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On the number of edges not covered by monochromatic copies of a matching-critical graph

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Abstract: Given a graph H, let $f(n,H)$ denote the maximum number of edges not contained in any monochromatic copy of H in a 2-edge-coloring of K_n. The Turán number of a graph H, denoted by $\mathrm{ex}(n,H)$, is the maximum number of edges in an n-vertex graph which does not contain H as a subgraph. It is easy to see that $f(n,H)\geqslant \mathrm{ex}(n,H)$ for any H and n. We show that this lower bound is tight for matching-critical graphs including Pertersen graph and vertex-disjoint union of copies of cliques with same order.

Key words: edge coloring; monochromatic copy; matching critical graph
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未被某个匹配临界图的所有单色复制覆盖的边数

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摘要:给定一个图 H,另 $f(n,H)$为完全图 K_n 的所有二色边染色中,包含未被单色图 H 的复制覆盖的边数的最大值.图的图兰数,记作 $\mathrm{ex}(n,H)$,是指所有的 n 个顶点的图中,不含有 H 作为子图的图的所含有边数的最大值.显然对于任意的 n 和 H,$f(n,H)$大于等于 $\mathrm{ex}(n,H)$.我们证明了,对于匹配临界图,以上两个数值当 n 是充分大的时候是相等的.

关键词:边染色;单色复制;匹配临界图

0 Introduction

Notations in this paper are standard. For a graph G with subgraph H, we use G-$E(H)$ to denote the spanning subgraph on $V(G)$ with edge set $E(G)\setminus E(H)$ and G-$V(H)$ to denote the induced subgraph of G on vertex set $V(G)\setminus V(H)$. If G and H are two disjoint subgraphs, we use $G+H$ to denote the graph obtained from $G\cup H$ by adding all edges between every vertex of G and every vertex of H. As usual, denote the balanced complete p-partite graph on n vertices by $T_p(n)$ and its number of edges by $t_p(n)$. Let $H(n,p,k)=K_{k-1}+T_p(n-k+1)$ and $h(n,p,k)=e(H(n,p,$

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k)). The Turán number of a graph H, denoted by $ex(n,H)$, is the maximum number of edges in an $n$vertex graph which does not contain H as a subgraph. If an n-vertex graph G has $ex(n,H)$ edges and does not contain H as a subgraph, we call G an extremal graph for H.

For a given graph H, let $f(n,H)$ denote the maximum number of edges not contained in any monochromatic copy of H in a 2-edge-coloring of K_n. It is easy to see that $f(n,H)\geqslant ex(n,H)$ for any H and n. In 2004, Keevash and Sudakov showed in Ref. [1] that this lower bound is tight for a sufficiently large n if H is edgecritical or a cycle of length 4. Here, we call a graph H with chromatic number $\chi(H)=p+1$ edgecritical if there exists some edge e in H such that $\chi(H-e)=p$.

Theorem 0.1[1] If H is an edgecritical graph with chromatic number $p+1\geqslant 3$, then

$$f(n,H)=ex(n,H)=t_p(n)$$

for a suffiently large n.

Moreover, Keevash and Sudakov asked whether $f(n,H)=ex(n,H)$ for any H and any sufficiently large n, and they obtained a general upper bound that $f(n,H)\leqslant ex(n,H)+o(n^2)$. In 2017, Ma[2] confirmed their problem for an infinite family of bipartite graphs. Later, Liu, Pikhurko and Sharifzadeh[3] extended his result to a larger family of bipartite graphs and proved a better upper bound for all bipartite graphs.

Theorem 0.2[3] If H is bipartite, then

$$f(n,H)\leqslant ex(n,H)+O(1)$$

for a sufficiently large n.

Denote by M_k the graph consisting of k independent edges and call it a matching with size k. Call a graph H with chromatic number $\chi(H)=p+1$ k-matching-critical if there exists a matching M_k in H such that $\chi(H-E(M_k))=p$ and $\chi(H-X)=p+1$ for any $X\subseteq V(H)$ with $|X|=k-1$. There are many interesting k-matching-critical graphs including the Pertersen graph, see Ref. [4]. In 1974, Simonovits[5] determined $ex(n,H)$ for matching-critical H and characterized its extremal graph.

Theorem 0.3[5] If H is a k-matching-critical graph with chromatic number $\chi(H)=p+1\geqslant 3$, then

$$ex(n,H)=h(n,p,k).$$

The unique extremal graph is $H(n,p,k)$.

In this paper, we will confirm Keevash and Sudakov's problem for matchingcritical H and sufficiently large n. In fact, we will prove the following theorem.

Theorem 0.4 If H is a k-matching-critical graph with chromatic number $\chi(H)=p+1\geqslant 3$, then

$$f(n,H)=h(n,p,k)$$

for a sufficiently large n.

1 Preliminaries

By the classical Turán's theorem, if an $n$vertex graph G does not contain K_p as a subgraph, then $e(G)\leqslant t_{p-1}(n)$, with the equality holds if and only if $G=T_{p-1}(n)$. Erdös and Stone[6] further proved that if an n-vertex graph G contains a little more than $t_{p-1}(n)$ edges, then G contains a copy of large complete p-partite graph.

Theorem 1.1[6] Let $p\geqslant 2$ and $t\geqslant 1$ be given integers. Then for any $\epsilon>0$, there exists an integer $n_0=n_0(p,t,\epsilon)$ such that: If a graph G on $n\geqslant n_0$ vertices contains more than $t_{p-1}(n)+\epsilon n^2$ edges, then it contains $T_p(tp)$ as a subgraph.

In 1968, Simonovits[7] introduced the socalled progressive induction which is similar to the mathematical induction and Euclidean algorithm and combined from them in a certain sense. It will be our main tool in the proof of Theorem 0.4.

Lemma 1.1[7] Let $\mathscr{U}=\bigcup_1^\infty \mathscr{U}_n$ be a set of given elements, such that $\mathscr{U}_n$ are disjoint subsets of $\mathscr{U}$. Let B be a condition or property defined on $\mathscr{U}$ (i. e. the elements of $\mathscr{U}_n$ may satisfy or not satisfy B). Let $\Delta(n)$ be a function defined also on $\mathscr{U}$ such that $\Delta(n)$ is a nonnegative integer and

(a) if a satisfies B, then $\Delta(a)$ vanishes.

(b) there is an M_0 such that if $n>M_0$ and $a\in\mathscr{U}_n$ then either a satisfies B or there exist an n' and an a' such that

$\frac{n}{2} < n' < n$, $a' \in \mathscr{U}_{n'}$ and $\Delta(a) < \Delta(a')$.

Then there exists an n_0 such that if $n > n_0$, from $a \in \mathscr{U}_n$ follows that a satisfies B.

From now on in this paper, we associate every graph we consider with a red/blue-edge-coloring. For any two vertices u and v, call u a red (blue) neighbor of v if the edge uv is colored red (blue). If an edge e is not contained in any monochromatic copy of a given graph H, then we call e NIM-H. If G consists of NIM-H edges, then we call G NIM-H. For $p \geqslant 2$ and disjoint vertex sets $V_1, \cdots, V_p \subseteq V(G)$, let $G[V_1]$ denote the induced subgraph of G on V_1 and $G[V_1, \cdots, V_p]$ denote the p-partite induced subgraph of G with parts $V_1, \cdots, V_p$. If G is a complete graph, then we use $K[V_1]$ and $(V_1, \cdots, V_p)$ instead of $G[V_1]$ and $G[V_1, \cdots, V_p]$.

The following lemma is the same as Lemma 3.2 in Ref. [1]. We include the proof for completeness.

Lemma 1.2 Let $p \geqslant 2$ and $t \geqslant 1$ be given integers. Let H be a given graph. Then there exists an integer $n_0 = n_0(p, t, H)$ such that for any $n > n_0$: If G is an NIM-H graph on n vertices containing at least $t_p(n)$ edges, then G contains a monochromatic copy of $T_p(tp)$.

Proof Let $|V(H)| = h$ and $t' = 2t(p!\ 4^{h(p-1)})$. Let $n \geqslant n_0 := n_0(p, t', 1/p(p-1))$ be sufficiently large, where $n_0(p, t', 1/p(p-1))$ is obtained from Theorem 1.1. Since

$$e(G) \geqslant t_p(n) = \left(\frac{p-1}{p} + o(1)\right)\binom{n}{2} \geqslant \left(\frac{p-2}{p-1} + \frac{1}{p(p-1)} + o(1)\right)\binom{n}{2},$$

G contains $T_p(t'p)$ as a subgraph by Theorem 1.1. Let $T_p(t'p) = (V_1, \cdots, V_p) \subseteq G$. Note that every edge between V_i and V_j is NIM-H for $1 \leqslant i \neq j \leqslant p$.

Choose a vertex $v_1 \in V_1$ arbitrarily. Assume without loss of generality that at least half of the edges between v_1 and V_p are red. Denote the red neighborhood of v_1 in V_p by R_p, then every vertex $v \in V_1 \cup \cdots \cup V_{p-1}$ has at most 4^h blue neighbors in R_p. Otherwise, let R' denote the blue neighborhood of $v \in V_1 \cup \cdots \cup V_{p-1}$ in R_p. By Ramsey's Theorem, $K[R']$ contains a monochromatic copy of K_h. Therefore, either $H \cup \{v\}$ forms a blue copy of K_{h+1} or $H \cup \{v_1\}$ forms a red copy of K_{r+1}, which implies that there exists a monochromatic copy of H using NIM-H edges, a contradiction. Now, choose $V_i^{(1)} \subseteq V_i$ of size $t(p-1)!\ 4^{h(p-2)}$ arbitrarily for every $1 \leqslant i \leqslant p-1$ and let $W_p = R_p \setminus B_p$, where B_p denotes the set of all blue neighbors of every vertex in $V_1^{(1)} \cup \cdots \cup V_{p-1}^{(1)}$. According to the previous analysis, we have

$$|W_p| = |R_p| - |B_p| \geqslant t'/2 - \left|\bigcup_{i=1}^{p-1} V_i^{(1)}\right| 4^h = t \cdot p!\ 4^{h(p-1)} - t(p-1) \cdot (p-1)!\ 4^{h(p-1)} = t(p-1)!\ 4^{h(p-1)} > t.$$

Note that $(V_1^{(1)}, \cdots, V_{p-1}^{(1)}, W_p) \subseteq G$ is an NIM-H graph and all edges between $V_1^{(1)} \cup \cdots \cup V_{p-1}^{(1)}$ and W_p are red. So by a similar argument as before, we have that every vertex in $V_i^{(1)}$ has at most 4^h blue neighbors in $V_j^{(1)}$ for any $1 \leqslant i \neq j \leqslant p-1$. Next, for $2 \leqslant j \leqslant p-1$, we define $(V_1^{(j)} \cup \cdots \cup V_{p-j}^{(j)} \cup W_{p-j+1} \cup \cdots \cup W_p) \subseteq G$ recursively as follows. Choose $V_i^{(j)} \subseteq V_i^{(j-1)}$ of size $t(p-j)! \cdot 4^{h(p-j-1)}$ arbitrarily for every $1 \leqslant i \leqslant p-j$ and let $W_{p-j+1} = V_{p-j+1}^{(j-1)} \setminus B_{p-j+1}$, where B_{p-j+1} denotes the set of all blue neighbors of every vertex in $V_1^{(j)} \cup \cdots \cup V_{p-j}^{(j)}$. Then we have

$$|W_{p-j+1}| = |V_{p-j+1}^{(j-1)}| - |B_{p-j+1}| = t \cdot (p-j+1)!\ 4^{h(p-j)} - t(p-j) \cdot (p-j)!\ 4^{h(p-j)} = t(p-j)!\ 4^{h(p-j)} \geqslant t \cdot 4^h > t$$

and all edges between $V_1^{(j)} \cup \cdots \cup V_{p-j}^{(j)}$ and W_{p-j+1} are red. At last, we get a p-partite graph $(V_1^{(p-1)}, W_2, \cdots, W_p) \subseteq G$ consisting of red edges. Note that $|V_1^{(p-1)}| = t$, so G contains a red copy of $T_p(tp)$. We are done.

2 Proof of Theorem 0.4

Let H be a $k$matchingcritical graph with chromatic number $\chi(H) = p + 1 \geqslant 3$. Since there exist $k-1$ vertices of $H(n, p, k)$ such that after deleting them the chromatic number of the obtained graph is p, it follows from the definition of k-matching-critical graphs that $\mathrm{ex}(n,H)\geqslant h(n,p,k)$. Let $|V(H)|=h$. Let n be a sufficiently large integer. Let G_n be the spanning subgraph of K_n consisting of NIM-H edges with $e(G_n)=f(n,H)$. If $e(G_n)<h(n,p,k)$, then we get a contradiction to the fact $f(n,H)\geqslant \mathrm{ex}(n,H)$. So we may assume that $e(G_n)\geqslant h(n,p,k)$.

Use progressive induction now. Let $\mathscr{U}_n$ be the set of NIM-H graphs G_n with $f(n,H)$ edges. Let property B be $e(G_n)\leqslant h(n,p,k)$. Let $\Delta(n)=e(G_n)-h(n,p,k)$. Then by our assumption, $\Delta(n)$ is a nonnegative integer. So by Lemma 1.1, we only need to find some n' such that $n/2<n'<n$ and $\Delta(n)<\Delta(n')$.

As $e(G_n)\geqslant h(n,p,k)\geqslant t_p(n)$, by Lemma 1.2, G_n contains a monochromatic copy of $T_p(n_1p)$ with n_1 being sufficiently large. Let $R=(R_1,\cdots,R_p)$ be a red copy of $T_p(n_1p)$ in G_n, then for any $1\leqslant i\leqslant p$, the maximum size of a red matching in $K[R_i]$ is at most $k-1$. Otherwise, let $\{e_1,\cdots,e_k\}$ be a red matching in $K[R_1]$, then these edges and $h-2k$ arbitrary other vertices in R_1, together with h vertices in every other $R_i(i\neq 1)$ will form a graph containing a red copy of H using NIM-H edges, a contradiction. So we can find a red copy of $T_p(n_2p)$ with $n_2\geqslant n_1-2k$ in R, say $T_0=(V_1^{(0)},\cdots,V_p^{(0)})$, such that $K[V_i^{(0)}]$ is a blue clique for every $1\leqslant i\leqslant p$. Therefore, all edges in $K[V_i^{(0)}]$ are not NIM-H for $1\leqslant i\leqslant p$.

Let $\epsilon\in\mathbb{R}^+$ be sufficiently small. We define a set of vertices $X=\{x_1,\cdots,x_l\}$ recursively as follows. If there exists some vertex $x_1\in V(G)\setminus V(T_0)$ such that x_1 has at least ϵ^2n_2 red neighbors in each part of T_0, then there exists a copy of $T_p(\epsilon^2n_2p)\subseteq T_0$, denoted by T_1, such that x_1 is joint to all vertices of T_1 by red edges; For $i\geqslant 2$, if there exists some vertex $x_i\in V(G_n)\setminus V(T_{i-1})$ such that x_i has at least $\epsilon^{2i}n_2$ red neighbors in each part of T_{i-1}, then there exists a copy of $T_p(\epsilon^{2i}n_2p)\subseteq T_{i-1}$, denoted by T_i, such that x_i is joint to all vertices of T_i by red edges. Since $T_i\subseteq T_{i-1}\cdots\subseteq T_1$, every vertex of $\{x_1,\cdots,x_i\}$ is joint to every vertex of T_i by a red edge. We claim that this process stops within $k-1$ steps. Otherwise, there exists a copy of $T_p(\epsilon^{2k}n_2p)$, denoted by $T_k=(V_1^{(k)},\cdots,V_p^{(k)})$, such that every vertex of $\{x_1,\cdots,x_k\}$ is joint to every vertex of T_k by a red edge. Then $\{x_1,\cdots,x_k\}$ and $h-k$ arbitrary other vertices in $V_1^{(k)}$, together with h vertices in every other $V_i^{(k)}(i\neq 1)$ will form a graph containing a red copy of H using NIM-H edges, a contradiction. Therefore, $0\leqslant l\leqslant k-1$. Let $T_l=(V_1^{(l)},\cdots,V_p^{(l)})$ and $W=V(G_n)\setminus(V(T_l)\cup X)$, then for any vertex $w\in W$, there exists some $1\leqslant i\leqslant p$ such that w has less than $\epsilon^{2l+2}n_2$ red neighbors in $V_i^{(l)}$. So we can get a partition of $W=C_1'\cup\cdots\cup C_p'\cup D$ as follows. For any $1\leqslant i\leqslant p$, if $w\in W$ has less than $\epsilon^{2l+2}n_2$ red neighbors in $V_i^{(l)}$ and at least $(1-\epsilon)\epsilon^{2l}n_2$ red neighbors in every $V_j^{(l)}$ with $j\neq i$, let $w\in C_i'$; Otherwise, i.e., there exist two indices $1\leqslant i\neq j\leqslant p$ such that $w\in W$ has less than $\epsilon^{2l+2}n_2$ red neighbors in $V_i^{(l)}$ and less than $(1-\epsilon)\epsilon^{2l}n_2$ red neighbors in $V_j^{(l)}$, let $w\in D$.

Claim 2.1 For any $1\leqslant i\leqslant p$, there exists $V_i'\subseteq V_i^{(l)}$ with $|V_i'|>(1-\epsilon^2k)|V_i^{(l)}|$ such that all edges between V_i' and C_i' are blue.

Proof Assume $i=1$ without loss of generality. Choose a maximal matching $\{x_1y_1,\cdots,x_ty_t\}$ between $V_1^{(l)}$ and C_1', where $x_s\in V_1^{(l)}$ and $y_s\in C_1'$ for $1\leqslant s\leqslant t$. Note that $t\leqslant k-1$. Otherwise, by the definition of C_1', y_s has at least $(1-\epsilon)\epsilon^{2l}n_2$ red neighbors in $V_j^{(l)}$ for any $1\leqslant s\leqslant k$ and $j\geqslant 2$. So $y_1,\cdots,y_k$ have at least $(1-k\epsilon)\epsilon^{2l}n_2$ common red neighbors in $V_j^{(l)}$ for any $j\geqslant 2$. Since n_2 is sufficiently large, we can choose ϵ sufficiently small such that $(1-k\epsilon)\epsilon^{2l}n_2>h$. For $j\geqslant 2$, let $N_j\subseteq V_j^{(l)}$ be a subset of the common red neighborhood of $y_1,\cdots,y_k$ with $|N_j|=h$. Then $\{x_1y_1,\cdots,x_ky_k\}$ and $h-2k$ arbitrary other vertices in $V_1^{(l)}$, together with $N_2,\cdots,N_p$ will form a graph containing a red copy of H using NIM-H edges, a contradiction. Let N be the set of all red neighbors of $y_1,\cdots,y_t$ in $V_1^{(l)}$. By the definition of C_1', we have $|N|<t\cdot\epsilon^{2l+2}n_2<k\cdot\epsilon^{2l+2}n_2=$

$k\epsilon^2|V_1^{(\ell)}|$. By the maximality of $\{x_1y_1,\cdots,x_ty_t\}$, all edges between $V_1^{(\ell)}\setminus N$ and $C_1'\setminus\{x_1,\cdots,x_t\}$ are blue. Moreover, by the definition of N, all edges between $V_1^{(\ell)}\setminus N$ and C'_1 are blue. Note that $|V_1^{(\ell)}\setminus N|>(1-k\epsilon^2)|V_1^{(\ell)}|$, we are done.

Since $V_i'\subseteq V_i^{(\ell)}\subseteq V_i^{(0)}$, $K[V_i']$ is a blue clique for any $1\leqslant i\leqslant p$. So by Claim 2.1, for every $1\leqslant i\leqslant p$, we can move $k\epsilon^2|V_i^{(\ell)}|$ vertices from $V_i^{(\ell)}$ to C_i' to get $V_i\subseteq V_i'\subseteq V_i^{(\ell)}$ and $C_i\supseteq C_i'$ such that all edges between V_i and C_i are blue. Since ϵ is sufficiently small, we have $(1-k\epsilon^2)\epsilon^{2\ell}n_2\geqslant(1-k\epsilon)$. $\epsilon^{2\ell}n_2>h$. So every edge between V_i and C_i is contained in a blue K_h, which implies that all edges between V_i and C_i are not NIM$-H$. Let $c=(1-k\epsilon^2)\epsilon^{2\ell}n_2$ and $T=(V_1,\cdots,V_p)$, then $T\subseteq T_\ell$ is a red copy of $T_p(cp)$ consisting of NIM-H edges. For any $1\leqslant i\leqslant p$, $K[V_i]$ is a blue clique with $|V_i|>h$, thus every edge in $K[V_i]$ is not NIM-H. Let $W'=V(G_n)\setminus(V(T)\cup X)$, then $C_1\cup\cdots\cup C_p\cup D$ is a partition of W'. Recall the definition of D, then for any $w\in D$, there exist two indices $1\leqslant i(w)\neq j(w)\leqslant p$ such that w has more than $c-\epsilon^{2\ell+2}n_2=(1-k\epsilon^2-\epsilon^2)\epsilon^{2\ell}n_2$ blue neighbors in $V_{i(w)}$ and more than $c-(1-\epsilon)\epsilon^{2\ell}n_2=(\epsilon-k\epsilon^2)\epsilon^{2\ell}n_2$ blue neighbors in $V_{j(w)}$. Since both of $K[V_{i(w)}]$ and $K[V_{j(w)}]$ are large blue cliques, every blue edge between w and $V_{i(w)}\cup V_{j(w)}$, is not NIM-H.

Combining all of the above, we know that every NIMH edge between $V(T)$ and $V(G_n)\setminus V(T)$ must belong to one of the following four sets.

(Ⅰ) The edges between X and $V(T)$, which are all red.

(Ⅱ) The edges between C_i and $\bigcup_{j\neq i}V_j$, where $1\leqslant i\leqslant p$.

(Ⅲ) The red edges between some $w\in D$ and $V_{i(w)}\cup V_{j(w)}$.

(Ⅳ) The edges between some $w\in D$ and $\bigcup_{t\neq i(w),j(w)}V_t$.

Let e_1 denote the number of NIM-H edges between $V(T)$ and $V(G_n)\setminus V(T)$, then we have

$$e_1\leqslant \ell\cdot cp+(n-cp-\ell-|D|)\cdot c(p-1)+|D|\cdot(\epsilon^{2\ell+2}n_2+(1-\epsilon)\epsilon^{2\ell}n_2+c(p-2)) \quad (1)$$

and

$$e(G_n)=e(T)+e_1+|E(W'\cup X)\cap E(G_n)| \quad (2)$$

Now we choose an induced copy of $T_p(cp)$ in $H(n,p,k)$. It is easy to see that the induced subgraph of $H(n,p,k)$ on the remaining vertices, i.e. on $V(H(n,p,k))\setminus V(T_p(cp))$, is a copy of $H(n-cp,p,k)$. Let e_2 be the number of edges between $T_p(cp)$ and $H(n-cp,p,k)$, then we have

$$e_2=(k-1)\cdot cp+(n-cp-k+1)\cdot c(p-1) \quad (3)$$

and

$$h(n,p,k)=t_p(cp)+e_2+h(n-cp,p,k)) \quad (4)$$

Since every edge in $E(W'\cup X)\cap E(G_n)$ is not contained in any monochromatic of H, we have $|E(W'\cup X)\cap E(G_n)|\leqslant f(n-cp,H)$, which implies that $|E(W'\cup X)\cap E(G_n)|-h(n-cp,p,k)\leqslant e(G_{n-cp})-h(n-cp,p,k)=\Delta(n-cp)$, where G_{n-cp} is an NIM-H graph on $n-cp$ vertices with $f(n-cp,H)$ edges. So by Eqs. (2), we have

$$\Delta(n)=e(G_n)-h(n,p,k)\leqslant (e_1-e_2)+\Delta(n-cp) \quad (5)$$

If $e_1<e_2$, then $\Delta(n)<\Delta(n-cp)$ and we are done. So assume $e_1\geqslant e_2$. However, by Eqs. (1) and (3), we have $e_1\leqslant e_2$. The equality holds if and only if $D=\emptyset$, $\ell=k-1$, all edges between X and $V(T)$ are NIM-H, and all edges between C_i and $\bigcup_{j\neq i}V_j$ are NIM-H for $1\leqslant i\leqslant p$. Since $e(G_n)\geqslant h(n,p,k)$ and $T_p(n-cp-k+1)$ has more edges than any other p-chromatic graph on $n-cp-k+1$ vertices, we have $G_n=H(n,p,k)$. So $\Delta(n)=0$ and we are done.

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