\partial f}{\partial y}\right\|\right],$$

where

$$\left\|\frac{\partial f}{\partial x}\right\| = \max_{(x,y)\in D_{ij}}\left|\frac{\partial f}{\partial x}\right|, \quad \left\|\frac{\partial f}{\partial y}\right\| = \max_{(x,y)\in D_{ij}}\left|\frac{\partial f}{\partial y}\right|.$$

Proof The Taylor formula of $f(x, y)$ at point $(x_{i+r}, y_{j+s}), r, s=0,1$, is that

$$f(x,y) = f(x_{i+r}, y_{j+s}) + \left[(x - x_{i+r})\frac{\partial}{\partial x} + (y - y_{j+s})\frac{\partial}{\partial y}\right] f(\mu_r, \upsilon_s), (x,y) \in D_{ij},$$

where μ_r is between x and x_{i+r}, υ_s is between y and y_{j+s}. It follows that

$$|f(x,y) - f(x_{i+r}, y_{j+s})| \leqslant h_i\left\|\frac{\partial f}{\partial x}\right\| + l_j\left\|\frac{\partial f}{\partial y}\right\|, r, s = 0, 1.$$

From Eqs.(24) and (25), we have

$$f(x,y) - P_{ij}(x,y) = \sum_{s=0}^{1}\sum_{r=0}^{1} a_{rs}(\theta,\eta)(f(x,y) - f_{i+r,j+s}) - \sum_{s=0}^{1}\sum_{r=0}^{1}[b_{rs}(\theta,\eta) h_i f^x_{i+r,j+s} + c_{rs}(\theta,\eta) l_j f^y_{i+r,j+s}],$$

then

$$|f(x,y) - P_{ij}(x,y)| \leqslant \sum_{s=0}^{1}\sum_{r=0}^{1} a_{rs}(\theta,\eta) |f(x,y) - f_{i+r,j+s}| + \sum_{s=0}^{1}\sum_{r=0}^{1}[h_i |b_{rs}(\theta,\eta)| \cdot |f^x_{i+r,j+s}| + l_j |c_{rs}(\theta,\eta)| \cdot |f^y_{i+r,j+s}|] \leqslant$$

$$h_i\left\|\frac{\partial f}{\partial x}\right\| + l_j\left\|\frac{\partial f}{\partial y}\right\| + \sum_{s=0}^{1}\sum_{r=0}^{1}[h_i |b_{rs}(\theta,\eta)| Q_2 + l_j |c_{rs}(\theta,\eta)| Q_3] \leqslant$$

$$h_i\left\|\frac{\partial f}{\partial x}\right\| + l_j\left\|\frac{\partial f}{\partial y}\right\| + 4(h_i Q_2 + l_j Q_3) \leqslant$$

$$5\left[h_i\left\|\frac{\partial f}{\partial x}\right\| + l_j\left\|\frac{\partial f}{\partial y}\right\|\right].$$

The proof is completed.

5 Numerical example

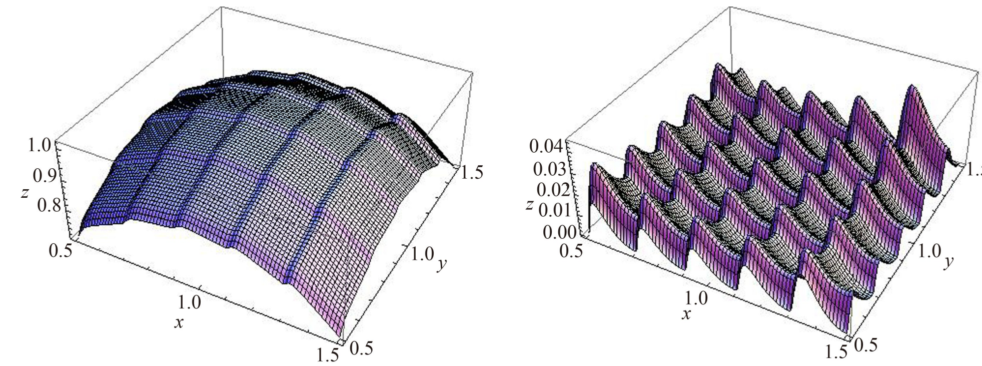
In this section, we utilize a numerical example to illustrate the effect of the shape parameters on the shape of BRIS surface. Because our method is generated form Ref.[25], we also compare our error with the error in Ref.[25]. The result shows that our method is better than the technique in Ref.[24].

Assume that

$$f(x,y) = \sqrt{1 - (1-x)^2 - (1-y)^2},$$

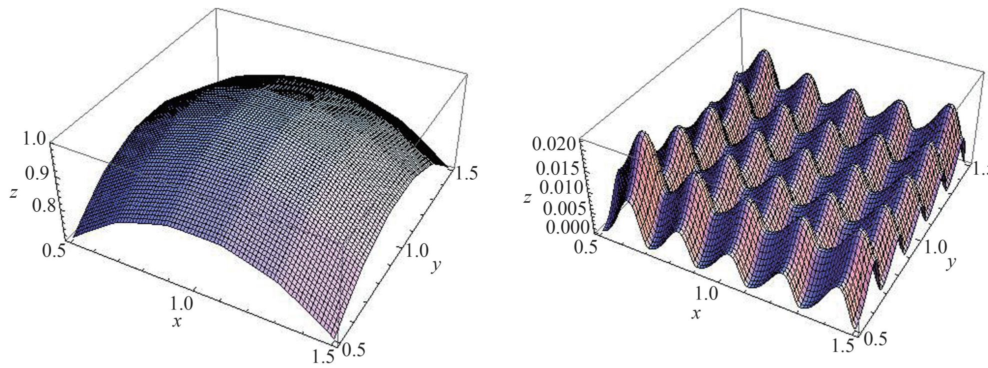
$$(x,y) \in D = [0.5, 1.5; 0, 5, 1.5].$$

BRIS P_{ij} defined by Eq.(7) of $f(x,y)$ can be constructed in D for the given shape parameters $\alpha^*_{ij}, \beta^*_{ij}, \gamma^*_{ij}, \alpha_{ij}, \beta_{ij}$ and γ_{ij}. The graphs of BRIS and corresponding error surfaces are shown in Figs.2～6. We can observe that P_{ij}'s approximate $f(x,y)$ well and that the approximation effect is different when the shape parameters vary. Especially, Fig.5(a) is the figure of Hermite interpolating spline of $f(x, y)$. After slightly changing the value of β^*_{ij} and β_{ij} in Fig.5, we can see from Fig.6 that the error is reduced by nearly a half. Comparing the error in Fig.6(b) with that in Ref.[25]. we can see that the error in Fig.7 is twice as much as our error in Fig.6(b).



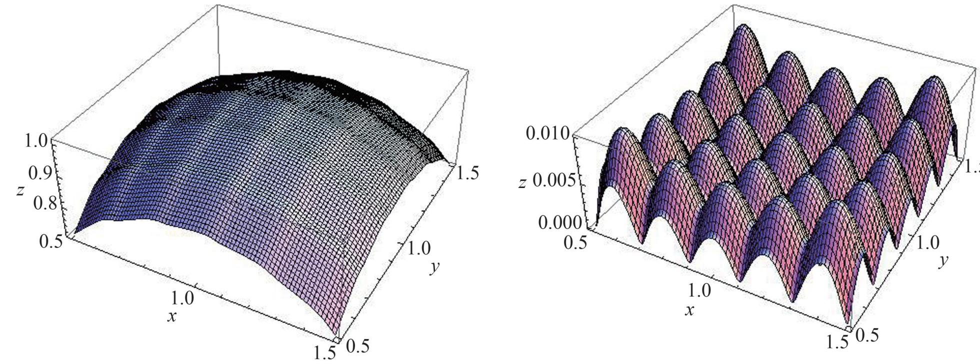
(a) BRIS　　(b) error surface

Fig.2　BRIS and the error surface when $\alpha^*_{ij}=\alpha_{ij}=1, \beta^*_{ij}=\beta_{ij}=1, \gamma^*_{ij}=\gamma_{ij}=100$



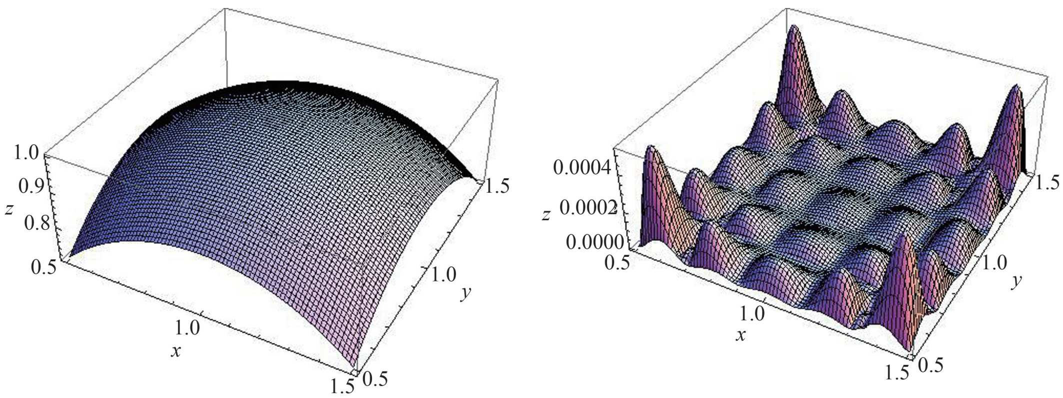
(a) BRIS　　(b) error surface

Fig.3　BRIS and the error surface when $\alpha^*_{ij}=\alpha_{ij}=10, \beta^*_{ij}=\beta_{ij}=1, \gamma^*_{ij}=\gamma_{ij}=1$



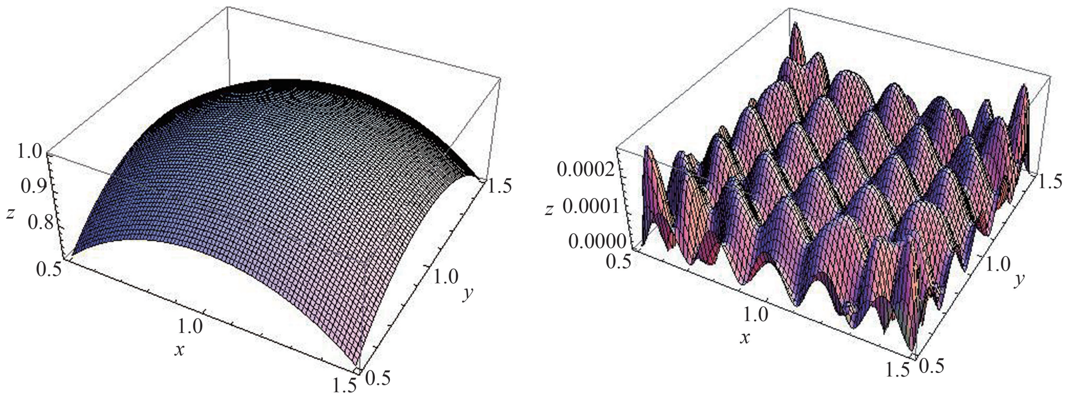
(a) BRIS　　(b) error surface

Fig.4　BRIS and the error surface when $\alpha^*_{ij}=\alpha_{ij}=1, \beta^*_{ij}=\beta_{ij}=10, \gamma^*_{ij}=\gamma_{ij}=1$



(a) BRIS　　(b) error surface

Fig.5　BRIS and the error surface when $\alpha^*_{ij}=\alpha_{ij}=1, \beta^*_{ij}=\beta_{ij}=2, \gamma^*_{ij}=\gamma_{ij}=1$



(a) BRIS (b) error surface

Fig.6 BRIS and the error surface when $\alpha_{ij}^{*}=\alpha_{ij}=1,\beta_{ij}^{*}=\beta_{ij}=2.1,\gamma_{ij}^{*}=\gamma_{ij}=1$

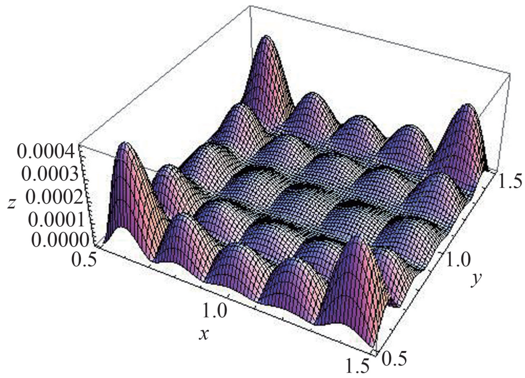


Fig.7 Error surface in Ref. [25] when $\alpha_{ij}^{*}=\alpha_{ij}=1.1,\beta_{ij}^{*}=\beta_{ij}=0.99$

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