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Poisson bracket method for obtaining normal coordinates of quadratic Hamiltonian

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Abstract: It was found that the problem of searching for normal coordinates W of quadratic Hamiltonian H can be ascribed to solving the newly established secular equation with two consecutive Poisson bracket operations. Solving W would simultaneously lead to the normal frequency. Some examples about quadratic Hamiltonian were presented to demonstrate the effectiveness of the presented method.

Key words: quadratic Hamiltonian; Poisson bracket operations; normal coordinates; normal frequency

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求二次型哈密顿量简正坐标的泊松括号方法

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摘要:研究发现寻求二次型哈密顿量 H 的简正坐标 W 可以归结于求解由两个相继的泊松括号运算组成的久期方程,从此方程解出 W 就能同时给出简正频率.通过几个二次型哈密顿量的例子说明了此方法的优点.

关键词:二次型哈密顿函数;泊松括号运算;简正坐标;简正频率

0 Introduction

In classical analytical mechanics the Poisson bracket is used to express Hamilton equation and Poisson theorem in a concise way[1-2]. In this paper we present a new application of Poisson bracket, e.g., we shall search for normal coordinates of quadratic Hamiltonian by establishing a secular equation with two consecutive Poisson bracket (PB) operations. Normal coordinates are such that make a Hamiltonian in separable form of independent oscillators.

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Generally, there are two methods for finding normal coordinates, one is by changing the coordinate scales and diagonalizing the new kinetic energy matrix[3-4], the other is by changing the coordinate scales and using arbitrary row of adjoint matrix of the frequency eigen matrix[5-6]. They are quite different from our new method to be presented in this paper. Our new idea is to establish the secular equation for normal coordinates with two consecutive Poisson bracket (PB) operations. Then solving this equation will also lead to normal frequencies. Some complicated examples of coupled oscillators are presented to show our new method's merit. This is a new approach to obtaining normal coordinates of classical dynamic systems, and therefore enriches the classical Lagranian-Hamiltonian theory.

1 The secular equation for normal coordinates

The secular equation we propose involves two Poisson bracket operations in uninterrupted succession

$$\{H,\{H,W\}\}=\lambda W \tag{1}$$

where H is a quadratic Hamiltonian, W is a classical dynamic variable we are searching for, it is not explicitly time dependent, and looks as if it is an "eigenvalue" of the two consecutive Poisson bracket operations.

We want to manifest that once W is found, it denotes normal coordinate which is capable of diagonalizing the Hamiltonian H.

According to time evolution of W

$$\frac{\mathrm{d}W}{\mathrm{d}t}=\sum_i\left(\frac{\partial W}{p_i}\dot{p}_i+\frac{\partial W}{q_i}\dot{q}_i\right) \tag{2}$$

and using the Hamilton canonical equation

$$\dot{q}_i=\frac{\partial H}{\partial p_i},\dot{p}_i=-\frac{\partial H}{\partial q_i} \tag{3}$$

as well as the definition of Poisson bracket

$$\{f,g\}=\sum_i\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right) \tag{4}$$

we can reform (2) as a Poisson bracket

$$\frac{\mathrm{d}W}{\mathrm{d}t}=\sum_i\left(\frac{\partial W}{\partial q_i}\frac{\partial H}{\partial p_i}-\frac{\partial W}{\partial p_i}\frac{\partial H}{\partial q_i}\right)=\{W,H\} \tag{5}$$

Differentiating this equation with time again we obtain

$$\frac{\mathrm{d}^2W}{\mathrm{d}t^2}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}W}{\mathrm{d}t}=\sum_i\left(\frac{\partial \dot{W}}{\partial q_i}\frac{\partial H}{\partial p_i}-\frac{\partial \dot{W}}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \tag{6}$$

and still using (5) we see

$$\frac{\mathrm{d}^2W}{\mathrm{d}t^2}=\sum_i\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}\{W,H\}-\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}\{W,H\}=\{\{W,H\},H\}=\{H,\{H,W\}\} \tag{7}$$

which involves two consecutive Poisson bracket operations. If we can find some W satisfying

$$\{H,\{H,W\}\}=\{\{W,H\},H\}=\lambda W \tag{8}$$

where λ is positive. Eq. (1) becomes

$$\frac{\mathrm{d}^2W}{\mathrm{d}t^2}=\lambda W \tag{9}$$

where W is qualified to be normal coordinates for H. To explain this, let us write down the Lagrangian for a quadratic physical system

$$\mathscr{L}=\frac{1}{2}\left[\sum_{i=1}^{l}m_i\dot{x}_i^2-\sum_{i,j=1}^{l}K_{ij}x_ix_j\right] \tag{10}$$

in terms of normal coordinates Q_i it just exhibits the form of l-independent oscillators[7]

$$\mathscr{L}=\frac{1}{2}\left[\sum_{i=1}^{l}\dot{Q}_i^2-\sum_{i=1}^{l}\omega_i^2Q_i^2\right] \tag{11}$$

By using the Lagrangian equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathscr{L}}{\partial\dot{Q}_i}-\frac{\partial\mathscr{L}}{\partial Q_i}=0 \tag{12}$$

we know that Q_i obeys the Newton equation

$$\frac{\mathrm{d}^2Q_i}{\mathrm{d}t^2}=\omega_i^2Q_i \tag{13}$$

Comparing Eq.(9) with Eq.(13), one can see that W really represents normal coordinates for H, and λ in Eq. (9) just corresponds to ω_i^2 in Eq.(13). Therefore, solving the secular equation (1) can lead to normal coordinates. This is a new approach to obtaining normal coordinates of classical dynamic systems.

2 Poisson bracket method for obtaining normal coordinates of three coupled oscillators

The Hamiltonian of three coupled oscillators is[8-9]

$$H=\frac{p_1^2}{2m}+\frac{p_2^2}{2M}+\frac{p_3^2}{2m}+$$

$$\frac{1}{2}k(x_2-x_1)^2+\frac{1}{2}k(x_2-x_3)^2 \quad (14)$$

where k is the spring constant. The mass in the middle is M, the other two masses are m. The fundamental Poisson brackets are

$$\left.\begin{aligned}\{x_1,H\}&=\frac{\partial H}{\partial p_1}=\frac{p_1}{m},\\ \{x_2,H\}&=\frac{\partial H}{\partial p_2}=\frac{p_2}{M},\ \{x_3,H\}=\frac{p_3}{m}\end{aligned}\right\} \quad (15)$$

And

$$\{p_1,H\}=-\frac{\partial H}{\partial x_1}=k(x_2-x_1) \quad (16)$$

$$\{p_2,H\}=-\frac{\partial H}{\partial x_2}=-k(x_2-x_1)+k(x_3-x_2) \quad (17)$$

$$\{p_3,H\}=-\frac{\partial H}{\partial x_3}=k(x_2-x_3) \quad (18)$$

Assuming the normal coordinate W for this H takes the form

$$W=x_1+fx_2+gx_3 \quad (19)$$

where f,g are to be determined. According to Eqs.(12) and (13), substituting (14) into the secular equations (1) and (8), we have

$$\begin{aligned}\lambda W&=\{\{W,H\},H\}=\\ &\{\frac{p_1}{m},H\}+f\left\{\frac{p_2}{m},H\right\}+g\left\{\frac{p_3}{m},H\right\}=\\ &\frac{1}{m}k(x_2-x_1)+\\ &f[-k(x_2-x_1)+(x_3-x_2)]+g=\\ &-k\left[\left(\frac{1}{m}-\frac{f}{M}\right)x_1+\left(\frac{-1}{m}+\frac{2f}{M}-\frac{g}{m}\right)x_2+\left(\frac{g}{m}-\frac{f}{M}\right)x_3\right]\end{aligned} \quad (20)$$

Comparing (20) with (19), we conclude that their corresponding coefficients must be proportional to each other, so

$$1:f:g=\left(\frac{1}{m}-\frac{f}{M}\right):\left(\frac{-1}{m}+\frac{2f}{M}-\frac{g}{m}\right):\left(\frac{g}{m}-\frac{f}{M}\right) \quad (21)$$

or equivalently

$$1:f:g=(M-fm):(-M+2fm-gM):(gM-fm) \quad (22)$$

from which we have two independent equations

$$\frac{M-fm}{gM-fm}=\frac{1}{g} \quad (23)$$

$$\frac{-M+2fm-gM}{gM-fm}=\frac{f}{g} \quad (24)$$

From Eq.(23) we see

$$g=1, \text{or } f=0 \quad (25)$$

When $f=0$, Eq.(24) reduces to $\frac{-M-gM}{gM}=0$, so $g=-1$; On the other hand, when $g=1$, Eq.(24) becomes to

$$-2M+2fm=f(M-fm) \quad (26)$$

in this case

$$f=-2, \text{or } f=\frac{M}{m} \quad (27)$$

In sum, we have three groups of solutions

$$f=0, g=-1 \quad (28)$$

$$f=\frac{M}{m}, g=1 \quad (29)$$

$$f=-2, g=1 \quad (30)$$

Substituting them into Eq.(19) respectively gives the normal coordinates

$$W=x_1-x_3 \quad (31)$$

$$W=x_1+\frac{M}{m}x_2+x_3 \quad (32)$$

$$W=x_1-2x_2+x_3 \quad (33)$$

On the other hand, respectively substituting the three solutions into (20) yields

$$\lambda W=-\frac{k}{m}(x_1-x_3) \quad (34)$$

$$\lambda W=0 \quad (35)$$

$$\lambda W=-k\left(\frac{1}{m}+\frac{2}{M}\right)(x_1-2x_2+x_3) \quad (36)$$

which indicates that the normal frequencies are

$$\omega_1=\sqrt{\frac{k}{m}}, \omega_2=0, \omega_3=\sqrt{\frac{k}{m}\left(1+\frac{2m}{M}\right)} \quad (37)$$

One may check our new method's correctness by other known methods.

3 Poisson bracket method for obtaining normal coordinates of linear bi-atomic chain with unequal masses

Now we search for normal coordinates of a linear chain composed of N-bi-atoms, each bi-atom containing two irons with unequal masses m and

m', whose Hamiltonian is[10]

$$H=\sum_{n=1}^{N}\Big[\frac{p_n^2}{2m}+\frac{p_n'^2}{2m'}+\frac{\beta}{2}(x_n-x_n')^2+\frac{\beta}{2}(x_n-x'_{n-1})^2+\frac{\beta}{2}\left(x_n'-x_{n+1}\right)^2\Big] \quad (38)$$

Supposing the normal coordinates take the form

$$W=\sum_{n=1}^{N}(f_n p_n+f_n' p_n') \quad (39)$$

where f_n , f_n' are to be determined, W should obey the secular equation (4). Using Eq.(38) we see

$$\{p_n,H\}=\left\{p_n,\frac{\beta}{2}(x_n-x_n')^2+\frac{\beta}{2}(x_n-x'_{n-1})^2\right\}=\beta(x_n'+x'_{n-1}-2x_n) \quad (40)$$

$$\{p_n',H\}=\beta(x_n+x_{n-1}-2x_n') \quad (41)$$

$$\{x_n,H\}=\left\{x_n,\frac{p_n^2}{2m}\right\}=\frac{p_n}{m},\{x_n',H\}=\frac{p_n'}{m'} \quad (42)$$

Using these we calculate

$$\{W,H\}=\sum_{n=1}^{N}\{(f_n p_n+f_n' p_n'),H\}=\beta\sum_{n=1}^{N}\Big[f_n(x_n'+x'_{n-1}-2x_n)\ p_n+f_n'\left(x_n+x_{n-1}-2x_n'\right)\Big]=\beta\sum_{n=1}^{N}\Big[\left(f_n'+f'_{n-1}-2f_n\right)x_n+(f_n+f_{n+1}-2f_n')x_n'\Big] \quad (43)$$

Further, we evaluate

$$\{\{W,H\},H\}=-\beta\sum_{n=1}^{N}\Big[\frac{1}{m}(f_n'+f'_{n-1}-2f_n)p_n+\frac{1}{m'}(f_n+f_{n+1}-2f_n')p_n'\Big] \quad (44)$$

Comparing the coefficients of (44) with those in (39) and using (8) we have

$$\lambda f_n=-\beta\frac{1}{m}(f_n'+f'_{n-1}-2f_n) \quad (45)$$

$$\lambda f_n'=-\beta\frac{1}{m'}(f_n+f_{n+1}-2f_n') \quad (46)$$

It then follows

$$\lambda=-\frac{\beta}{mf_n}(f_n'+f'_{n-1}-2f_n)=-\frac{\beta}{m'f_n'}(f_n+f_{n+1}-2f_n') \quad (47)$$

which is

$$\left.\begin{aligned}&\frac{1}{m}\left(\frac{f_n'+f'_{n-1}}{2f_n}-1\right)=\frac{1}{m'}\left(\frac{f_n+f_{n+1}}{2f_n'}-1\right),\\&n=1,2,\cdots,N\end{aligned}\right\} \quad (48)$$

and leads us to

$$f_n=\mu\cos 2n\theta_l,f_n'=\nu\cos(2n+1)\theta_l \quad (49)$$

where

$$\theta_l=\frac{\pi}{N},1,2,\cdots,2N \quad (50)$$

Substituting Eq.(49) into Eq.(47) leads to

$$\lambda=\frac{2\beta}{m}\left(1-\frac{\nu}{\mu}\cos\theta_l\right)=\frac{2\beta}{m'}\left(1-\frac{\mu}{\nu}\cos\theta_l\right) \quad (51)$$

and

$$\frac{\nu}{\mu}=\frac{2\beta\cos\theta_l}{2\beta-m'\lambda}=\frac{2\beta-m\lambda}{2\beta\cos\theta_l} \quad (52)$$

thus

$$m'\lambda^2-2\beta(m+m')\lambda+4\beta^2\sin^2\theta_l=0 \quad (53)$$

whose solution gives the normal frequency

$$\lambda=\beta\left(\frac{1}{m}+\frac{1}{m'}\right)\pm\beta\left[\left(\frac{1}{m}+\frac{1}{m'}\right)^2-\frac{4\sin^2\theta}{mm'}\right]=\omega^2 \quad (54)$$

and the normal coordinates for bi-atomic linear chain is

$$W=\sum_{n=1}^{N}[\mu p_n\cos 2n\theta_l+\nu p_n'\cos(2n+1)\theta_l] \quad (55)$$

4 Poisson bracket method for obtaining normal coordinates of an impeller model

In this section we employ the Poisson bracket method to find normal coordinates of an impeller model. A practical impeller is a rotating component of a centrifugal pump which transfers energy from the motor that drives the pump to the fluid being pumped by accelerating the fluid outwards from the center of rotation. Here we write down the Hamiltonian of an impeller with two types of vanes

$$H''=\sum_{n=1}^{N}\Big[\frac{p_{2n}^2}{2m_1}+\frac{p_{2n+1}^2}{2m_2}+\frac{\lambda}{2}(X_{2n}-X_{2n+1})^2+\frac{\lambda}{2}(X_{2n+1}-X_{2n+2})^2+\frac{1}{2}\eta_1X_{2n}^2+\frac{1}{2}\eta_2X_{2n+2}^2\Big] \quad (56)$$

where N is impeller vane number, λ is the spring coupling constant between two neighbouring vanes, η_i ($i=1,2$) is the coupling constant between two types of vanes with the rotating center, respectively. Using (56) we calculate the following Poisson brackets:

$$\{P_{2m}, H''\} = \mathrm{i}\{\lambda X_{2m+1} + \lambda X_{2m-1} - X_{2m}(2\lambda+\eta_1)\} \tag{57}$$

$$\{P_{2m+1}, H''\} = \{\lambda X_{2m} + \lambda X_{2m+2} - X_{2m+1}(2\lambda+\eta_2)\} \tag{58}$$

$$\{X_{2n}, H''\} = \frac{\mathrm{i}}{m_1}P_{2n}, \{X_{2n+1}, H\} = \frac{\mathrm{i}}{m_2}P_{2n+1} \tag{59}$$

Supposing the invariant eigenvector for H'' is

$$F = \sum_{m=1}^{N}(f_{2m}P_{2m} + f_{2m+1}P_{2m+1}) \tag{60}$$

then using Eqs.(57)～(59) we evaluate

$$\{F, H''\} = \sum_{m=1}^{N}\{\mathrm{i}X_{2m}[f_{2m+1}\lambda + f_{2m-1}\lambda - f_{2m}(2\lambda+\eta_1)] + \mathrm{i}X_{2m+1}[f_{2m}\lambda + f_{2m+2}\lambda - f_{2m+1}(2\lambda+\eta_2)]\} \tag{61}$$

It follows

$$\{\{F, H''\}, H''\} = \sum_{m=1}^{N}\left\{\frac{P_{2m}[f_{2m}(2\lambda+\eta) - \lambda(f_{2m+1}+f_{2m-1})]}{m_1} + \frac{P_{2m+1}[f_{2m+1}(2\lambda+\eta) - \lambda(f_{2m}+f_{2m+2})]}{m_2}\right\} = \omega^2\sum_{m=1}^{N}(f_{2m}P_{2m} + f_{2m+1}P_{2m+1}) \tag{62}$$

Comparing the two sides of Eq. (62) we have

$$f_{2m}(2\lambda+\eta_1) - \lambda(f_{2m+1}+f_{2m-1}) = f_{2m}m_1\omega^2 f_{2m+1}(2\lambda+\eta_2) - \lambda(f_{2m}+f_{2m+2}) = f_{2m+1}m_2\omega^2 \tag{63}$$

The condition that the solution of ω^2 exists is

$$\frac{1}{m_1}\left\{(2\lambda+\eta_1) - \frac{\lambda(f_{2m+1}+f_{2m-1})}{f_{2m}}\right\} = \frac{1}{m_2}\left\{(2\lambda+\eta_2) - \frac{\lambda(f_{2m}+f_{2m+2})}{f_{2m+1}}\right\} \tag{64}$$

Let

$$f_{2m} = \xi\cos 2m\theta, f_{2m+1} = \xi'\cos(2m+1)\theta \tag{65}$$

and substitute Eq.(65) into Eq.(63) we have

$$\left.\begin{aligned}\frac{1}{m_1}\left\{(2\lambda+\eta_1) - \frac{2\lambda\xi'\cos\theta}{\xi}\right\} = \omega^2,\\ \frac{1}{m_2}\left\{(2\lambda+\eta_2) - \frac{2\lambda\xi\cos\theta}{\xi'}\right\} = \omega^2\end{aligned}\right\} \tag{66}$$

Therefore

$$\frac{\xi'}{\xi} = \frac{(2\lambda+\eta_1) - m_1\omega^2}{2\lambda\cos\theta} = \frac{2\lambda\cos\theta}{(2\lambda+\eta_2) - m_2\omega^2} \tag{67}$$

It then follows

$$m_1m_2\omega^4 - \omega^2[(2\lambda+\eta_1)m_2 + (2\lambda+\eta_2)m_1] + (2\lambda+\eta_1)(2\lambda+\eta_2) - 4\lambda^2\cos^2\theta = 0 \tag{68}$$

with the solution

$$\omega_{\pm}^2 = \frac{1}{2}\left\{\frac{2\lambda+\eta_1}{m_1} + \frac{2\lambda+\eta_2}{m_2} \pm \left[\left(\frac{2\lambda+\eta_1}{m_1} - \frac{2\lambda+\eta_2}{m_2}\right)^2 + \frac{16\lambda^2\cos^2\theta}{m_1m_2}\right]^{\frac{1}{2}}\right\} \tag{69}$$

which is the normal coordinates of the impeller model.

5 Conclusion

In summary, in the context of classical mechanics we have proposed a new approach to finding normal coordinates of quadratic Hamiltonian by establishing the secular equation with two consecutive Poisson bracket operations. This new method develops the role of Poisson bracket in material mechanics and mechanical engineering, since it paves an effective route leading to normal frequency of dynamical systems. Moreover, we may combine this method with the technique of integration within ordered product of operators[11] as well as the invariant eigenvector method in quantum theory[12] to find diagonalizing unitary operators and quantum eigenstates. The analogy between classical Poisson brackets and quantum mechanical commutators has been demonstrated through this paper.

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