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New criteria of supersolubility of finite groups

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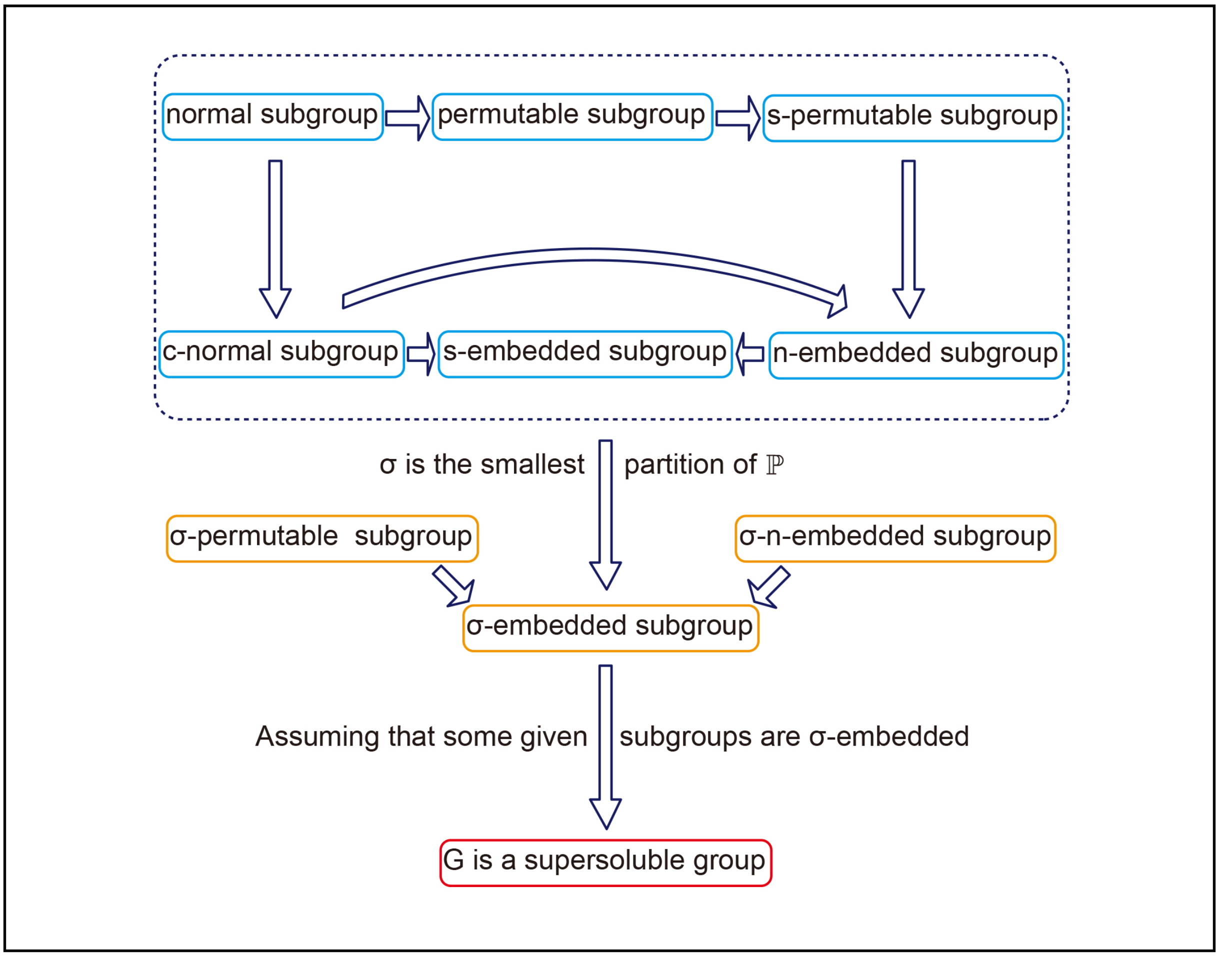
Graphical abstract



Some new criteria of supersolubility of finite groups are obtained by assuming that some given subgroups are σ-embedded.

Public summary

■ Some new properties of σ-embedded subgroups are established.

■ Some new criteria of supersolubility of finite groups are obtained.

■ Some known results in this research field are generalized.

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Abstract: We study the structure of finite groups in which some given subgroups are σ-embedded. In particular, we obtain some new criteria for the supersolubility of finite groups, which generalize some known results.

Keywords: finite group; p-nilpotent group; supersoluble group; σ-permutable subgroup; σ-embedded subgroup

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1 Introduction

Throughout this paper, all groups are finite, and G always denotes a group. Moreover, n is an integer, and $\mathbb{P}$ is the set of all primes. The symbol $\pi(n)$ denotes the set of all primes dividing n and $\pi(G)=\pi(|G|)$, the set of all primes dividing the order of G.

In what follows, $\sigma=\{\sigma_i|i\in I\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P}=\bigcup_{i\in I}\sigma_i$ and $\sigma_i\cap\sigma_j=\emptyset$ for all $i\neq j$. Π is always supposed to be a nonempty subset of the set σ and $\Pi'=\sigma\backslash\Pi$. We write $\sigma(n)=\{\sigma_i|\sigma_i\cap\pi(n)\neq\emptyset\}$ and $\sigma(G)=\sigma(|G|)$.

Following Refs. [1–4], G is said to be σ-primary if $|\sigma(G)|\leqslant 1$. An integer n is a Π-number if $\pi(n)\subseteq\bigcup_{\sigma_i\in\Pi}\sigma_i$. A subgroup H of G is called a Π'-subgroup of G if $|H|$ is a Π'-number. A set $\mathcal{H}$ of subgroups of G is said to be a complete Hall σ-set of G if every nonidentity member of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $i\in I$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every $\sigma_i\in\sigma(G)$. G is said to be σ-full if G possesses a complete Hall σ-set; a σ-full group of Sylow type if every subgroup of G is a D_{σ_i}-group for all $\sigma_i\in\sigma(G)$. A subgroup H of G is called σ-subnormal in G if there exists a subgroup chain $H=H_0\leqslant H_1\leqslant\cdots\leqslant H_t=G$ such that either H_{i-1} is normal in H_i or $H_i/(H_{i-1})_{H_i}$ is σ-primary for all $i=1,\cdots,t$.

It is well known that embedded subgroups and supplemented subgroups play an important role in the finite group theory. For example, a subgroup H of G is said to be c-normal[5] in G if G has a normal subgroup T such that $G=HT$ and $H\cap T\leqslant H_G$, where H_G denotes the maximal normal subgroup of G contained in H. A subgroup H of G is called n-embedded[6] in G if G has a normal subgroup T such that $HT=H^G$ and $H\cap T\leqslant H_{sG}$, where H^G is the largest normal subgroup of G containing H and H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G (note that subgroup A of G is said to be s-permutable in G if $AP=PA$ for any Sylow subgroup P of G). A subgroup H of G is called s-embedded[6] in G if G has an s-permutable subgroup T such that $HT=H^{sG}$ and $H\cap T\leqslant H_{sG}$, where H^{sG} is the intersection of all s-permutable subgroups of G containing H. A subgroup H of G is called σ-n-embedded[7] in G if there exists a normal subgroup T of G such that $HT=H^G$ and $H\cap T\leqslant H_{\sigma G}$, where $H_{\sigma G}$ is the subgroup of H generated by all those subgroups of H that are σ-permutable in G (note that a subgroup H of G is called σ-permutable[2] in G if G possesses a complete Hall σ-set $\mathcal{H}$ such that $HA^x=A^xH$ for all $A\in\mathcal{H}$ and all $x\in G$). A subgroup H of G is called σ-embedded[7] in G if there exists a σ-permutable subgroup T of G such that $HT=H^{\sigma G}$ and $H\cap T\leqslant H_{\sigma G}$, where $H^{\sigma G}$ is the σ-permutable closure of H, that is, the intersection of all σ-permutable subgroups of G containing H. By using the above embedded subgroups and supplemented subgroups, people have obtained a series of interesting results (see, for example, Refs. [5–13]).

Some properties of σ-embedded subgroups were analyzed in Refs. [7, 8]. In this paper, we continue the study of σ-embedded subgroups and use them to determine the structure of finite groups. In particular, we obtain some new criteria for the supersolubility of finite groups.

Theorem 1.1. Let G be a σ-full group of Sylow type and $\mathcal{H}=\{H_1,\cdots,H_t\}$ a complete Hall σ-set of G such that H_i is a nilpotent σ_i-subgroup for all $i=1,\cdots,t$. If every cyclic subgroup of any non-cyclic H_i of prime order and order 4 (if $2\in\pi(H_i)$ and the Sylow 2-subgroup of H_i is nonabelian) is σ-embedded in G, then G is supersoluble.

Theorem 1.2. Let G be a σ-full group of Sylow type and $\mathcal{H}=\{H_1,\cdots,H_t\}$ a complete Hall σ-set of G such that H_i is a nilpotent σ_i-subgroup for all $i=1,\cdots,t$. Suppose that E is a normal subgroup of G with G/E supersoluble. If every cyclic subgroup of any noncyclic $H_i\cap E$ of prime order and order 4 (if $2\in\pi(H_i\cap E)$ and the Sylow 2-subgroup of $H_i\cap E$ is nonabelian) is σ-embedded in G, then G is supersoluble.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries which will be used in this paper. In Section 3, we give the proofs of Theorems 1.1 and 1.2. In Section 4, we give some applications of our results.

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All unexplained terminologies and notations are standard. The reader is referred to Refs. [14–16] if necessary.

2 Preliminaries

Let $\mathcal{L}$ be some nonempty set of subgroups of G and $E \leqslant G$. Following Ref. [2], a subgroup A of G is called $\mathcal{L}$-permutable if $AH = HA$ for all $H \in \mathcal{L}$; $\mathcal{L}^E$-permutable if $AH^x = H^xA$ for all $H \in \mathcal{L}$ and all $x \in E$. In particular, a subgroup H of G is σ-permutable in G if G possesses a complete Hall σ-set $\mathcal{H}$ such that H is $\mathcal{H}^G$-permutable.

Lemma 2.1. [2, Lemma 2.8] Let H, K and N be subgroups of a σ-full group G. Let $\mathcal{H} = \{H_1, \cdots, H_t\}$ be a complete Hall σ-set of G and $\mathcal{L} = \mathcal{H}^K$. Suppose that H *is* $\mathcal{L}$-permutable and N is normal in G. Then, HN/N is $\mathcal{L}^*$-permutable, where $\mathcal{L}^* = \{H_1N/N, \cdots, H_tN/N\}^{KN/N}$. In particular, if H is σ-permutable in G, then HN/N is σ-permutable in G/N.

Following Refs. [1, 2], we use $O^{\Pi}(G)$ to denote the subgroup of G generated by all its Π'-subgroups of G. Instead of $O^{\{\sigma_i\}}(G)$, we write $O^{\sigma_i}(G)$.

Lemma 2.2. [2, Lemma 3.1] Let H be a σ_1-subgroup of a σ-full group G. Then, H *is* σ-permutable in G if and only if $O^{\sigma_1}(G) \leqslant N_G(H)$.

Lemma 2.3. [2, Theorem C] Let G be a σ-full group of Sylow type. Then, the set of all σ-permutable subgroups of G forms a sublattice of the lattice of all σ-subnormal subgroups of G. In particular, for any subgroup A of G, $A^{\sigma G}$ and $A_{\sigma G}$ are both σ-permutable in G.

Lemma 2.4. Let G be a σ-full group of Sylow type and H a subgroup of G. Then, H is σ-embedded in G if and only if there exists a σ-permutable subgroup T of G such that HT is σ-permutable in G and $H \cap T \leqslant H_{\sigma G}$.

Proof. The necessity is evident. Now, we prove the sufficiency. Let $T_1 = T \cap H^{\sigma G}$. Then, by Lemma 2.3, T_1 is σ-permutable in G. Clearly, $HT_1 = H(T \cap H^{\sigma G}) = HT \cap H^{\sigma G} \leqslant H^{\sigma G}$. Then, by the hypothesis and Lemma 2.3, HT_1 is σ-permutable in G. Moreover, since $H^{\sigma G} \leqslant H \leqslant HT_1$, we have that $HT_1 = H^{\sigma G}$. It is clear that $H \cap T_1 \leqslant H \cap T \leqslant H_{\sigma G}$. This shows that H is σ-embedded in G. The lemma is proved.

Lemma 2.5. [6, Lemma 3.1] Let $G = P \rtimes Q$, where P is the Sylow 2-subgroup of G and $|Q| = q$ for some prime number $q \neq 2$. If every cyclic subgroup of P of order 2 and order 4 (if P is a nonabelian group) is s-embedded in G, then G is 2-nilpotent.

Lemma 2.6. [7, Lemma 2.7] Let G be a σ-full group of Sylow type, N a normal subgroup of G and $H \leqslant K \leqslant G$.

① If H is σ-embedded in G, then H is σ-embedded in K.

② Suppose that H is σ-embedded in G and $N \leqslant H$ or $(|H|, |N|) = 1$. Then, HN/N is σ-embedded in G/N.

Lemma 2.7. [17] Let G be a nonsupersoluble group such that all its proper subgroups are supersoluble. Then:

① G has a unique normal Sylow subgroup, says G_p.

② $G_p/\Phi(G_p)$ is a minimal normal subgroup of $G/\Phi(G_p)$.

③ If $p > 2$, then G_p is of exponent p.

3 Proofs of Theorems 1.1 and 1.2

To prove Theorems 1.1 and 1.2, we first prove the following results.

Proposition 3.1. Let G be a σ-full group of Sylow type and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. Suppose that $p \in \sigma_i$ for some $\sigma_i \in \sigma(G)$ and a Hall σ_i-subgroup H_i of G is nilpotent. Then, G is p-nilpotent if and only if all subgroups of P that have order p and order 4 (if P is a nonabelian 2-group) is σ-embedded in G.

Proof. ($\Longrightarrow$) Since G is p-nilpotent, there exists a normal Hall p'-subgroup K of G. Let H be a subgroup of P with order p or order 4 (if P is a nonabelian 2-group). Then, clearly, HK is σ-permutable in G and $H \cap K = 1$. By Lemma 2.4, we have that H is σ-embedded in G.

($\Longleftarrow$) Suppose that this is false and let G be a counter-example of minimal order. We now proceed via the following steps:

(1) Let E be a proper subgroup of G. Then, E is p-nilpotent.

If $p \notin \pi(E)$, then, of course, E is p-nilpotent. If $p \in \pi(E)$, then E satisfies the hypothesis of the theorem by Lemma 2.6①. Hence, E is p-nilpotent by the choice of G.

(2) $G = P \rtimes Q$ is a minimal nonnilpotent group, where Q is a Sylow q-subgroup of G with $p < q$, $P/\Phi(P)$ is a chief factor of G, and if $p \neq 2$, the exponent of P is p.

This directly follows from Claim (1) and Ref. [18, Chapter Ⅳ, Theorem 5.4].

(3) $|Q| = q$.

Assume that $|Q| > q$. Let K be a maximal subgroup of Q. Then, $K \neq 1$ and K is normal in G by Claim (2). Now, we consider the quotient group G/K. Clearly, the hypothesis holds for G/K by Lemma 2.6②. Hence, G/K is p-nilpotent by the choice of G, and so G is p-nilpotent, a contradiction. Hence, (3) holds.

(4) There exists some $\sigma_j \in \sigma(G)$ such that $\sigma_j \cap \pi(G) = \{q\}$, where $j \neq i$.

Assume that this is false. Then, $\sigma_i \cap \pi(G) = \{p, q\}$, and thus, G is a σ_i-group. It follows from the hypothesis that G is nilpotent, a contradiction. Hence, we have (4).

(5) $p \neq 2$.

Assume that $p = 2$. In view of Lemma 2.5, G is 2-nilpotent by Claims (2)–(4), a contradiction. Hence, (5) holds.

(6) Final contradiction.

We claim that $|P/\Phi(P)| = p$. Assume that there exists a minimal subgroup $X/\Phi(P)$ of $P/\Phi(P)$ such that $X/\Phi(P)$ is not σ-permutable in $G/\Phi(P)$. Let $x \in X \backslash \Phi(P)$ and $L = \langle x \rangle$. Then, $X = L\Phi(P)$ and $|L| = p$ by Claims (2) and (5). If $L = L_{\sigma G}$, then by Lemmas 2.1 and 2.3, $X/\Phi(P) = L\Phi(P)/\Phi(P)$ is σ-permutable in $G/\Phi(P)$, contrary to the choice of $X/\Phi(P)$. Hence, $1 = L_{\sigma G} < L$. Then, by the hypothesis and Lemma 2.4, there exists a σ-permutable subgroup T of G such that LT is σ-permutable in G and $L \cap T \leqslant L_{\sigma G} = 1$. Let $T_1 = T \cap P$. Then, T_1 and $LT_1 = L(T \cap P) = LT \cap P$ are both σ-permutable in G by Lemma 2.3. Hence, $O^{\sigma_i}(G) \leqslant N_G(T_1)$ by Lemma 2.2 since T_1 is a σ_i-subgroup. Note that $Q^G \leqslant O^{\sigma_i}(G)$. If $Q^G \neq G$, then Q *char* $Q^G \trianglelefteq G$ by Claim (2), so $Q \trianglelefteq G$, a contradiction. Hence, $Q^G = G$, and so $T_1 \trianglelefteq G$. This shows that $T_1\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$, which implies that $T_1\Phi(P)/\Phi(P) = 1$ or $P/\Phi(P)$ by Claim (2). If $T_1\Phi(P)/\Phi(P) = P/\Phi(P)$, then $T_1 = P$, and so $L = L \cap T_1 \leqslant L \cap T \leqslant L_{\sigma G} = 1$, a contradiction. Hence, $T_1 \leqslant \Phi(P)$. In addition, since LT_1 is σ-permutable in G, we have that $X/\Phi(P) = L\Phi(P)/\Phi(P) = LT_1\Phi(P)/\Phi(P)$ is σ-permutable in $G/\Phi(P)$ by Lemma 2.1, a contradiction. The

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contradiction shows that every minimal subgroup of $P/\Phi(P)$ is σ-permutable in $G/\Phi(P)$. It follows that every minimal subgroup of $P/\Phi(P)$ is s-permutable in $G/\Phi(P)$ since $q\notin\sigma_i$ and $\pi(G)=\{p,q\}$ by Claim (2). Hence, $|P/\Phi(P)|=p$ by Ref. [19, Lemma 2.11]. It follows that P is cyclic of exponent p, which implies that $Q\trianglelefteq G$ since p is the smallest prime dividing $|G|$, a contradiction. This completes the proof.

Proposition 3.2. Let G be a σ-full group of Sylow type and $\mathcal{H}=\{H_1,\cdots,H_t\}$ a complete Hall σ-set of G such that H_i is a nilpotent σ_i-subgroup for all $i=1,\cdots,t$. Let P be a normal p-group of G with G/P supersoluble. If every cyclic subgroup H of P of order p and order 4 (if P is a nonabelian 2-group) is σ-embedded in G, then G is supersoluble.

Proof. Assume that this is false and let (G,P) be a counterexample with $|G|+|P|$ minimal. Without loss of generality, we may assume that $P\leqslant H_1$.

(1) Let E be a proper subgroup of G. Then, E is supersoluble.

It is clear that $(E,E\cap P)$ satisfies the hypothesis by Lemma 2.6①. Hence, E is supersoluble by the choice of (G,P).

(2) p is the largest prime number dividing $|G|$ and $p>2$.

Assume that this is false and let q be the largest prime number dividing $|G|$. Let Q be a Sylow q-subgroup of G. Then, $QP/P\trianglelefteq G/P$ since G/P is supersoluble. It follows that $QP\trianglelefteq G$. Hence, by Lemma 2.6①, every cyclic subgroup H of P of order p and order 4 (if P is a nonabelian 2-group) is σ-embedded in QP. In view of Proposition 3.1, we have that QP is p-nilpotent, and so $Q\trianglelefteq G$. Next, we consider the quotient group G/Q. It is easy to see that $(G/Q,PQ/Q)$ satisfies the hypothesis by Lemma 2.6②. Hence, G/Q is supersoluble by the choice of (G,P). Moreover, since G/P is supersoluble, we obtain that G is supersoluble, a contradiction. Hence, p is the largest prime number dividing $|G|$. It is also clear that $p>2$.

(3) Let P_1 be a Sylow p-subgroup of G. Then, $P_1\trianglelefteq G$.

Since G/P is supersoluble and p is the largest prime number dividing $|G|$ by Claim (2), we have that $P_1/P\trianglelefteq G/P$. This implies that $P_1\trianglelefteq G$.

(4) P_1 is the unique normal Sylow subgroup of G, $P_1/\Phi(P_1)$ is a minimal normal subgroup of $G/\Phi(P_1)$, and the exponent of P_1 is p.

This directly follows from Claims (1)–(3) and Lemma 2.7.

(5) $\Phi(P_1)\neq 1$.

Assume that $\Phi(P_1)=1$. Then, P_1 is a minimal normal subgroup of G by Claim (4), and so $P=P_1$. Let H be a subgroup of P with order p. Then, by the hypothesis and Lemma 2.4, there exists a σ-permutable subgroup T of G such that HT is σ-permutable in G and $H\cap T\leqslant H_{\sigma G}$. Let $T_1=T\cap P$. Then, T_1 and $HT_1=H(T\cap P)=HT\cap P$ are both σ-permutable in G by Lemma 2.3. It follows from Lemma 2.2 that $O^{\sigma_1}(G)\leqslant N_G(T_1)$ since T_1 is a σ_1-subgroup of G. In addition, since $T_1\trianglelefteq P$ and the Hall σ_1-subgroup H_1 of G is nilpotent by the hypothesis, we obtain that $T_1\trianglelefteq H_1$, and thus $T_1\trianglelefteq H_1O^{\sigma_1}(G)=G$. Hence, $T_1=1$ or P by Claim (4). In the former case, we have that $H=HT_1$ is σ-permutable in G. In the later case, we obtain that $H=H\cap T_1\leqslant H\cap T\leqslant H_{\sigma G}$, and thus, $H=H_{\sigma G}$ is σ-permutable in G by Lemma 2.3. It follows that $O^{\sigma_1}(G)\leq N_G(H)$ for H is a subgroup of order p. Then, by a similar argument as above, we have that $H\trianglelefteq G$. Therefore, $H=P$ by Claim (4) since $P=P_1$. This shows that P is a group of order p. It follows from G/P is supersoluble that G is supersoluble, a contradiction. Hence, we have (5).

(6) $P=P_1$.

It is clear that $P\Phi(P_1)/\Phi(P_1)\trianglelefteq G/\Phi(P_1)$. By Claim (4), we have that $P\Phi(P_1)/\Phi(P_1)=1$ or $P_1/\Phi(P_1)$. The former case shows that $P\leqslant\Phi(P_1)$, so $G/\Phi(P_1)\cong(G/P)/(\Phi(P_1)/P)$ is supersoluble. Ref. [18, Chapter Ⅵ, Theorem 8.6(a)] and Claim (5) show that G is supersoluble, a contradiction. Hence, $P\Phi(P_1)/\Phi(P_1)=P_1/\Phi(P_1)$, and thus, $P_1=P\Phi(P_1)$. This shows that $P_1=P$, as desired.

(7) Final contradiction.

Claims (6) and (4) show that the exponent of P is p. Hence, there exists a subgroup H of P with order p such that $H\nleq\Phi(P)$ by Claims (5) and (6). By the hypothesis and Lemma 2.4, there exists a σ-permutable subgroup T of G such that HT is σ-permutable in G and $H\cap T\leqslant H_{\sigma G}$.

We claim that $H_{\sigma G}=1$. If not, then $H=H_{\sigma G}$ is σ-permutable in G by Lemma 2.3. It follows from Lemma 2.1 that $H\Phi(P)/\Phi(P)$ is σ-permutable in $G/\Phi(P)$. Hence, by Lemma 2.2, we have $O^{\sigma_1}(G/\Phi(P))\leqslant N_{G/\Phi(P)}(H\Phi(P)/\Phi(P))$. Moreover, since $H\Phi(P)/\Phi(P)\trianglelefteq P/\Phi(P)$ and the Hall σ_1-subgroup of $G/\Phi(P)$ is nilpotent by the hypothesis, we obtain that $H\Phi(P)/\Phi(P)\trianglelefteq G/\Phi(P)$. In view of Claims (4) and (6), we have that $P=H\Phi(P)=H$, and so P is a cyclic group of order p. This shows that G is supersoluble since G/P is supersoluble, a contradiction. Hence, $H_{\sigma G}=1$.

Let $T_1=T\cap P$. Then, T_1 is σ-permutable in G by Lemma 2.3. Note that T_1 is a σ_1-subgroup of G. Hence, $O^{\sigma_1}(G/\Phi(P))\leqslant N_{G/\Phi(P)}(T_1\Phi(P)/\Phi(P))$ by Lemmas 2.1 and 2.2. With a similar argument as above, we have that $T_1\Phi(P)/\Phi(P)\trianglelefteq G/\Phi(P)$. Hence, $T_1\Phi(P)=\Phi(P)$ or $T_1\Phi(P)=P$ by Claims (4) and (6). If $T_1\Phi(P)=P$, then $P=T_1$, and thus, $H=H\cap T_1\leqslant H\cap T\leqslant H_{\sigma G}=1$, a contradiction. Hence, $T_1\Phi(P)=\Phi(P)$. It is easy to see that $HT_1=H(T\cap P)=HT\cap P$ is also σ-permutable in G and it is a σ_1-subgroup of G. As arguments above, we obtain that $HT_1\Phi(P)=\Phi(P)$ or $HT_1\Phi(P)=P$. It follows from $T_1\Phi(P)=\Phi(P)$ that $HT_1\Phi(P)=H\Phi(P)=\Phi(P)$ or $HT_1\Phi(P)=H\Phi(P)=P$. However, as $H\nleq\Phi(P)$, we have that $H\Phi(P)=P$, and so $P=H$ is a cyclic group of order p. In addition, since G/P is supersoluble, we obtain that G is supersoluble, a contradiction. This contradiction completes the proof.

Proof of Theorem 1.1. We prove this theorem by induction on $|G|$. By Proposition 3.1 and Ref. [18, Chapter Ⅳ, Theorem 2.8], we know that G possesses an ordered Sylow tower of the supersoluble type. Let p be the largest prime dividing $|G|$ and P a Sylow p-subgroup of G. Then, P is normal in G. If $G=P$, then, clearly, G is supersoluble. If $P<G$, then we consider the quotient group G/P. It follows from Lemma 2.6② that G/P satisfies the hypothesis, and thus, G/P is supersoluble by induction on $|G|$. In view of Proposition 3.2, we have that G is supersoluble. This completes the proof.

Proof of Theorem 1.2. Suppose that this theorem is false and let (G,E) be a counterexample with $|G|+|E|$ minimal. Then, $E\neq 1$.

(1) E is supersoluble.

By Lemma 2.6①, E satisfies the hypothesis of Theorem

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1.1. Hence, E is supersoluble by Theorem 1.1.

(2) Let p be the largest prime number dividing $|E|$ and P a Sylow p-subgroup of E. Then, $P \trianglelefteq G$ and $P < E$.

Since E is supersoluble, we have that P *char* $E \trianglelefteq G$, and so $P \trianglelefteq G$. If $P = E$, then G is supersoluble by Proposition 3.2, a contradiction. Hence, $P < E$.

(3) Final contradiction.

We now consider the quotient group G/P. It follows from Lemma 2.6② that the hypothesis holds for $(G/P, E/P)$ since $G/E \cong (G/P)/(E/P)$ is supersoluble. Hence, G/P is supersoluble by the choice of (G, E). Therefore, G is supersoluble by Proposition 3.2. The final contradiction completes the proof.

4 Some applications of our results

It is clear that every σ-n-embedded subgroup and every σ-permutable subgroup of G are σ-embedded in G. Note that if σ is the smallest partition of $\mathbb{P}$, that is, σ_i is a one-element set for any $i \in I$, then every normal subgroup, every permutable subgroup, every c-normal subgroup, every s-permutable subgroup, every s-embedded subgroup and every n-embedded subgroup of G are also σ-embedded in G. However, the converse is not true in general (see Refs. [6, Example 1.2] and [7, Example 1.2]). Hence, we may directly obtain the following results from Theorems 1.1 and 1.2 and Propositions 3.1 and 3.2.

Corollary 4.1. [20, Theorem 3] Let G be a group of odd order. If all subgroups of G of prime order are normal in G, then G is supersoluble.

Corollary 4.2. [9, Theorem 3.1] If every subgroup of prime order and every cyclic subgroup of order 4 are permutable in G, then G is supersoluble.

Corollary 4.3. If every subgroup of prime order and every cyclic subgroup of order 4 are s-permutable in G, then G is supersoluble.

Corollary 4.4. [13, Theorem 3.2] Let P be a Sylow p-subgroup of G, where p is the smallest prime belonging to $\pi(G)$. If every cyclic subgroup of P of order p and order 4 (if P is a nonabelian 2-group) is s-permutable in G, then G is p-nilpotent.

Corollary 4.5. [13, Theorem 3.1] Suppose that P is a normal p-subgroup of G with G/P is supersoluble. Suppose further that every subgroup of P of order p and order 4 (if P is a nonabelian 2-group) is s-permutable in G, then G is supersoluble.

Corollary 4.6. [13, Theorem 3.4] Let N be a proper normal subgroup of G with G/N is supersoluble. Suppose that every subgroup of N of prime order and order 4 (if $2 \in \pi(N)$ and the Sylow 2-subgroup of N is nonabelian) is s-permutable in G, then G is supersoluble.

Corollary 4.7. [10, Lemma 3.1] Let P be a Sylow p-subgroup of G, where p is the smallest prime belonging to $\pi(G)$. If every cyclic subgroup of P of order p and order 4 is c-normal in G, then G is p-nilpotent.

Corollary 4.8. [5, Theorem 4.2] Let G be a group. If every cyclic subgroup of G of prime order and order 4 is c-normal in G, then G is supersoluble.

Corollary 4.9. [12, Theorem 3.4] Let N be a normal subgroup of G with G/N being supersoluble. Suppose that every subgroup of N of prime order and order 4 is c-normal in G, then G is supersoluble.

Corollary 4.10. [13, Corollary 3.9] Let H be a subgroup of G with $G' \leq H$. Suppose that every subgroup of H of prime order and order 4 (if $2 \in \pi(H)$ and the Sylow 2-subgroup of H is nonabelian) is s-permutable in G, then G is supersoluble.

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Conflict of interest

The authors declare that they have no conflict of interest.

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