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On the multiplicatively weighted Harary index of composite graphs

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Abstract: Let $H_M(G)$ be the multiplicatively weighted Harary index of the molecular graph G, which is defined as $H_M(G)=\sum_{\{u,v\}\subseteq V(G)}\frac{d_G(u)d_G(v)}{d_G(u,v)}$, where $d_G(u)$ is the degree of a vertex $u\in V(G)$, and the $d_G(u,v)$ denotes the distance between u and v in G. We introduce four graph operations and obtain explicit formulas for the values of multiplicatively weighted Harary index of composite graphs generated by the four graph operations. Based on this, a lower and an upper bound is determined for the multiplicatively weighted Harary index among graphs in each of the four classes of composite graphs.

Key words: multiplicatively weighted Harary index; composite graph; Graph operations; Regular graph

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复合图的乘权 Harary 指数

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摘要:分子图的乘权 Harary 指数 $H_M(G)$,被定义为 $H_M(G)=\sum_{\{u,v\}\subseteq V(G)}\frac{d_G(u)d_G(v)}{d_G(u,v)}$,其中 $d_G(u)$表示图 G 中顶点 u 的度,$d_G(u,v)$表示图 G 中顶点 u 和 v 之间的距离. 本文主要研究 4 种图操作下得到的复合图的乘权 Harary 指数,以及在 4 种图操作下一些特殊复合图的乘权 Harary 指数的上下界.

关键词:乘权 Harary 指数;复合图;图操作;正则图

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0 Introduction

For terminology and notations not defined here, we refer to Ref. [1]. Throughout this paper, we consider finite and simple connected graphs. We denote the vertex set of a graph G by $V(G)$, and the edge set by $E(G)\}$, the order of G by n_G and the size by m_G.

In history, graph theory is closely related to physics, chemistry and other disciplines. In chemical graph theory, some simple molecular properties are directly mapped to the corresponding topological indices. A topological index is a molecular descriptor associated with the structure of a graph, which has been usually used in physical chemistry, toxicology, pharmacology, chemical synthesis and other fields. Among various kinds of topological indices, there are several types of some indices, especially those based on the distance, which are very important in the structure of the graph}. As is known to all, distance is an crucial parameter in graph theory[2]. This parameter not only affects the structure of the graph, but also derives some important distance parameters, such as diameter, radius, average distance, distance matrix, resistance distance, wiener index, etc.[3-5]

The Wiener index is considered to be one of the most famous distance-based topological indices[6], which is defined as

$$W(G)=\sum_{\{u,v\}\subseteq V(G)} d_G(u,v),$$

where $d_G(u,v)$ is the distance from vertices u to v in G, i. e. the length of a shortest path between vertices u and v in G. The researchers who are interested in the Wiener index of the graph may be referred to Refs. [7-9]. Based on Wiener index, the first and the second Zagreb indices of a graph G are defined as follows[10]:

$$M_1(G)=\sum_{\{uv\in E(G)} (d_G(u)+d_G(v)),$$

$$M_2(G)=\sum_{uv\in E(G)} d_G(u)d_G(v).$$

Furthermore, the first and the second Zagreb coindices of a graph G are defined as follows[11]:

$$\overline{M}_1(G)=\sum_{uv\notin E(G)} (d_G(u)+d_G(v)),$$

$$\overline{M}_2(G)=\sum_{uv\notin E(G)} d_G(u)d_G(v).$$

The Wiener index values the contributions of distant pairs of vertices far more than the contributions from close pairs, in direct contradiction with chemical intuition.

To overcome the above contradiction caused by the Wiener index, some experts consider the sum of reciprocal values of distances between pairs of different vertices. Hence, the Harary index, one of the distance-based topological indices, was introduced[12-13], was defined as

$$H(G)=\sum_{\{u,v\}\subseteq V(G)} \frac{1}{d_G(u,v)}.$$

The Wiener index and Harary index all can reflect the property of graph, so they have some nice interpretations and applications in chemical graph theory. Devillers and Todeschini shown the Harary index is better than the Wiener index in characterizing the molecular structure of chemical graphs, respectively[14-15] for details. Hence, Harary index has been of great interest and extensively studied. For more results on Harary index one may be referred to Refs. [16,19] and the references therein. However, the performance of the Harary index in QSAR/QSPR studies turned out to be quite modest[16]. Hence, the fundamental paradox of the distance-based indices has not been successfully resolved. As Ref. [20] suggested, the performance of the Harary-type indices was improved by taking into consideration the contributions of vertex pairs with their distance. This paved the way for defining another index called additively weighted Harary index of G[21], defined as

$$H_A(G)=\sum_{\{u,v\}\subseteq V(G)} \frac{d_G(u)+d_G(v)}{d_G(u,v)}.$$

At the end of Ref. [21], a variation of additively weighted harary index was suggested, in which the additive weighting $d_G(u)+d_G(v)$ of pairs is replaced with the multiplicative one $d_G(u) d_G(v)$. The multiplicatively weighted Harary index[21] is defined as

$$H_M(G)=\sum_{\{u,v\}\subseteq V(G)}\frac{d_G(u)d_G(v)}{d_G(u,v)}.$$

The intuitive idea of pairs of close atoms contributing more than the distant ones has been difficult to capture in topological indices[21]. However, a possibly useful approach could be the multiplicatively weighted Harray index. After the concept of multiplicativelyHarary index was proposed, it was studied by some experts and scholars, and a lot of progress was made. Deng and Krishnakumari in Ref. [22] first proved that the multiplicatively weighted Harary index of a graph is monotonic on some transformations, and then determined the extremal values of the multiplicatively weighted Harary indices for some familiar classes of graphs and characterized the corresponding extremal graphs. Li and Zhang in Ref. [23] determined sharp upper bounds on the multiplicatively weighted Harary index of trees with given parameters. For a list of new results about the additively and multiplicatively weighted Harary index[24-25].

In this paper, we first introduce four graph operations[25] and obtain explicit formulas for the values of multiplicatively weighted Harary index of composite graphs generated by the four graph operations. Then determine the extremal values of the multiplicatively weighted Harary indices for the four classes of composite graphs and characterize the corresponding extremal graphs. In addition, based on the four graph operations of regular graphs, we obtain explicit formulas for the values of multiplicatively weighted Harary index and determine a lower and an upper bound for the multiplicatively weighted Harary index for each of four classes of composite graphs. We adopt the main techniques in Ref. [25] while obtaining our results.

Note that for a given graph G and $u\in V(G)$, let

$$P(G)=\sum_{u,v\in V(G)}\frac{1}{d_G(u,v)+1},$$

$$P_G(v)=\sum_{u\in V(G)}\frac{1}{d_G(u,v)+1},$$

$$H_G(v)=\sum_{u\in V(G)\setminus\{v\}}\frac{1}{d_G(u,v)}.$$

And the form of summation that occurs in the article satisfies

$$\sum_{\{u,v\}\subseteq V(G)}f(u,v)=\sum_{u\in V(G)}f(u,u)+2\sum_{\{u,v\}\subseteq V(G)}f(u,v).$$

1 Multiplicatively weighted Harary index and operations of graphs

In this section, we introduce the four graph operations of graph G and H, then we will obtain that the multiplicatively weighted Harary index of $G\{H\}$, $G\circ H$, $(G\cdot H)(x;y)$, and $(G\sim H)(x;y)$ on these operations.

1.1 Multiplicatively weighted Harary index of the composite graph $G\{H\}$

Definition 1.1 Let G be a graph with n_G vertices and e_G edges, H be a rooted graph with w as its root and on n_H vertices, e_H edges. Then the rooted product $G\{H\}$ is obtained by taking one copy of G and n_G copies of a rooted graph H, and by identifying the root of the ith copy of H with the ith vertex of G, $i=1,2,\cdots,n_G$.

For the rooted product $G\{H\}$, we have

$|V(G\{H\})|=n_G n_H$, $|E(G\{H\})|=e_G+n_G e_H$.

An example of rooted product can be seen in Fig. 1.

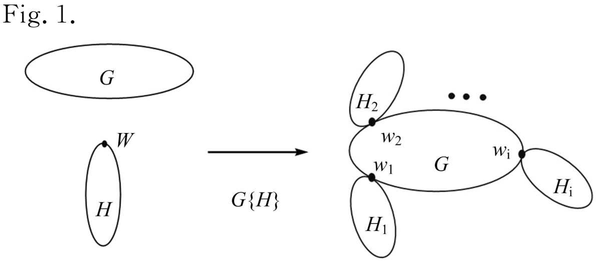

Fig. 1 $G\{H\}$

Lemma 1.1[25] Let G be a simple graph and H be a rooted graph with w as its root. Then for a vertex u of $G\{H\}$ such that $u\in V(G)$, we have $d_{G\{H\}}(u)=d_G(u)+d_H(w)$, and for a vertex v of $G\{H\}$ such that $v\in V(H)\setminus\{w\}$ we have $d_{G\{H\}}(v)=d_H(v_0)$, where v_0 is the corresponding vertex in H as v of H_i. Also

(Ⅰ) if u, $v\in V(G)$, then $d_{G\{H\}}(u,v)\}=d_G(u,v)$;

(Ⅱ) if $u\in V(G)$, $v\in V(H_i)$, where $i=1,2,3,\cdots,n_G$, then $d_{G\{H\}}(u,v)=d_G(u,w_i)+d_{H_i}(w_i,v)=d_G(u,w_i)+d_H(w,v_0)$, where w_i is the root of H_i and v_0 is the corresponding vertex in H as v of H_i;

(Ⅲ) if $u,v\in V(H_i)$, where $i=1,2,3,\cdots,n_G$, then $d_{G\{H\}}(u,v)=d_H(u_0,v_0)$, where u_0 and v_0 are the corresponding vertices in H as u and v of H_i;

(Ⅳ) if $u\in V(H_i)$, $v\in V(H_j)$ and $1\leqslant i<j\leqslant n_G$, then $d_{G\{H\}}(u,v)=d_{H_i}(u,w_i)+d_{H_j}(w_j,v)+d_G(w_i,w_j)=d_H(u_0,w)+d_H(w,v_0)+d_G(w_i,w_j)$, where w_i is the root of H_i and w_j is the root of H_j. Also, u_0 and v_0 are the corresponding vertices in H as u of H_i and v of H_j, respectively.

Theorem 1.1 Let G be a simple graph and H be a rooted graph with w as its root. Then

$$H_M(G\{H\})=H_M(G)+n_GH_M(H)+d_H(w)H_A(G)+d_H^2(w)H(G)+2e_G\sum_{v\in V(H)\setminus\{w\}}\frac{d_H(v)}{d_H(w,v)}+2\sum_{\{u,t\}\subseteq V(G)}\sum_{v\in V(H)\setminus\{w\}}\frac{d_G(u)d_H(v)+d_H(w)d_H(v)}{d_G(u,t)+d_H(w,v)}+\sum_{\{t,l\}\subseteq V(G)}\sum_{\{u,v\}\subseteq V(H)\setminus\{w\}}\frac{d_H(u)d_H(v)}{d_G(t,l)+d_H(w,v)+d_H(u,w)}.$$

Proof From the definition we have

$$H_M(G\{H\})=\sum_{\{u,v\}\subseteq V(G\{H\})}\frac{d_{G\{H\}}(u)d_{G\{H\}}(v)}{d_{G\{H\}}(u,v)}.$$

First, we divide the vertices of the composite graph into four parts by Lemma 1.1, which are denoted as A_1, A_2, A_3, and A_4, where $A_1=\{\{u,v\}\mid u,v\in V(G),1\leqslant i\leqslant n_G\}$, $A_2=\{\{u,v\}\mid u\in V(G),\ v\in V(H_i)\setminus\{w_i\},\ 1\leqslant i\leqslant n_G\}$, $A_3=\{\{u,v\}\mid u,v\in V(H_i)\setminus\{w_i\},\ 1\leqslant i\leqslant n_G\}$, $A_4=\{\{u,v\}\mid u\in V(H_i)\setminus\{w_i\},\ v\in V(H_j)\setminus\{w_j\},\ 1\leqslant i<j\leqslant n_G\}$.

Then, according to the partition of vertex set, we partition the sum into four sums $S_i(i=1,2,3,4)$, where

$$S_1=\sum_{\{u,v\}\subseteq V(G)}\frac{d_{G\{H\}}(u)d_{G\{H\}}(v)}{d_{G\{H\}}(u,v)}=\sum_{\{u,v\}\subseteq V(G)}\frac{d_G(u)d_G(v)+d_H(w)(d_G(u)+d_G(v))+d_H^2(w)}{d_G(u,v)}=H_M(G)+d_H(w)H_A(G)+d_H^2(w)H(G),$$

$$S_2=\sum_{i=1}^{n_G}\sum_{\substack{u\in V(G)\\ v\in V(H_i)\setminus\{w_i\}}}\frac{d_{G\{H\}}(u)d_{G\{H\}}(v)}{d_{G\{H\}}(u,v)}=\sum_{\substack{\{u,t\}\subseteq V(G)\\ v\in V(H)\setminus\{w\}}}\frac{(d_G(u)+d_H(w))d_H(v)}{d_G(u,t)+d_H(w,v)}=$$

$$2\sum_{\substack{\{u,t\}\subseteq V(G)\\ v\in V(H)\setminus\{w\}}}\frac{d_G(u)d_H(v)+d_H(w)d_H(v)}{d_G(u,t)+d_H(w,v)}+\sum_{\substack{u=t\\ v\in V(H)\setminus\{w\}}}\frac{d_G(u)d_H(v)+d_H(w)d_H(v)}{d_H(w,v)}=$$

$$2\sum_{v\in V(H)\setminus\{w\}}\sum_{\{u,t\}\setminus V(G)}\frac{d_G(u)d_H(v)+d_H(w)d_H(v)}{d_G(u,t)+d_H(w,v)}+2e_G\sum_{v\in V(H)\setminus\{w\}}\frac{d_H(v)}{d_H(w,v)}+n_G\sum_{v\in V(H)\setminus\{w\}}\frac{d_H(w)d_H(v)}{d_H(w,v)},$$

$$S_3=\sum_{i=1}^{n_G}\sum_{\{u,v\}\subseteq V(H_i)\setminus\{w_i\}}\frac{d_{G\{H\}}(u)d_{G\{H\}}(v)}{d_{G\{H\}}(u,v)}=\sum_{i=1}^{n_G}\sum_{\{u,v\}\subseteq V(H_i)\setminus\{w_i\}}\frac{d_H(u)d_H(v)}{d_H(u,v)}=n_G\sum_{\{u,v\}\subseteq V(H)\setminus\{w\}}\frac{d_H(u)d_H(v)}{d_H(u,v)}=n_GH_M(H)-n_Gd_H(w)\sum_{v\in V(H)\setminus\{w\}}\frac{d_H(v)}{d_H(w,v)},$$

$$S_4=\sum_{1\leqslant i<j\leqslant n_G}\sum_{\substack{u\in V(H_i)\setminus\{w_i\}\\ v\in V(H_j)\setminus\{w_j\}}}\frac{d_{G\{H\}}(u)d_{G\{H\}}(v)}{d_{G\{H\}}(u,v)}=\sum_{\{t,l\}\subseteq V(G)}\sum_{\{u,v\}\subseteq V(H)\setminus\{w\}}\frac{d_H(u)d_H(v)}{d_H(u,w)+d_H(v,w)+d_G(t,l)}.$$

Finally, we add up the sum of the four parts and simplify it, the Theorem 1.1 is complete.

Based on Theorem 1.1, we obtain the next corollary immediately.

Corollary 1.1 Let G be a r-regular graph and H be a k-regular rooted graph with w as its root. Then

$$H_M(G\{H\})=(r+k)^2H(G)+n_Gk^2H(H)+n_GkrH_H(w)+ 2k(r+k)\sum_{v\in V(H)\setminus\{w\}}\sum_{\{u,t\}\subseteq V(G)}\frac{1}{d_G(u,t)+d_H(w,v)}+ k^2\sum_{\{t,l\}\subseteq V(G)}\sum_{\{u,v\}\subseteq V(H)\setminus\{w\}}\frac{1}{d_H(u,w)+d_H(v,w)+d_G(t,l)}.$$

We can determine a lower and an upper bound for $H_M(G\{H\})$, where G is a r-regular graph and H is a k-regular rooted graph.

We know that $1\leqslant d_G(u,v)\leqslant D(G)$, where $\{u,v\}\subseteq V(G)$, $u\neq v$ and $D(G)$ is the diameter of G.

Similarly, we have $1\leqslant d_H(u,v)\leqslant D(H)$, where $\{u,v\}\subseteq V(H)$, $u\neq v$ and $D(H)$ is the diameter of H.

Hence, we obtain

$$H_M(G\{H\})\geqslant(r+k)^2H(G)+n_Gk^2H(H)+n_GkrH_H(w)+n_G(n_G-1)(n_H-1)\frac{k(r+k)}{D(G)+D(H)}+ n_G(n_G-1)(n_H-1)\frac{n_Hk^2}{4(2D(H)+D(G))},$$

$$H_M(G\{H\})\leqslant(r+k)^2H(G)+n_Gk^2H(H)+n_GkrH_H(w)+n_G(n_G-1)(n_H-1)\frac{k(r+k)}{2}+ n_G(n_G-1)(n_H-1)\frac{n_Hk^2}{12}.$$

1.2 Multiplicatively weighted Harary index of the composite graph $G\circ H$

Definition 1.2 Let G be a graph with the order by n_G and the size by e_G, and H also be a graph with the order is n_H and the size is e_H. The corona product $G\circ H$ is obtained by taking one copy of G and n_G copies of H; and by joining each vertex of the ith copy of H to the ith vertex of G, $i=1,2,\cdots,n_G$.

For the corona product $G\circ H$, we have

$$|V(G\circ H)|=n_G(n_H+1),$$
$$|E(G\circ H)|=e_G+n_G(e_H+n_H).$$

An example of rooted product can be seen in Fig. 2.

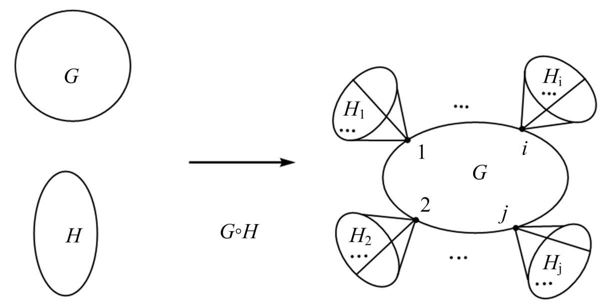

Fig. 2 $G\circ H$

Lemma 1.2[25] Let G and H be two simple connected graphs. For a vertex u of $G\circ H$ such that $u\in V(G)$, we have $d_{G\circ H}(u)=d_G(u)+n_H$, and for a vertex v of $G\circ H$ such that $v\in V(H)$, we have $d_{G\circ H}(v)=d_H(u)+1$. Also:

(Ⅰ) if $u,v\in V(G)$, then $d_{G\circ H}(u,v)=d_G(u,v)$;

(Ⅱ) if $u\in V(G)$, $V\in V(H_i)$, then $d_{G\circ H}=d_G(u,w_i)+1$, where w_i is the ith vertex in G and $i=1,2,3,\cdots,n_G$;

(Ⅲ) if u, $v\in V(H_i)$, where $i=1,2,3,\cdots,n_G$, then

$$d_{G\circ H}(u,v)=\begin{cases}1, & uv\in E(H_i),\\ 2, & uv\notin E(H_i);\end{cases}$$

(Ⅳ) if $u\in V(H_i)$, $v\in V(H_j)$ and $1\leqslant i<j\leqslant n_G$, then $d_{G\circ H}(u,v)=d_G(w_i,w_j)+2$, where w_i is the ith and w_j is the j-th vertices in G.

Lemma 1.3[26] Let G be a graph of order n and size e. Then

(Ⅰ $\overline{M}_1(G)=2e(n-1)-M_1(G)$,

(Ⅱ) $\overline{M}_2(G)=2e^2-M_2(G)-\frac{1}{2}M_1(G)$.

Lemma 1.4[25] Let G be a simple graph and K_2 be the complete graph of order 2. Then

$$H(G\{K_2\})=H(G)+P(G)+\sum_{\{u,v\}\subseteq V(G)}\frac{1}{d_G(u,v)+2}.$$

Theorem 1.2 Let G and H be two simple graphs. Then

$$H_M(G\circ H)=H_M(G)+n_H H_A(G)+n_H^2 H(G)+\frac{1}{4}n_G M_1(H)+\frac{1}{2}n_G M_2(H)+n_G e_H^2+n_G(n_H-\frac{1}{2})e_H+\frac{(n_H-1)n_H n_G}{4}+(2e_H+n_H)\sum_{\{u,v\}\subseteq V(G)}\frac{d_G(u)}{d_G(u,v)+1}+n_H(2e_H+n_H)\sum_{i=1}^{n_G}P_G(w_i)+(2e_H+n_H)^2[H(G\{K_2\})-H(G)-P(G)].$$

Proof By definition we have

$$H_M(G\circ H)=\sum_{\{u,v\}\subseteq V(G\circ H)}\frac{d_{G\circ H}(u)d_{G\circ H}(v)}{d_{G\circ H}(u,v)},$$

By Lemma 1.2, we partition the sum into four sums S_i ($i=1,2,3,4$). We consider four sums S_1, S_2, S_3, S_4 as follows:

$$S_1=\sum_{\{u,v\}\subseteq V(G)}\frac{d_{G\circ H}(u)d_{G\circ H}(v)}{d_{G\circ H}(u,v)}=\sum_{\{u,v\}\subseteq V(G)}\frac{(d_G(u)+n_H)(d_G(v)+n_H)}{d_G(u,v)}=\sum_{\{u,v\}\subseteq V(G)}\frac{d_G(u)d_G(v)+n_H(d_G(u)+d_G(v))+n_H^2}{d_G(u,v)}=H_M(G)+n_H H_A(G)+n_H^2 H(G).$$

$$S_2=\sum_{i=1}^{n_G}\sum_{\substack{u\in V(G)\\ v\in V(H_i)}}\frac{d_{G\circ H}(u)d_{G\circ H}(v)}{d_{G\circ H}(u,v)}=\sum_{i=1}^{n_G}\sum_{\substack{u\in V(G)\\ v\in V(H_i)}}\frac{(d_G(u)+n_H)(d_H(v)+1)}{d_G(u,w_i)+1}.$$

Now we consider the following relation

$$\sum_{\substack{u\in V(G)\\ v\in V(H_i)}}\frac{(d_G(u)+n_H)(d_H(v)+1)}{d_G(u,w_i)+1}=\sum_{\substack{u\in V(G)\\ v\in V(H_i)}}\frac{d_G(u)d_H(v)+d_G(u)+n_H d_H(v)+n_H}{d_G(u,w_i)+1}=$$

$$\sum_{\substack{u\in V(G)\\ v\in V(H_i)}}\frac{d_G(u)d_H(v)}{d_G(u,w_i)+1}+\sum_{\substack{u\in V(G)\\ v\in V(H_i)}}\frac{d_G(u)}{d_G(u,w_i)+1}+\sum_{\substack{u\in V(G)\\ v\in V(H_i)}}\frac{n_H d_H(v)}{d_G(u,w_i)+1}+n_H\sum_{\substack{u\in V(G)\\ v\in V(H_i)}}\frac{1}{d_G(u,w_i)+1}=(2e_H+n_H)\sum_{u\in V(G)}\frac{d_G(u)}{d_G(u,w_i)+1}+n_H(2e_H+n_H)\sum_{u\in V(G)}\frac{1}{d_G(u,w_i)+1}.$$

Hence, we get

$$S_2=(2e_H+n_H)\sum_{\{u,v\}\subseteq V(G)}\frac{d_G(u)}{d_G(u,v)+1}+n_H(2e_H+n_H)\sum_{i=1}^{n_G}\sum_{u\in V(G)}\frac{1}{d_G(u,w_i)+1}=(2e_H+n_H)\sum_{\{u,v\}\subseteq V(G)}\frac{d_G(u)}{d_G(u,v)+1}+n_H(2e_H+n_H)\sum_{i=1}^{n_G}P_G(w_i),$$

$$S_3=\sum_{i=1}^{n_G}\sum_{\{u,v\}\subseteq V(H_i)}\frac{d_{G\circ H}(u)d_{G\circ H}(v)}{d_{G\circ H}(u,v)}=\sum_{i=1}^{n_G}\sum_{\{u,v\}\subseteq V(H_i)}\frac{(d_H(u)+1)(d_H(v)+1)}{d_{G\circ H}(u,v)}.$$

Next we consider the following relation

$$\sum_{\{u,v\}\subseteq V(H_i)}\frac{d_H(u)d_H(v)+d_H(u)+d_H(v)+1}{d_H(u,v)}=\sum_{uv\in E(H)}(d_H(u)d_H(v)+d_H(u)+d_H(v)+1)+\sum_{uv\notin E(G)}\frac{d_H(u)d_H(v)+d_H(u)+d_H(v)+1}{2}=M_2(H)+M_1(H)+e_H+\frac{1}{2}\overline{M}_2(H)+\frac{1}{2}\overline{M}_1(H)+\frac{1}{2}(\frac{n_H(n_H-1)}{2}-e_H).$$

So, we obtain

$$S_3=n_G[\frac{1}{4}M_1(H)+\frac{1}{2}M_2(H)+(n_H-\frac{1}{2})e_H+e_H^2+\frac{n_H(n_H-1)}{4}],$$

$$S_4=\sum_{1\leqslant i<j\leqslant n_G}\sum_{\substack{u\in V(H_i)\\ v\in V(H_j)}}\frac{d_{G\circ H}(u)d_{G\circ H}(v)}{d_{G\circ H}(u,v)}=\sum_{1\leqslant i<j\leqslant n_G}\sum_{\substack{u\in V(H_i)\\ v\in V(H_j)}}\frac{d_H(u)d_H(v)+d_H(u)+d_H(v)+1}{d_G(w_i,w_j)+2}.$$

Finally we consider the following relation, where $1 \leqslant i < j \leqslant n_G$

$$\sum_{\substack{u\in V(H_i)\\ v\in V(H_j)}} \frac{d_H(u)d_H(v)+d_H(u)+d_H(v)+1}{d_G(w_i,w_j)+2} = \frac{1}{d_G(w_i,w_j)+2}\sum_{\substack{u\in V(H_i)\\ v\in V(H_j)}} d_H(u)d_H(v) + \frac{1}{d_G(w_i,w_j)+2}\sum_{\substack{u\in V(H_i)\\ v\in V(H_j)}} (d_H(u)+d_H(v)+1) = (4e_H^2+4e_Hn_H+n_H^2)\frac{1}{d_G(w_i,w_j)+2}.$$

Therefore, we have

$$S_4 = (4e_H^2+4e_Hn_H+n_H^2)\sum_{\{u,v\}\subseteq V(G)}\frac{1}{d_G(u,v)+2}.$$

The result now follows by adding the four contributions and simplifying the expression.

Based on Theorem 1.2, we get the two following corollaries.

Corollary 1.2 Let G be a simple graph and H be a r-regular graph. Then

$$H_M(G\circ H) = H_M(G)+n_HH_A(G)+n_H^2H(G)+ n_G(r+1)^2\left(\frac{1}{2}e_H+\frac{n_H(n_H-1)}{4}\right)+ n_H(r+1)\sum_{\{u,v\}\subseteq V(G)}\frac{d_G(u)}{d_G(u,v)+1}+ (r+1)^2n_H^2\sum_{i=1}^{n_G}P_G(w_i)+ (r+1)^2n_H^2[H(G\{K_2\})-H(G)-P(G)].$$

Corollary 1.3 Let G be a k-regular graph and H be a r-regular graph. Then

$$H_M(G\circ H) = (k+n_H)^2H(G)+ n_G(r+1)^2\left(\frac{1}{2}e_H+\frac{n_H(n_H-1)}{4}\right)+ n_H(r+1)(k+n_H)\sum_{\{u,v\}\subseteq V(G)}\frac{1}{d_G(u,v)+1}+ n_H^2(r+1)^2[H(G\{K_2\})-H(G)-P(G)].$$

1.3 Multiplicatively weighted Harary index of the composite graph $(G \cdot H)(y;z)$

Definition 1.3 Let G and H be two simple graphs. For given vertices $y\in V(G)$, and $z\in V(H)$, the splice of G and H by vertices y and z, which is denoted by $(G\cdot H)(y;z)$, is defined by identifying the vertices y and z in the union of G and H.

Then for the splice of G and H by vertices y and z, we have

$$|V((G\cdot H)(y;z))| = n_G+n_H-1,$$
$$|E((G\cdot H)(y;z))| = e_G+e_H.$$

An example of rooted product can be seen in Fig 3.

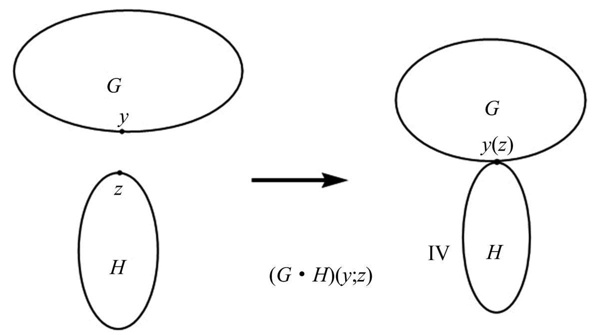

Fig. 3 G • H

Lemma 1.5[25] Let G and H be simple graphs with disjoint vertex sets. For given vertices $y\in V(G)$ and $z\in V(H)$, suppose that the splice of G and H by vertices y and z is denoted by $(G\cdot H)(y;z)$ for convenience. Then for a vertex u of $(G\cdot H)(y;z)$ such that $u\in V(G)\backslash\{y\}$, we have $d_{G\cdot H}(u)=d_G(u)$; and for a vertex v of $(G\cdot H)(y;z)$ such that $v\in V(H)\backslash\{z\}$, we have $d_{G\cdot H}(v)=d_G(v)$, $d_{G\cdot H}(y)=d_{G\cdot H}(z)=d_G(y)+d_H(z)$. Also:

(Ⅰ) if u, $v\in V(G)$, then $d_{G\cdot H}(u,v)=d_G(u,v)$;

(Ⅱ) if u, $v\in V(H)$, then $d_{G\cdot H}(u,v)=d_H(u,v)$;

(Ⅲ) if $u\in V(G)$, $v\in V(H)$, then $d_{G\cdot H}(u,v)=d_G(u,y)+d_H(z,v)$.

Theorem 1.3 Let G and H be two simple graphs. For vertices $y\in V(G)$ and $z\in V(H)$, consider $(G\cdot H)(y;z)$. Then

$$H_M(G\cdot H) = H_M(G)+H_M(H)+ d_H(z)\sum_{u\in V(G)\backslash\{y\}}\frac{d_G(v)}{d_G(y,v)}+ d_G(y)\sum_{u\in V(H)\backslash\{z\}}\frac{d_H(v)}{d_G(z,v)}+ \sum_{\substack{u\in V(G)\backslash\{y\}\\ v\in V(H)\backslash\{z\}}}\frac{d_G(u)d_H(v)}{d_G(u,y)+d_H(z,v)}.$$

Proof For convenience we denote $(G\cdot H)(y;z)$ by $G\cdot H$. By definition we have

$$H_M(G\cdot H) = \sum_{\{u,v\}\subseteq V(G\cdot H)}\frac{d_{G\cdot H}(u)d_{G\cdot H}(v)}{d_{G\cdot H}(u,v)}.$$

By Lemma 1.5, we partition the sum into

three sums S_i ($i=1,2,3$). Hence we obtain

$$S_1=\sum_{\{u,v\}\subseteq V(G)}\frac{d_{G\cdot H}(u)d_{G\cdot H}(v)}{d_{G\cdot H}(u,v)}=\sum_{\{u,v\}\subseteq V(G)\setminus\{y\}}\frac{d_G(u)d_G(v)}{d_G(u,v)}+\sum_{v\in V(G)\setminus\{y\}}\frac{(d_G(y)+d_H(z))d_G(v)}{d_G(y,v)}=H_M(G)+d_H(z)\sum_{u\in V(G)\setminus\{y\}}\frac{d_G(v)}{d_G(y,v)}.$$

Similarly, we have

$$S_2=H_M(H)+d_G(y)\sum_{v\in V(H)\setminus\{z\}}\frac{d_H(v)}{d_H(z,v)},$$

$$S_3=\sum_{\substack{u\in V(G)\setminus\{y\}\\ v\in V(H)\setminus\{z\}}}\frac{d_{G\cdot H}(u)d_{G\cdot H}(v)}{d_{G\cdot H}(u,v)}=\sum_{\substack{u\in V(G)\setminus\{y\}\\ v\in V(H)\setminus\{z\}}}\frac{d_G(u)d_H(v)}{d_G(u,y)+d_H(z,v)}.$$

The result now follows by adding the three sums S_i, $i=1,2,3$.

Corollary 1.4 Let G be a r-regular graph and H be a k-regular graph. For vertices $y\in V(G)$ and $z\in V(H)$, consider $(G\cdot H)(y;z)$. Then

$$H_M(G\cdot H)=r^2H(G)+k^2H(H)+kr(H_G(y)+H_H(z))+kr\sum_{\substack{u\in V(G)\setminus\{y\}\\ v\in V(H)\setminus\{z\}}}\frac{1}{d_G(u,y)+d_H(v,z)}.$$

We can determine a lower and an upper bound for $H_M(G\cdot H)$, where G and H are r-regular and k-regular graphs, respectively.

We know that $1\leqslant d_G(u,y)\leqslant D(G)$, where $u\in V(G)\setminus\{y\}$ and $D(G)$ is the diameter of G. Similarly, we get $1\leqslant d_H(v,z)\leqslant D(H)$, where $v\in V(H)\setminus\{z\}$ and $D(H)$ is the diameter of H.

Therefore, we have

$$H_M(G\cdot H)\geqslant r^2H(G)+kr(H_G(y)+H_H(z))+k^2H(H)+kr\frac{(n_G-1)(n_H-1)}{D(G)+D(H)},$$

$$H_M(G\cdot H)\leqslant r^2H(G)+kr(H_G(y)+H_H(z))+k^2H(H)+kr\frac{(n_G-1)(n_H-1)}{2}.$$

1.4 Multiplicatively weighted Harary index of the composite graph $(G\sim H)(y;z)$

Definition 1.4 Let G and H be two simple graphs, and $y\in V(G)$, $z\in V(H)$. A link of G and H by vertices y and z, denoted by $(G\sim H)(y;z)$, which is defined as the graph obtained by joining y and z with an edge in the union of these graphs.

For a link of G and H by vertices y and z, we have

$$|V((G\sim H)(y;z))|=n_G+n_H,$$
$$|E((G\sim H)(y;z))|=e_G+e_H+1.$$

An example of rooted product can be seen in Fig 4.

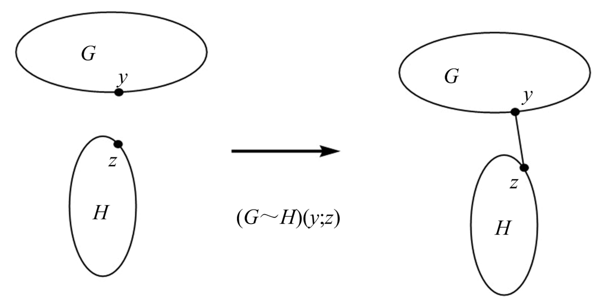

Fig. 4 $G\sim H$

Lemma 1.6[25] Let G and H be two simple graphs with disjoint vertex sets. For given vertices $y\in V(G)$ and $z\in V(H)$, suppose a link of G and H by vertices y and z is denoted by $G\sim H$ for convenience. Then for a vertex u of $G\sim H$ such that $u\in V(G)\setminus\{y\}$, we have $d_{G\sim H}(u)=d_G(u)$ and for a vertex v of $G\sim H$ such that $v\in V(H)\setminus\{z\}$, we have $d_{G\sim H}(v)=d_G(v)$; $d_{G\sim H}(y)=d_G(y)+1$, $d_{G\sim H}(z)=d_H(z)+1$.

(Ⅰ) if u, $v\in V(G)$, then $d_{G\sim H}(u,v)=d_G(u,v)$;

(Ⅱ) if u, $v\in V(H)$, then $d_{G\sim H}(u,v)=d_H(u,v)$;

(Ⅲ) if $u\in V(G)$, $v\in V(H)$, then $d_{G\sim H}(u,v)=d_G(u,y)+d_H(z,v)+1$.

Theorem 1.4 Let G and H be two simple graphs. For vertices $y\in V(G)$ and $z\in V(H)$, consider $(G\sim H)(y;z)$. Then

$$H_M(G\sim H)=H_M(G)+H_M(H)+d_G(y)d_H(z)+d_G(y)+d_H(z)+1+\sum_{v\in V(G)\setminus\{y\}}\frac{d_G(v)}{d_G(y,v)}+\sum_{v\in V(H)\setminus\{z\}}\frac{d_H(v)}{d_H(z,v)}+\sum_{\substack{u\in V(G)\setminus\{y\}\\ v\in V(H)\setminus\{z\}}}\frac{d_G(u)d_H(v)}{d_G(u,y)+d_H(z,v)+1}+\sum_{u\in V(G)\setminus\{y\}}\frac{d_G(u)d_H(z)+d_G(u)}{d_G(u,y)+1}+$$

$$\sum_{v\in V(H)\backslash\{z\}}\frac{d_G(y)d_H(v)+d_H(v)}{d_G(z,v)+1}.$$

Proof By definition we have

$$H_M((G\sim H)(y;z))=\sum_{\{u,v\}\subseteq V(G)}\frac{d_{G\sim H}(u)d_{G\sim H}(v)}{d_{G\sim H}(u,v)}.$$

Similarly to the proof of Theorem 1.3, we partition the sum into three sums S_i, $i=1,2,3$.

We consider three sums S_1, S_2, S_3 as follows:

$$S_1=\sum_{\{u,v\}\subseteq V(G)}\frac{d_{G\sim H}(u)d_{G\sim H}(v)}{d_{G\sim H}(u,v)}=\sum_{\{u,v\}\subseteq V(G)\backslash\{y\}}\frac{d_G(u)d_G(v)}{d_G(u,v)}+\sum_{v\in V(G)\backslash\{y\}}\frac{(d_G(y)+1)d_G(v)}{d_G(y,v)}=H_M(G)+\sum_{v\in V(G)\backslash\{y\}}\frac{d_G(v)}{d_G(y,v)}.$$

Similarly, we have

$$S_2=H_M(H)+\sum_{v\in V(H)\backslash\{z\}}\frac{d_H(v)}{d_H(z,v)}$$

and

$$S_3=\sum_{\substack{u\in V(G)\\ v\in V(H)}}\frac{d_{G\sim H}(u)d_{G\sim H}(v)}{d_{G\sim H}(u,v)}=d_G(y)d_H(z)+d_G(y)+d_H(z)+1+\sum_{\substack{u\in V(G)\backslash\{y\}\\ v\in V(H)\backslash\{z\}}}\frac{d_G(u)d_H(v)}{d_G(u,y)+d_H(z,v)+1}+\sum_{u\in V(G)\backslash\{y\}}\frac{d_G(u)(d_H(z)+1)}{d_G(u,y)+1}+\sum_{v\in V(H)\backslash\{z\}}\frac{(d_G(y)+1)d_H(v)}{d_G(z,v)+1}.$$

We obtain the result by adding the three sums S_i, $i=1,2,3$.

In a similar way, we have the two following corollaries.

Corollary 1.5 Let G be a r-regular graph and H be a k-regular graph. For vertices $y\in V(G)$ and $z\in V(H)$, consider $(G\sim H)(y;z)$. Then

$$H_M(G\sim H)=r^2H(G)+k^2H(H)+rH_G(y)+kH_H(z)+r(k+1)P_G(y)+1-rk+k(r+1)P_H(z)+rk\sum_{\substack{u\in V(G)\backslash\{y\}\\ v\in V(H)\backslash\{z\}}}\frac{1}{d_G(u,y)+d_H(z,v)+1}.$$

Similarly, we can determine a lower and an upper bound for $H_M(G\sim H)$, where G and H are r-regular and k-regular graphs, respectively.

We see that $1\leqslant d_G(u,y)\leqslant D(G)$, where $u\in V(G)\backslash\{y\}$ and $D(G)$ is the diameter of G. Similarly, we have $1\leqslant d_H(v,z)\leqslant D(H)$, where $v\in V(H)\backslash\{z\}$ and $D(H)$ is the diameter of H.

Hence, we obtain

$$H_M(G\sim H)\geqslant r^2H(G)+k^2H(H)+rH_G(y)+kH_H(z)+1-rk+r(k+1)P_G(y)+k(r+1)P_H(z)+rk\frac{(n_G-1)(n_H-1)}{D(G)+D(H)+1}$$

and

$$H_M(G\sim H)\leqslant r^2H(G)+k^2H(H)+rH_G(y)+kH_H(z)+1-rk+r(k+1)P_G(y)+k(r+1)P_H(z)+rk\frac{(n_G-1)(n_H-1)}{3}.$$

Remark 1.1 From the definition of $H(G)$ and $P(G)$, it is obvious that the complete graph has the largest $H(G)$ and $P(G)$ among all graphs on the same number of vertices. So, for any graph G on n vertices we have $H(G)\leqslant\frac{n(n-1)}{2}$, $P(G)\leqslant\frac{n(n-1)}{2}+n$. Also, from the fact that adding an edge to G will increase its multiplicatively weighted Harary index, it immediately follows that the complete graph has the largest $H_M(G)$ among all graph on the same number of vertices. Hence, for any graph G on n vertices we have $H_M(G)\leqslant\frac{n(n-1)^3}{2}$.

From the above remark, we obtain the next corollaries immediately.

Corollary 1.6 Let G be a r-regular graph and H be a k-regular rooted graph. Then

$$H_M(G\{H\})\leqslant(r+1)\left[\frac{r(r+k)^2}{2}+\frac{k^3(k+1)}{2}\right]+r(r+1)k^2\left[1+\frac{r+k}{2}+\frac{k(k+1)}{12}\right].$$

Corollary 1.7 Let G be a r-regular graph and H be a k-regular graph. Then

$$H_M((G\cdot H)(y;z))\leqslant$$

$$\frac{r^3(r+1)}{2}+\frac{k^3(k+1)}{2}+kr(k+r+\frac{kr}{2}),$$

$$H_M((G\sim H)(y;z))\leqslant$$
$$\frac{r^2k^2}{3}+\frac{(r+k)(r+1)(k+1)}{2}+$$
$$r^2(\frac{r(r+1)}{2}+1)+k^2(\frac{k(k+1)}{2}+1).$$

2 Conclusion

In this paper we have investigated the multiplicatively weighted Harary index for some graph products such as splice, link, corona and rooted product. Also we have determined lower and upper bounds for some of them.

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