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Positive solutions to boundary value problems for fractional differential equations with time delays

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Abstract: The existence of positive solutions for a class of singular delay fractional differential equations with multi-point boundary value conditions was studied. By means of the Krasnosel'skii fixed point theorem, sufficient conditions of existence results of positive solutions were presented.

Key words: singular fractional differential equations; boundary value problems; time-delays; positive solutions; fixed point theorem

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带有时滞的分数阶微分方程的边值问题的正解

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摘要:研究了一类带有多点边值条件的奇异时滞分数阶微分方程的正解的存在性.利用 Krasnosel'skii 不动点定理,给出了边值问题正解存在的充分条件.

关键词:奇异分数阶微分方程;边值问题;时滞;正解;不动点定理

0 Introduction

We are concerned with the boundary value problem (BVP) for the following fractional functional differential equations

$$\left.\begin{aligned} & D_{0^+}^{\alpha}x(t)+f(t,x(t-\tau))=0,\ t\in(0,1)\backslash\{\tau\}; \\ & x(t)=\eta(t),\ t\in[-\tau,0]; \\ & x'(0)=0,\ x(1)=0 \end{aligned}\right\} \tag{1}$$

where $2<\alpha\leqslant 3$ is a real number, $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative, the time

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delay τ is given and satisfies $0 < \tau < 1$, $\eta(t) \in C([-\tau, 0], \mathbb{R}^+)$, $\eta(t) > 0$ for $t \in [-\tau, 0)$ and $\eta(0) = 0$, $\eta'_-(0) = 0$. $f(t, x) \in C((0,1) \times (0, +\infty), \mathbb{R})$ is continuous, may be singular at $t = 0$, $t = 1$ and $x = 0$, and can take negative values. A function x is called a positive solution of BVP (1) if $x(t)$ is nonnegative on $[-\tau, 1]$, $x(t) > 0$ for $t \in [-\tau, 0) \cup (0, 1]$ and it satisfies BVP (1).

Fractional differential equations have played a significant role in engineering, science, economy, and other fields. For details, see Refs. [1-8] and the references cited therein. As an important branch of nonlinear analysis, boundary value problems of fractional functional differential equations have also gained considerable attention simultaneously (see e. g. Refs. [9-19]). Using the Krasnosel'skii fixed point theorem, Su[15] studied the positive solutions to the following singular boundary value problems for fractional functional differential equations

$$\left.\begin{aligned} &D^\alpha x(t) + f(t, x(t-\tau)) = 0,\ t \in (0,1)\backslash\{\tau\};\\ &x(t) = \eta(t),\ t \in [-\tau, 0];\\ &x(1) = 0, \end{aligned}\right\} \tag{1}$$

where $1 < \alpha \leqslant 2$ is a real number, D^α is the Riemann-Liouville fractional derivative. Motivated by the above work, we discuss the existence of positive solutions for BVP (1), which aims to extend the results in Ref. [15] from the low order to the high order case. By means of the Krasnosel'skii fixed point theorem, two existence results of positive solutions of fractional differential equations for BVP (1) are established, which involves not only the past time delays but also the fractional derivative with the order.

The paper is organized as follows. In Section 1, we recall some definitions, lemmas, and theorems. The main results are stated in Section 2.

1 Preliminaries

In this section, several definitions and lemmas are reviewed which will be used in Section 2.

Definition 1.1 The Riemann-Louville fractional integral of order $\alpha > 0$ of a function $u: (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) \mathrm{d}s,$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 1.2 The Riemann-Louville fractional derivative of order $\alpha > 0$ of a function $u: (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^\alpha u(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n I_{0^+}^{n-\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s) \mathrm{d}s,$$

where $n = [\alpha] + 1$, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

More details on fractional calculus and related properties can be found in Refs. [1-4].

The following is an existence and uniqueness result of solution for a linear boundary value problem, which is important for us in the following analysis.

Lemma 1.1[12] Let $a \in \mathbb{R}$, $y \in C(0,1) \cap L(0,1)$, $2 < \alpha \leqslant 3$. Then the unique solution of the following boundary value problem

$$\left.\begin{aligned} &D_{0^+}^\alpha x(t) + y(t) = 0,\ t \in (0,1);\\ &x(0) = x'(0) = 0,\ x(1) = a \end{aligned}\right\} \tag{2}$$

is given by

$$x(t) = a t^{\alpha-1} + \int_0^1 G(t,s) y(s) \mathrm{d}s$$

where $G(t,s)$ is Green function given by

$$G(t,s) = \begin{cases} \dfrac{(1-s)^{\alpha-1} t^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leqslant s \leqslant t \leqslant 1;\\ \dfrac{(1-s)^{\alpha-1} t^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leqslant t \leqslant s \leqslant 1 \end{cases} \tag{3}$$

Remark 1.1 If $G(t,s)$ is defined by (3), then $G(t,s) \geqslant 0$ for $t, s \in (0,1)$ and $G(0,s) = 0$, $G_t(0,s) = 0$.

Lemma 1.2[12] The Green function defined by Eq. (3) satisfies the following properties:

① $\dfrac{t^{\alpha-1}(1-t)s(1-s)^\alpha}{\Gamma(\alpha)} \leqslant G(t,s) \leqslant$

$\frac{(\alpha-1)s(1-s)^{\alpha-1}}{\Gamma(\alpha)}=\frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha-1)}$, $t,s\in(0,1)$;

② $\frac{t^{\alpha-1}(1-t)s(1-s)^{\alpha}}{\Gamma(\alpha)}\leqslant G(t,s)\leqslant\frac{(\alpha-1)t^{\alpha-1}(1-t)}{\Gamma(\alpha)}=\frac{t^{\alpha-1}(1-t)}{\Gamma(\alpha-1)}$, $t,s\in(0,1)$.

The following fixed point theorem is the main tool in this paper.

Lemma 1.3[20-21] Let X be a Banach space and K be a cone in X. Suppose that Ω_1 and Ω_2 are open subsets of X such that $0\in\overline{\Omega}_1\subset\Omega_2$ and T: $K\cap\overline{\Omega}_2\backslash\Omega_1\to K$ is a completely continuous operator such that either

(i) $\|Tu\|\leqslant\|u\|$ for $u\in\partial\Omega_1$ and $\|Tu\|\geqslant\|u\|$ for $u\in\partial\Omega_2$

or

(ii) $\|Tu\|\geqslant\|u\|$ for $u\in\partial\Omega_1$ and $\|Tu\|\leqslant\|u\|$ for $u\in\partial\Omega_2$.

Then T has a fixed point in $K\cap\overline{\Omega}_2\backslash\Omega_1$.

2 The main results

In this section, by means of the Krasnosel'skii fixed point theorem, sufficient conditions of existence of positive solutions for BVP (1) are established.

Define the Banach space

$$X=\{x(t):x\in C([-\tau,1],\mathbb{R}),\ x(t)=0\text{ for }t\in[-\tau,0],x(1)=0\}$$

with the norm $\|x\|=\sup_{t\in[-\tau,1]}|x(t)|=\sup_{t\in[0,1]}|x(t)|$.

Define the cone

$$P=\{x\in X:x(t)\geqslant 0\text{ for }t\in[-\tau,1]\}.$$

Let

$$\bar{\eta}(t)=\begin{cases}\eta(t),\ t\in[-\tau,0];\\ 0,\ t\in[0,1];\end{cases}$$

$$\omega(t)=\begin{cases}0,\ t\in[-\tau,0];\\ \int_0^1 G(t,s)\rho(s)\mathrm{d}s,t\in[0,1];\end{cases}$$

and, for any $x\in P$, define

$$x^*(t)=\max\{x(t)+\bar{\eta}(t)-\omega(t),0\}=\begin{cases}\eta(t),\ t\in[-\tau,0];\\ \max\{x(t)-\omega(t),0\},t\in[0,1].\end{cases}$$

Remark 2.1 According to Lemma 1.1, the restriction $\omega|_{[0,1]}=\int_0^1 G(t,s)\rho(s)\mathrm{d}s$ of ω on $[0,1]$ is exactly the solution of BVP

$$\begin{cases}D_{0^+}^{\alpha}x(t)+\rho(t)=0,t\in(0,1);\\ x(0)=x'(0)=0,\ x(1)=0.\end{cases}$$

For any $x\in P$, define the operator as

$$(Tx)(t)=\begin{cases}0,\ t\in[-\tau,0];\\ \int_0^1 G(t,s)[f(s,x^*(s-\tau)+\rho(s)]\mathrm{d}s,\ t\in(0,1]\end{cases}\tag{4}$$

Lemma 1.1 and Remark 2.1 imply that if $\tilde{x}$ is a fixed point of the operator T, $\tilde{x}$ satisfies the following BVP:

$$\begin{cases}D_{0^+}^{\alpha}\tilde{x}(t)+f(t,\tilde{x}^*(t-\tau))+\rho(t)=0,\\ \qquad t\in(0,1)\backslash\{\tau\};\\ \tilde{x}(t)=0,\ t\in[-\tau,0];\\ \tilde{x}'(0)=0,\tilde{x}(1)=0.\end{cases}$$

Let $x(t)=\tilde{x}(t)+\bar{\eta}(t)-\omega(t)$. Thus, if

$$\tilde{x}(t-\tau)+\bar{\eta}(t-\tau)-\omega(t-\tau)\geqslant 0\text{ for }t\in[0,1]\tag{5}$$

$\tilde{x}^*(t-\tau)=\tilde{x}(t-\tau)+\bar{\eta}(t-\tau)-\omega(t-\tau)$. By Remark 2.1, we have the following lemma.

Lemma 2.1 Assume that $\tilde{x}(t)$ is a fixed point of the operator T, then $x(t)=\tilde{x}(t)+\bar{\eta}(t)-\omega(t)$ is a positive solution of BVP (1).

Proof From Lemma 1.1 and Remark 2.1, for $t\in(0,1)\backslash\{\tau\}$, we have

$$D_{0^+}^{\alpha}x(t)=D_{0^+}^{\alpha}\tilde{x}(t)+D_{0^+}^{\alpha}\bar{\eta}(t)-D_{0^+}^{\alpha}\omega(t)=[-f(t,\tilde{x}^*(t-\tau))-\rho(t)]-(-\rho(t))=-f(t,x(t-\tau)).$$

For $t\in[-\tau,0]$, we can obtain

$$x(t)=\tilde{x}(t)+\bar{\eta}(t)-\omega(t)=\eta(t).$$

Next, we show that $x(t)$ satisfies the boundary conditions of BVP (1). Owing to $G(0,s)=0$, $G_t(0,s)=0$, we can deduce

$$x(1)=\tilde{x}(1)+\bar{\eta}(1)-\omega(1)=-\int_0^1 G(0,s)\rho(s)\mathrm{d}s=0,$$

and

$$\omega'_+(0)=\lim_{\Delta t\to 0^+}\frac{1}{\Delta t}\left(\int_0^1 G(\Delta t,s)\rho(s)\mathrm{d}s-\int_0^1 G(0,s)\rho(s)\mathrm{d}s\right)=\lim_{\Delta t\to 0^+}\int_0^1\frac{G(\Delta t,s)-G(0,s)}{\Delta t}\rho(s)\mathrm{d}s=0,$$

which implies that $\omega'(0)=0$. In view of the condition $\eta'_{-}(0)=0$, by the definition of $\bar{\eta}(t)$, we can easily deduce $\eta'(0)=0$. Therefore, we have $x'(0)=\widetilde{x}'(0)+\bar{\eta}'(0)-\omega'(0)=0$. The proof is completed.

To proceed, we need the following assumptions.

(H_1) There exists a nonnegative function $\rho\in C(0,1)\cap L(0,1)$ such that

$$\varphi_2(t)h_2(y)\leqslant f(t,y)+\rho(t)\leqslant \varphi_1(t)[g(y)+h_1(y)]$$

for all $(t,y)\in(0,1)\times(0,+\infty)$, where $\varphi_1,\varphi_2\in L(0,1)$ are nonnegative for $t\in(0,1)$, $h_1,h_2\in C([0,+\infty),[0,+\infty))$ are non-decreasing, and $g\in C([0,+\infty),[0,+\infty))$ is non-increasing.

(H_2) $0<\int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)g(\eta(s-\tau))\mathrm{d}s<+\infty$, and there exists a constant $k>0$ such that

$$\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)\cdot g\left(\frac{k}{2(\alpha-1)}(s-\tau)^{\alpha-1}(1-s+\tau)\right)\mathrm{d}s<+\infty.$$

(H_3) There exists a subinterval $[\beta,\gamma]\subset(\tau,1)$ and a constant $r_1=\max\{k,2c\}$ such that

$$\frac{\xi}{\Gamma(\alpha)}h_2\left(\frac{\sigma r_1}{2(\alpha-1)}\right)\int_\beta^\gamma s(1-s)^{\alpha-1}\varphi_2(s)\mathrm{d}s\geqslant r_1,$$

where

$$\xi=\min_{t\in[\beta,\gamma]}t^{\alpha-1}(1-t),$$
$$\sigma=\min_{s\in[\beta,\gamma]}(s-\tau)^{\alpha-1}(1-s+\tau),$$
$$c=\frac{\alpha-1}{\Gamma(\alpha-1)}\int_0^1\rho(s)\mathrm{d}s<+\infty.$$

In what follows, we will concentrate our study on finding the fixed points of the operator T. Define the cone as follows

$$K=\{x\in P: x(t)\geqslant\frac{(1-t)t^{\alpha-1}}{\alpha-1}\|x\| \text{ for } t\in[0,1]\}.$$

Let $\Omega_1=\{x:\|x\|<r_1\}$, $\Omega_2=\{x:\|x\|<r_2\}$, where $r_2>r_1\geqslant\max\{k,2c\}$. Then, we can prove the following property of the operator T.

Lemma 2.2 Assume that the conditions (H_1) and (H_2) hold, then the operator $T:K\cap(\bar{\Omega}_2\backslash\Omega_1)\to K$ is completely continuous.

Proof We divide the proof into the following four steps.

Step 1 We show that the operator T is well defined on $K\cap(\bar{\Omega}_2\backslash\Omega_1)$.

For any $x\in K\cap(\bar{\Omega}_2\backslash\Omega_1)$, we know that $r_1\leqslant\|x\|\leqslant r_2$, and

$$x(t)\geqslant\frac{1}{\alpha-1}t^{\alpha-1}(1-t)\|x\|\geqslant \frac{1}{\alpha-1}t^{\alpha-1}(1-t)r_1,\ \text{for } t\in[0,1],$$

and on the other hand, from Lemma 1.2, we can obtain

$$\omega(t)\leqslant\frac{\alpha-1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-t)\rho(s)\mathrm{d}s= \frac{1}{\alpha-1}t^{\alpha-1}(1-t)c,\ \text{for } t\in[0,1].$$

Thus,

$$x(t)-\omega(t)\geqslant\frac{1}{\alpha-1}t^{\alpha-1}(1-t)(r_1-c)\geqslant \frac{r_1}{2(\alpha-1)}t^{\alpha-1}(1-t),\ \text{for } t\in[0,1] \quad (6)$$

which together with Lemma 1.1 and conditions (H_1) and (H_2) shows that for $t\in[0,1]$ we have

$$(Tx)(t)=\int_0^\tau G(t,s)[f(s,\eta(s-\tau))+\rho(s)]\mathrm{d}s+\int_\tau^1 G(t,s)[f(s,x(s-\tau)-\omega(s-\tau))+\rho(s)]\mathrm{d}s\leqslant$$
$$\frac{\alpha-1}{\Gamma(\alpha)}\{\int_0^\tau s(1-s)^{\alpha-1}[f(s,\eta(s-\tau))+\rho(s)]\mathrm{d}s+\int_\tau^1 s(1-s)^{\alpha-1}[f(s,x(s-\tau)-\omega(s-\tau))+\rho(s)]\mathrm{d}s\}\leqslant$$
$$\frac{1}{\Gamma(\alpha-1)}\int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)[g(\eta(s-\tau))+h_1(\eta(s-\tau))]\mathrm{d}s+$$
$$\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)[g(x(s-\tau)-\omega(s-\tau))+h_1(x(s-\tau)-\omega(s-\tau))]\mathrm{d}s\}\leqslant$$
$$\frac{1}{\Gamma(\alpha-1)}\int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)[g(\eta(s-\tau))+h_1(\eta(s-\tau))]\mathrm{d}s+$$

$$\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)[g(\frac{r_1}{2(\alpha-1)}(s-\tau)^{\alpha-1}(1-s+\tau))+h_1(x(s-\tau))]\mathrm{d}s\leqslant$$

$$\frac{1}{\Gamma(\alpha-1)}\int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)[g(\eta(s-\tau))+h_1(\eta(s-\tau))]\mathrm{d}s+$$

$$\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)[g(\frac{k}{2(\alpha-1)}(s-\tau)^{\alpha-1}(1-s+\tau))+h_1(r_1))]\mathrm{d}s<+\infty \tag{7}$$

Therefore, the operator T is well defined.

Step 2 We show that $T:K\cap(\overline{\Omega}_2\backslash\Omega_1)\to K$.

For any $y\in K\cap(\overline{\Omega}_2\backslash\Omega_1)$, by Lemma 1.2, we have

$$(Tx)(t)=\int_0^1 G(t,s)[f(s,x^*(s-\tau))+\rho(s)]\mathrm{d}s\leqslant\int_0^1\frac{\alpha-1}{\Gamma(\alpha)}s(1-s)^{\alpha-1}[f(s,x^*(s-\tau))+\rho(s)]\mathrm{d}s,$$

and this implies

$$\| Tx\|\leqslant\frac{\alpha-1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}[f(s,x^*(s-\tau))+\rho(s)]\mathrm{d}s.$$

Thus, for any $t\in[0,1]$, we deduce that

$$(Tx)(t)\geqslant\int_0^1\frac{t^{\alpha-1}(1-t)}{\Gamma(\alpha)}s(1-s)^{\alpha-1}[f(s,x^*(s-\tau))+\rho(s)]\mathrm{d}s\geqslant$$

$$\frac{t^{\alpha-1}(1-t)}{\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha)}{\alpha-1}\| Tx\|=\frac{1}{\alpha-1}t^{\alpha-1}(1-t)\| Tx\|.$$

Therefore, $T:K\cap(\overline{\Omega}_2\backslash\Omega_1)\to K$.

Step 3 We show that $T:K\cap(\overline{\Omega}_2\backslash\Omega_1)\to K$ is continuous.

Let $x_n\to x$ as $n\to\infty$ in $K\cap(\overline{\Omega}_2\backslash\Omega_1)$. Then $x_n(s)\rightrightarrows x(s)$ as $n\to\infty$, and $r_1\leqslant\| x_n\|\leqslant r_2$, $r_1\leqslant\| x\|\leqslant r_2$. Like for Eq.(6), for any $t\in[0,1]$, we know that

$$x_n(t)-\omega(t)\geqslant\frac{r_1}{2(\alpha-1)}t^{\alpha-1}(1-t),x(t)-\omega(t)\geqslant\frac{r_1}{2(\alpha-1)}t^{\alpha-1}(1-t).$$

Together with (H_1) and (H_2), we get

$$|(Tx_n(t))-(Tx)(t)|=$$

$$|\int_\tau^1 G(t,s)[f(s,x_n(s-\tau)-\omega(s-\tau))+\rho(s)]\mathrm{d}s-\int_\tau^1 G(t,s)[f(s,x(s-\tau)-\omega(s-\tau))+\rho(s)]\mathrm{d}s|\leqslant$$

$$\int_\tau^1 G(t,s)\,|f(s,x_n(s-\tau)-\omega(s-\tau))-f(s,x(s-\tau)-\omega(s-\tau))|\,\mathrm{d}s\leqslant$$

$$\frac{1}{\Gamma(\alpha-1)}\int_\tau^1 2s(1-s)^{\alpha-1}\varphi_1(s)[g(x_n(s-\tau)-\omega(s-\tau))+h_1(x_n(s-\tau)-\omega(s-\tau))]\mathrm{d}s\leqslant$$

$$\frac{2}{\Gamma(\alpha-1)}\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)[g(\frac{k}{2(\alpha-1)})(s-\tau)^{\alpha-1}(1-s+\tau)+h_1(r_2)]\mathrm{d}s<+\infty.$$

With the help of Lebesgue's dominated convergence theorem and the continuity of f, we have $\| Tx_n-Tx\|\to 0$ as $n\to\infty$, that is, T is continuous.

Step 4 We prove $T:K\cap(\overline{\Omega}_2\backslash\Omega_1)\to K$ is compact.

First, Eq.(7) indicates that $T(K\cap(\overline{\Omega}_2\backslash\Omega_1))$ is uniformly bounded. Next, we show that $T(K\cap(\overline{\Omega}_2\backslash\Omega_1))$ is equicontinuous. Since $G(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$, for any $\varepsilon>0$, there exists $\eta>0$, for any $t_1,t_2\in[0,1]$ and $|t_1-t_2|<\eta$, we have

$$|G(t_1,s)-G(t_2,s)|<\frac{\varepsilon}{2}\{\int_0^\tau\varphi_1(s)[g(\eta(s-\tau))+h_1(\eta(s-\tau))]\mathrm{d}s+$$

$$\int_\tau^1 \varphi_1(s)[g(\frac{k}{2(\alpha-1)}(s-\tau)^{\alpha-1}(1-s+\tau))+h_1(r_2)]\mathrm{d}s\}^{-1},$$

which is well defined by condition (H_2). Thus, for any $x \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$, we obtain

$$|(Tx)(t_1)-(Tx)(t_2)| \leqslant \int_0^\tau |G(t_1,s)-G(t_2,s)||f(s,\eta(s-\tau))+\rho(s)|\mathrm{d}s+$$
$$\int_\tau^1 |G(t_1,s)-G(t_2,s)||f(s,x(s-\tau)-\omega(s-\tau))+\rho(s)|\mathrm{d}s \leqslant$$
$$\int_0^\tau |G(t_1,s)-G(t_2,s)|\varphi_1(s)[g(\eta(s-\tau))+h_1(\eta(s-\tau))]\mathrm{d}s+$$
$$\int_\tau^1 |G(t_1,s)-G(t_2,s)|\varphi_1(s)[g(\frac{k}{2(\alpha-1)}(s-\tau)^{\alpha-1}(1-s+\tau))+h_2(r_2)]\mathrm{d}s < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

As a result, $T(K \cap (\overline{\Omega}_2 \backslash \Omega_1))$ is equicontinuous on $[0,1]$. Evidently, $T(K \cap (\overline{\Omega}_2 \backslash \Omega_1))$ is also equicontinuous on $[-\tau,0]$. Consequently, by means of Arzela-Ascoli theorem, $T: K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ is completely continuous, which completes the proof.

Now we give the first result of this work.

Theorem 2.1 Assume that conditions (H_1)～(H_3) hold. Suppose that

$$\lim_{x\to+\infty} \frac{h_1(x)}{x}=0 \tag{8}$$

then BVP (1) has at least one positive solution.

Proof For any $x \in \partial\Omega_1$, we have $\|x\|=r_1$. Like for Eq. (6), we get

$$x(t)-\omega(t) \geqslant \frac{r_1}{2(\alpha-1)}t^{\alpha-1}(1-t),\ t \in [0,1].$$

Then, we can deduce from Lemma 1.2 and conditions (H_1),(H_3) that

$$(Tx)(t)=\int_0^\tau G(t,s)[f(s,\eta(s-\tau))+\rho(s)]\mathrm{d}s+\int_\tau^1 G(t,s)[f(s,x(s-\tau)-\omega(s-\tau))+\rho(s)]\mathrm{d}s \geqslant$$
$$\int_\beta^\gamma G(t,s)[f(s,x(s-\tau))+\rho(s)]\mathrm{d}s \geqslant \int_\beta^\gamma \frac{t^{\alpha-1}(1-t)s(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_2(s)h_2(x(s-\tau)-\omega(s-\tau))\mathrm{d}s \geqslant$$
$$\frac{\xi}{\Gamma(\alpha)}\int_\beta^\gamma s(1-s)^{\alpha-1}\varphi_2(s)h_2(\frac{r_1}{2(\alpha-1)}(s-\tau)^{\alpha-1}(1-s+\tau))\mathrm{d}s \geqslant$$
$$\frac{\xi}{\Gamma(\alpha)}h_2(\frac{\sigma r_1}{2(\alpha-1)})\int_\beta^\gamma s(1-s)^{\alpha-1}\varphi_2(s)\mathrm{d}s \geqslant r_1=\|x\|,$$

that is, $\|Tx\| \geqslant \|x\|$ for any $x \in \partial\Omega_1$.

On the other hand, let $\varepsilon>0$ such that $\varepsilon\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)\mathrm{d}s < \Gamma(\alpha-1)$, then (8) yields that there exists $M>0$ such that

$$h_1(x) \leqslant \varepsilon x,\ \text{as } x>M \tag{9}$$

Choose

$$r_2=\max\{\frac{\int_0^\tau s(1-s)^{\alpha-1}[g(\eta(s-\tau))+h_1(\eta(s-\tau))]\mathrm{d}s+\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)g(\frac{k}{2(\alpha-1)}(s-\tau)^{\alpha-1}(1-s+\tau))\mathrm{d}s}{\Gamma(\alpha-1)-\varepsilon\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)\mathrm{d}s},$$
$$M+1,r_1+1\};$$

then for $x \in \partial\Omega_2$, that is $\|x\|=r_2$. As in the case of Eq. (6), we have

$$x(t)-\omega(t) \geqslant \frac{1}{\alpha-1}t^{\alpha-1}(1-t)\|x\|-\frac{1}{\alpha-1}t^{\alpha-1}(1-t)c \geqslant \frac{r_2}{2(\alpha-1)}t^{\alpha-1}(1-t),\ t \in [0,1].$$

As a result, from Eq. (9) and condition (H_1), we can arrive at

$$(Tx)(t)=\int_0^\tau G(t,s)[f(s,\eta(s-\tau))+\rho(s)]\mathrm{d}s+\int_\tau^1 G(t,s)[f(s,x(s-\tau)-\omega(s-\tau))+\rho(s)]\mathrm{d}s\leqslant$$

$$\frac{1}{\Gamma(\alpha-1)}\{\int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)[g(\eta(s-\tau))+h_1(\eta(s-\tau))]\mathrm{d}s+$$

$$\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)[g(\frac{r_2}{2(\alpha-1)}(s-\tau)^{\alpha-1}(1-s+\tau))+h_1(r_2))]\mathrm{d}s\}\leqslant$$

$$\frac{1}{\Gamma(\alpha-1)}\{\int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)[g(\eta(s-\tau))+h_1(\eta(s-\tau))]\mathrm{d}s+$$

$$\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)[g(\frac{k}{2(\alpha-1)}(s-\tau)^{\alpha-1}(1-s+\tau))+\varepsilon r_2]\mathrm{d}s\}\leqslant$$

$$r_2=\|x\|,\ t\in[0,1],$$

that is, $\|Tx\|\leqslant\|x\|$ for any $x\in\partial\Omega_2$.

Therefore, Lemma 1.3 together with Lemma 2.2, asserts that operator T has a fixed point $\tilde{x}\in K\cap(\overline{\Omega}_2\backslash\Omega_1)$. Owing to

$$\tilde{x}(t)-\omega(t)\geqslant\frac{r_1}{2(\alpha-1)}t^{\alpha-1}(1-t)>0,t\in[0,1],$$

it is easy to see that Eq. (5) is satisfied. Hence, Lemma 2.1 guarantees that BVP (1) has at least a positive solution and our conclusion follows.

In order to present another result of this work, we present the following assumptions:

(H_4) There exists a subinterval $[a,b]\subset(\tau,1)$ such that $\int_a^b s(1-s)^{\alpha-1}\varphi_2(s)\mathrm{d}s>0$.

(H_5) There exists $R_1\geqslant\max\{k,2c\}$ such that

$$\frac{\Gamma(\alpha-1)R_1}{G+\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)h_1(R_1)\mathrm{d}s}\geqslant 1,$$

where

$$G:=\int_0^\tau s(1-s)^{\alpha-1}[g(\eta(s-\tau))+h_1(\eta(s-\tau))]\mathrm{d}s+$$

$$\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)g(\frac{k}{2(\alpha-1)}(s-\tau)^{\alpha-1}(1-s+\tau))\mathrm{d}s.$$

Theorem 2.2 Assume that conditions (H_1),(H_2),(H_4) and (H_5) hold. Suppose that

$$\lim_{x\to+\infty}\frac{h_2(x)}{x}=+\infty \tag{10}$$

then BVP (1) has at least one positive solution.

Proof Let $\Omega_1=\{x:\|x\|<R_1\}$, then for any $x\in\partial\Omega_1$, we have $\|x\|=R_1$. Like for Eq. (6), we get

$$x(t)-\omega(t)\geqslant\frac{R_1}{2(\alpha-1)}t^{\alpha-1}(1-t),\ t\in[0,1].$$

In accordance with (H_5) we have

$$(Tx)(t)\leqslant\frac{1}{\Gamma(\alpha-1)}\{\int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)[g(\eta(s-\tau))+h_1(\eta(s-\tau))]\mathrm{d}s+$$

$$\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)[g(\frac{k}{2(\alpha-1)}(s-\tau)^{\alpha-1}(1-s+\tau))+h_1(x(s-\tau))]\mathrm{d}s\}\leqslant$$

$$\frac{G}{\Gamma(\alpha-1)}+\frac{1}{\Gamma(\alpha-1)}\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)h_1(R_1)\mathrm{d}s\leqslant R_1=\|x\|,$$

and hence, $\| Tx \| \leqslant \| x \|$ for any $x \in \partial\Omega_1$.

On the other hand, define $A := \min_{s\in[a,b]} (s-\tau)^{\alpha-1}(1-s+\tau)$, $B := \min_{t\in[a,b]} t^{\alpha-1}(1-t)$. Choose $M^* = \frac{2(\alpha-1)\Gamma(\alpha)}{AB\int_a^b s(1-s)^{\alpha-1}\varphi_2(s)\mathrm{d}s} > 0$, it follows from Eq. (10) that there exists $M > 0$ such that

$$h_2(x) \leqslant M^* x\text{, as } x > M \tag{11}$$

Choose $R_2 = \max\{R_1+1, \frac{2M(\alpha-1)}{A}+1\}$. Let $\Omega_2 = \{x: \| x \| < R_2\}$, then for $x \in \partial\Omega_2$, that is $\| x \| = r_2$. As in the case of Eq. (6), we have

$$x(t)-\omega(t) \geqslant \frac{R_2}{2(\alpha-1)} t^{\alpha-1}(1-t), t \in [0,1].$$

As a result, from Eq. (11) and condition (H_1), we can deduce

$$\begin{aligned}(Tx)(t) = \int_0^\tau G(t,s)[f(s,\eta(s-\tau))+\rho(s)]\mathrm{d}s + \int_\tau^1 G(t,s)[f(s,x(s-\tau)-\omega(s-\tau))+\rho(s)]\mathrm{d}s \geqslant \\ \int_a^b G(t,s)[f(s,x(s-\tau)-\omega(s-\tau))+\rho(s)]\mathrm{d}s \geqslant \\ \int_a^b \frac{t^{\alpha-1}(1-t)s(1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_2(s)h_2(x(s-\tau)-\omega(s-\tau))\mathrm{d}s \geqslant \\ \frac{B}{\Gamma(\alpha)}\int_a^b s(1-s)^{\alpha-1}\varphi_2(s)h_2(\frac{R_2}{2(\alpha-1)}(s-\tau)^{\alpha-1}(1-s+\tau))\mathrm{d}s \geqslant \\ \frac{B}{\Gamma(\alpha)}h_2(\frac{AR_2}{2(\alpha-1)})\int_a^b s(1-s)^{\alpha-1}\varphi_2(s)\mathrm{d}s \geqslant \frac{B}{\Gamma(\alpha)}M^* \frac{AR_2}{2(\alpha-1)}\int_a^b s(1-s)^{\alpha-1}\varphi_2(s)\mathrm{d}s \geqslant R_2 = \| x \|,\end{aligned}$$

that is, $\| Tx \| \leqslant \| x \|$ for any $x \in \partial\Omega_2$.

Consequently, from Lemma 2.2, T has a fixed point $\tilde{x} \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$. Similar to the end of the proof of Theorem 2.1, BVP (1) has at least a positive solution and our conclusion follows. The proof is completed.

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