第48卷第11期 中国科学技术大学学报 Vol. 48, No. 11
2018年11月 JOURNAL OF UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA Nov. 2018

文章编号:0253-2778(2018)11-0902-04

Planar order on vertex poset

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Abstract: A planar order is a special linear extension of the edge poset (partially ordered set) of a processive plane graph. The definition of a planar order makes sense for any finite poset and is equivalent to the one of a conjugate order. Here it was proved that there is a planar order on the vertex poset of a processive planar graph naturally induced from the planar order of its edge poset.

Key words: edge poset; vertex poset; planar order

CLC number: O157.5 **Document code**: A doi:10.3969/j.issn.0253-2778.2018.11.006

2010 Mathematics Subject Classification: 05C99

Citation: LU Xuexing. Planar order on vertex poset[J]. Journal of University of Science and Technology of China, 2018,48(11):902-905.

鲁学星. 顶点偏序集上的平面序[J]. 中国科学技术大学学报,2018,48(11):902-905.

顶点偏序集上的平面序

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摘要:平面序是渐进平面图的边偏序的一类特殊线性扩张,平面序的定义对一般的有限偏序集都有意义,并且事实上等价于共轭序的概念。这里证明了一个渐进平面图的边偏序集上平面序可以自然诱导其顶点偏序上的一个平面序.

关键词:边偏序集;顶点偏序集;平面序

0 Introduction

The notion of a processive plane graph, a special case of Joyal and Street′s progressive plane graph[1], was introduce in Ref.[2] as a graphical tool for tensor calculus in semi-groupal categories. Ref.[2] gave a totally combinatorial characterization of an equivalence class of processive plane graphs in terms of the notions of a POP-graph which is a processive graph (a special kind of acyclic directed graph) equipped with a planar order (a special linear order of the edges).

However, it turns out that the notion of a planar order can be defined for a general finite poset (partially ordered set) and essentially equivalent to the one of a conjugate order[1], which is an important notion in the study of planar posets. So this raises an interesting question: for a processive graph, are there some relations between planar orders on its edges and planar orders on its

Received: 2017-10-23; **Revised**: 2018-04-24

Foundation item: Supported by the Fundamental Research Funds for the Central Universities.

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vertices? In this paper, we will give a positive answer to this question by showing that any planar order of edges of a processive graph naturally induces a planar order of vertices.

1 processive plane graph

Definition 1.1 A processive plane graph is an acyclic directed graph drawn in a plane box with the properties that: ① all edges monotonically decrease in the vertical direction; ②all sources and sinks are of degree one; and ③ all sources and sinks are placed on the horizontal boundaries of the plane box.

Fig. 1 shows an example box.

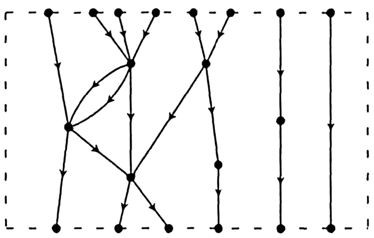

Fig. 1 A processive plane graph

processive plane graphs can also be defined in terms of processive graphs[2] and their boxed planar drawings.

Definition 1.2 A processive graph is an acyclic directed graph with all its sinks and sources of degree one.

A planar drawing of processive graph G is called boxed[1] if G is drawn in a plane box with all sinks of G on one horizontal boundary of the plane box and all sources of G on the other horizontal boundary of the plane box. A planar drawing of an acyclic directed graph is called upward if all edges increases monotonically in the vertical direction (or other fixed direction). Thus a processive plane graph is exactly a boxed and upward planar drawing of a processive graph.

Definition 1.3 Two processive plane graphs are equivalent if they are connected by a planar isotopy such that each intermediate planar drawing is boxed (not necessarily upward).

Equivalence classes of processive plane graphs are mainly used to construct free strict tensor categories in Ref. [1].

2 Planar order and POP-graph

Ref. [2] gave a combinatorial characterization of an equivalence classes of a processive plane graph in terms of a planar order on its underlying processive graph. In this paper, we define planar order for any poset.

Definition 2.1 A planar order on a poset $(X, \rightarrow)$ is a linear order $\prec$ on X, such that

(P_1) for any $x_1, x_2 \in X$, $x_1 \rightarrow x_2$ implies $x_1 \prec x_2$;

(P_2) for any $x_1, x_2, x_3 \in X$, $x_1 \prec x_2 \prec x_3$ and $x_1 \rightarrow x_3$ imply that either $x_1 \rightarrow x_2$ or $x_2 \rightarrow x_3$.

(P_1) says that $\prec$ is a linear extension of $\rightarrow$.

Recall that two partial orders on a set are conjugate if each pair of elements are comparable by exactly one of them. It is easy to see that (P_2) is equivalent to the condition that if $e_1 \prec e_2 \prec e_3$, then $e_1 \not\rightarrow e_2$ and $e_2 \not\rightarrow e_3$ imply that $e_1 \not\rightarrow e_3$. Thus (P_2) enables us to define a transitive binary relation: $e_1 < e_2$ if and only if $e_1 \prec e_2$ and $e_1 \not\rightarrow e_2$; moreover, if (P_1) is satisfied, then the linearity of $\prec$ implies that $<$ is a conjugate order of $\rightarrow$. So the planar order $\prec$ is a reformulation of the conjugate order of $\rightarrow$.

In a directed graph, we denote $e_1 \rightarrow e_2$ if there is a directed path starting from edge e_1 and ending with edge e_2. Similarly, $v_1 \rightarrow v_2$ denotes that there is a directed path starting from vertex v_1 and ending with vertex v_2. For any acyclic directed graph, its edge set and vertex set are posets with the relation $e_1 \rightarrow e_2$ and $v_1 \rightarrow v_2$. We call them edge poset and vertex poset of the acyclic directed graph, respectively.

The following is a key notion in Ref. [2].

Definition 2.2 A planarly ordered processive graph or POP-graph[2], is a processive graph G together with a planar order $\prec$ on its edge poset $(E(G), \rightarrow)$.

We simply denote a POP-graph as $(G, \prec)$, see Fig. 2 for an example.

A basic result is the following.

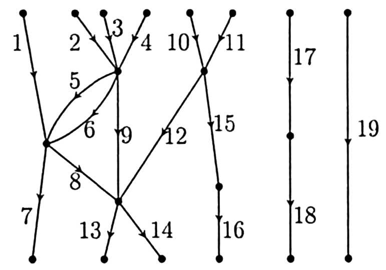

Fig. 2 A POP-graph

Theorem 2. 1[2] There is a bijection between POP-graphs and equivalence classes of processive plane graphs.

The POP-graph in Fig. 2 corresponds to the processive plane graph in Fig. 1.

3 Planar order on verties

In this section, we will prove our main result. Before that we need some preliminaries.

From now on, we fix be a POP-graph $(G,\prec)$. For a vertex v of $(G,\prec)$, the set $I(v)$ of incoming edges and the set $O(v)$ of outgoing edges are linearly ordered by $\prec$. We introduce some notations when $I(v)$ or $O(v)$ are not empty:

$$i^-(v)=\min I(v),$$
$$i^+(v)=\max I(v),$$
$$o^-(v)=\min O(v),$$
$$o^+(v)=\max O(v).$$

The following lemma is a result first proved in Ref. [4].

Lemma 3. 1 Let v be a vertex of $(G,\prec)$. If the degree of v is not one, then $o^-(v)=i^+(v)+1$ under the linear order $\prec$.

Proof Notice that G is a processive graph, then $\deg(v)\neq 1$ implies that $I(v)\neq\emptyset$ and $O(v)\neq\emptyset$. Thus both $i^+(v)$ and $o^-(v)$ exist. Now we prove $o^-(v)=i^+(v)+1$ by contradiction. Suppose there exists an edge e, such that $i^+(v)\prec e\prec o^-(v)$. Since $i^+(v)\to o^-(v)$, then by (P_2) we have $i^+(v)\to e$ or $e\to o^-(v)$. If $i^+(v)\to e$, then there must exists an edge $e'\in O(v)-\{o^-(v)\}$, such that $e'\to e$ or $e'=e$. Thus $e'\preceq e$, which contradicts with $e\prec o^-(v)$. Otherwise, $e\to o^-(v)$, then there must exist an edge $e''\in I(v)-\{i^+(v)\}$ such that $e\to e''$ or $e''=e$. Then $e\preceq e''$, which contradicts with $i^+(v)\prec e$.

Lemma 3. 1 shows that for any vertex v, $\overline{E(v)}=\overline{I(v)}\sqcup\overline{O(v)}$, where $E(v)$ is the set of incident edges of v and $\overline{X}$ denotes the interval of subset X in a poset. Due to Lemma 3. 1, we can define a linear order $\prec_V$ on the vertex set $V(G)$. For any two different vertices v_1, v_2 of G, $v_1\prec_V v_2$ if and only if one of the following conditions is satisfied:

① $I^+(v_1)\prec I^+(v_2)$, ② $I^+(v_1)\prec O^-(v_2)$,
③ $O^-(v_1)\preceq I^+(v_2)$, ④ $O^-(v_1)\prec O^-(v_2)$.

We write $v_1\preceq_V v_2$ if $v_1=v_2$ or $v_1\prec_V v_2$. The following theorem is our main result.

Theorem 3. 1 For any POP-graph $(G,\prec)$, $\preceq_V$ defines a planar order on the vertex poset $(V(G),\to)$.

Proof ① $\prec_V$ satisfies (P_1). If $v_1\to v_2$, then there exist $e_i\in E(G)$ $(1\leqslant i\leqslant n)$ such that $v_1=s(e_1)$, $v_2=t(e_n)$ and $t(e_i)=s(e_{i+1})$ for $(1\leqslant i\leqslant n-1)$, which implies that $o^-(v)\preceq e_1\preceq e_n\preceq i^+(v_2)$. Thus $o^-(v_1)\preceq i^+(v_2)$, then by definition of $\prec_V$, we have $v_1\prec_V v_2$.

② $\prec_V$ satisfies (P_2). Suppose $v_1\prec_V v_2\prec_V v_3$ and $v_1\to v_3$, then $o^-(v_1)$ and $i^+(v_3)$ exist and $o^-(v_1)\prec i^+(v_3)$. We have four cases:

Case 1 v_1 is a source and v_3 is a sink. In this case, by Definition 1. 2, $\{o^-(v_1)\}=O(v_1)$ and $\{i^+(v_3)\}=I(v_3)$. So $v_1\to v_3$ implies that $o^-(v_1)\to i^+(v_3)$. Let $e=i^+(v_2)$ or $o^-(v_2)$, then $v_1\prec_V v_2\prec_V v_3$ implies that $o^-(v_1)\prec e\prec i^+(v_3)$ or $o^-(v_1)=e$ or $e=i^+(v_3)$. In the first case, by (P_2), we have $o^-(v_1)\to e$ or $e\to i^+(v_3)$, which implies that $v_1\to v_2$ or $v_2\to v_3$. In the second case, we have $v_1\to v_2$, and in the third case, we have $v_2\to v_3$.

Case 2 v_1 is not a source and v_3 is a sink. In this case, $i^+(v_1)$ exists and by Definition 1. 2 $\{i^+(v_3)\}=I(v_3)$. So $v_1\to v_3$ implies that $i^+(v_1)\to i^+(v_3)$. Let $e=i^+(v_2)$ or $o^-(v_2)$, then $v_1\prec_V v_2\prec_V v_3$ implies that $i^+(v_1)\prec e\prec i^+(v_3)$ or $e=i^+(v_3)$. In the first case, by (P_2), we have $i^+(v_1)\to e$ or $e\to i^+(v_3)$, which implies that $v_1\to$

v_2 or $v_2 \rightarrow v_3$. In the second case, we have $v_2 \rightarrow v_3$.

Case 3 v_1 is a source and v_3 is not a sink. This case is similar to Case 2.

Case 4 v_1 is not a source and v_3 is not a sink. In this case, both $i^+(v_1)$ and $o^-(v_2)$ exist and $v_1 \rightarrow v_3$ implies that $i^+(v_1) \rightarrow o^-(v_3)$. Let $e = i^+(v_2)$ or $o^-(v_2)$, then $v_1 \prec_V v_2 \prec_V v_3$ implies that $i^+(v_1) \prec e \prec i^+(v_3)$. By ($P_2$), we have $i^+(v_1) \rightarrow e$ or $e \rightarrow o^-(v_3)$, which implies that $v_1 \rightarrow v_2$ or $v_2 \rightarrow v_3$.

Fig. 3 shows the planar order on the vertex poset of the POP-graph in Fig. 2.

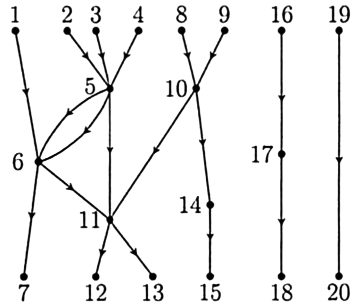

Fig. 3 Induced planar order on vertices

Theorem 3. 1 shows that for any processive graph, each conjugate order of its edge poset induces a conjugate order of its vertex poset. However, in general, the converse is not true. Therefore, together with Theorem 2. 1, Theorem 3. 1 demonstrates that edge poset is more effective tool than vertex poset in the study of upward planarity. It is worth to mention that Fraysseix and Mendez, in a different but essentially equivalent context, also showed a similar judgment in their final remark of Ref. [3]. In our subsequent work, we will show that for a transitive reduced processive graph, a planar order on its vertex set can naturally induce a planar order on its edge set, which is essentially related the work in Ref. [3].

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