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Harnack inequality for polyharmonic equations

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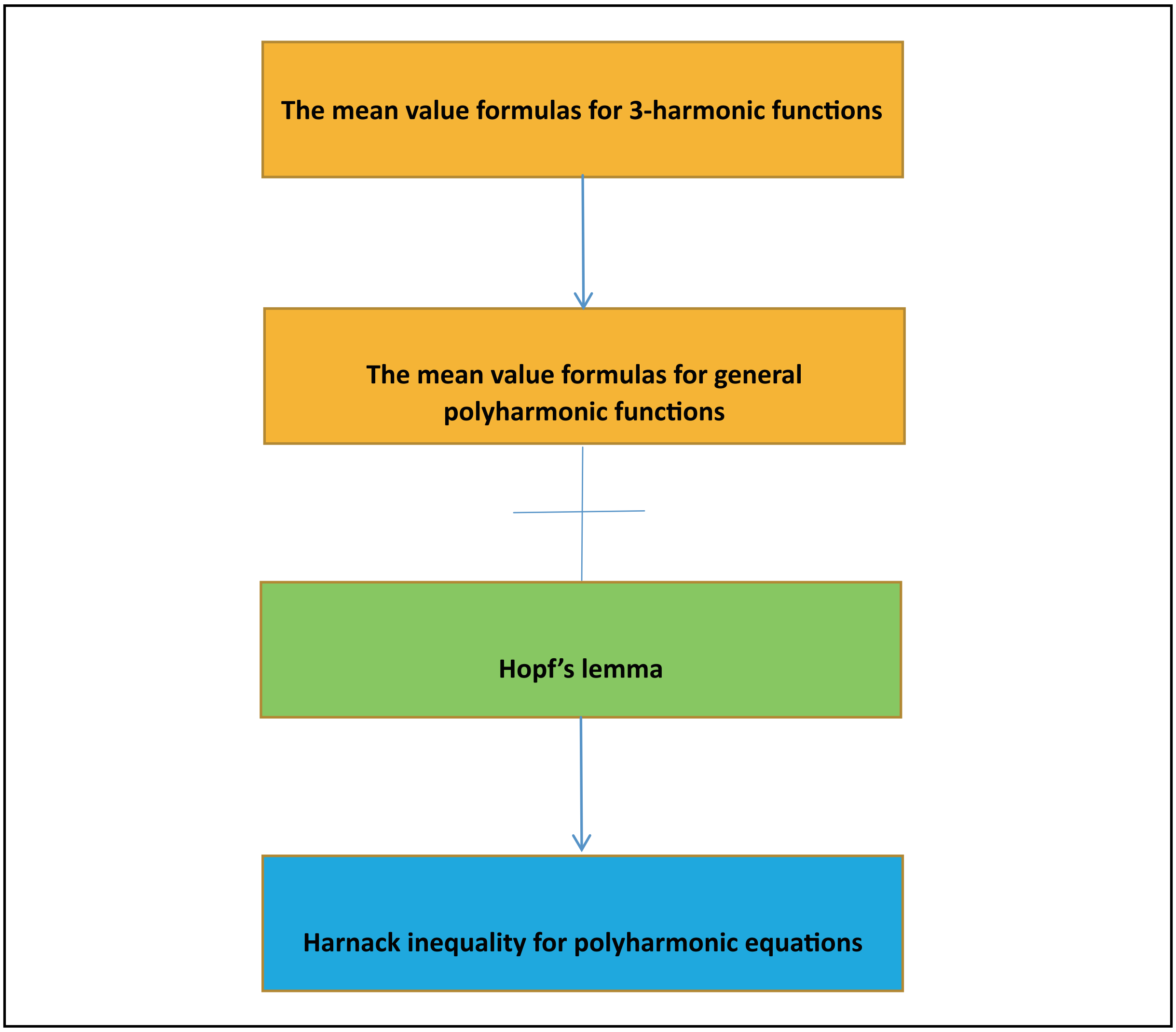
Graphical abstract



Harnack inequality.

Public summary

■ Some new type mean value formulas for polyharmonic functions were established.

■ The Harnack inequality for polyharmonic functions was proved.

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Abstract: Some new types of mean value formulas for the polyharmonic functions were established. Based on the formulas, the Harnack inequality for the nonnegative solutions to the polyharmonic equations was proved.

Keywords: Harnack inequality; mean value formulas; polyharmonic equations; polyharmonic functions; Hopf's lemma

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2020 Mathematics Subject Classification: Primary 35J30; Secondary 35J05, 35B45, 35D30

1 Introduction

The Harnack estimates for the harmonic equation have been investigated profoundly[1, 2]. Let $\Omega\subset\mathbb{R}^n (n\geqslant 2)$ be a connected domain. In 2006, Caristi and Mitidieri[3] considered the Harnack inequality for nonnegative solutions to the biharmonic equation

$$(-\Delta)^2 u(x)=0,\ \ \text{in } \Omega. \tag{1}$$

They used the mean value formulas for the biharmonic functions, which are the solutions to the biharmonic equations and the maximum principle, to prove the Harnack inequality. Motivated by the approaches and results in their work, we will consider the Harnack inequality for the nonnegative weak solutions to the k-harmonic ($k\geqslant 3$) equation

$$(-\Delta)^k u(x)=0,\ \ \text{in } \Omega. \tag{2}$$

The function u that satisfies Eq. (2) is called k-harmonic or polyharmonic. We shall focus on the case $k=3$ and prove the mean value formulas for the 3-harmonic function and for the general k-harmonic function cases by induction argument. Then, we will give the proof of the Harnack inequality for Eq. (2).

Theorem 1.1. Assume that u is a nonnegative weak solution of Eq. (2) such that $-\Delta u\geqslant 0$ in Ω. Then there exists $C=C(n)>0$, such that for any $x\in\Omega$ and each R satisfying $0<2R<\text{dist}(x,\partial\Omega)$ and $B_{2R}(x)\subset\subset\Omega$, it holds that

$$\sup_{B_{R/2}(x)} u\leqslant C\inf_{B_{R/2}(x)} u. \tag{3}$$

Remark 1.1. The assumption $-\Delta u\geqslant 0$ in Ω is necessary. Since if we let $u(x)=x_1^2$ for $x=(x_1,\cdots,x_n)\in\mathbb{R}^n$, then $(-\Delta)^k u(x)=0$ for $k\geqslant 3$, $-\Delta u(x)=-2<0$. However, for any $R>0$, u does not satisfy the Harnack inequality in $B_R(0)$.

The remaining part of this paper is organized as follows: In Section 2, we figure out the mean value formulas for polyharmonic functions, and in Section 3, we give the proof of Theorem 1.1.

2 The mean value formulas

In this section, we first prove the mean value formulas for 3-harmonic functions, and then extend the mean value formulas to the general polyharmonic function cases by the induction argument. The mean value formulas we consider here are different from those in Refs. [4, 5] and references therein.

Definition 2.1. For any $x\in\mathbb{R}^n$, $r>0$, the spherical average of u is defined as

$$\bar{u}(x,r)=\frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)\,\mathrm{d}\sigma_y.$$

Remark 2.1. When there is no ambiguity, we will simply write $\bar{u}(r)$ instead of $\bar{u}(x,r)$. Obviously, we have

$$\bar{u}(r)=\frac{1}{\omega_n}\int_{\partial B_1(0)} u(x+r\xi)\,\mathrm{d}\sigma_\xi,$$

where ω_n is the measure of the unit sphere $\partial B_1(0)$.

The following lemma was very useful in the study of higher-order conformally invariant elliptic equations[6]. For the convenience of the readers, we will give a proof.

Lemma 2.1. For any integer $k\geqslant 1$, it holds that $\Delta^k\bar{u}(r)=\overline{\Delta^k u}(r)$.

Proof. Since $\bar{u}(r)$ is radially symmetric, it follows that

$$\Delta\bar{u}(r)=\bar{u}''(r)+\frac{n-1}{r}\bar{u}'(r)=\frac{1}{r^{n-1}}(r^{n-1}\bar{u}'(r))'.$$

Notice that

$$\bar{u}'(r)=\frac{1}{\omega_n}\int_{\partial B_1(0)}\nabla u(x+r\xi)\cdot\xi\mathrm{d}\sigma_\xi,$$

where ξ is the outward unit normal vector to the boundary $\partial B_1(0)$. Then the divergence theorem[2] implies that

$$r^{n-1}\bar{u}'(r)=\frac{1}{\omega_n}\int_{\partial B_r(x)}\nabla u(y)\cdot\xi\mathrm{d}\sigma_y=\frac{1}{\omega_n}\int_{B_r(x)}\Delta u(y)\mathrm{d}y.$$

Since

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$$\frac{\partial}{\partial r}\int_{B_r(x)}\Delta u(y)\mathrm{d}y=\lim_{t\to r}\frac{\int_{B_t(x)}\Delta u(y)\mathrm{d}y-\int_{B_r(x)}\Delta u(y)\mathrm{d}y}{t-r}=$$
$$\lim_{t\to r}\frac{\int_0^t\int_{\partial B_s(x)}\Delta u(y)\mathrm{d}\sigma_y\mathrm{d}s-\int_0^r\int_{\partial B_s(x)}\Delta u(y)\mathrm{d}\sigma_y\mathrm{d}s}{t-r}=$$
$$\int_{\partial B_r(x)}\Delta u(y)\mathrm{d}\sigma_y,$$

then

$$\Delta\bar{u}(r)=\frac{1}{r^{n-1}}(r^{n-1}\bar{u}'(r))'=\frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)}\Delta u(y)\mathrm{d}\sigma_y=\overline{\Delta u}(r).$$

Therefore, for any integer $k\geq 1$, it can be easily concluded that $\Delta^k\bar{u}(r)=\overline{\Delta^k u}(r)$ by induction.

Now, we give the proof of the mean value formula for 3-harmonic functions by the approach in Ref. [7].

Lemma 2.2. Assume that u is a weak solution to $(-\Delta)^3u=0$ in Ω. For any $x\in\Omega$, denoted by $d_x=\mathrm{dist}(x,\partial\Omega)$, then for any $0<R<d_x$, the following mean value formula holds

$$u(x)=\frac{(n+1)(n+2)}{16}\frac{1}{|B_R(x)|}\int_{B_R(x)}u(y)\cdot$$
$$\left[(n+2)(n+4)-2(n+3)(n+5)\frac{|y-x|}{R}+(n+4)(n+6)\frac{|y-x|^2}{R^2}\right]\mathrm{d}y.$$

Proof. By Weyl's lemma[7], we can prove that $u\in C^\infty(\Omega)$. For any fixed point $x\in\Omega$, we denote by

$$\bar{u}(r)=\frac{1}{\omega_n}\int_{\partial B_1(0)}u(x+r\xi)\,\mathrm{d}\sigma_\xi$$

and

$$\overline{\Delta^2 u}(r)=\frac{1}{\omega_n}\int_{\partial B_1(0)}\Delta^2u(x+r\xi)\,\mathrm{d}\sigma_\xi.$$

Since $\bar{u}(r)$ is radially symmetric, then by Lemma 2.1, for $0<r<d_x$, we have

$$\Delta^2\bar{u}(r)=\bar{u}^{(4)}(r)+\frac{2(n-1)}{r}\bar{u}'''(r)+\frac{n^2-4n+3}{r^2}\bar{u}''(r)-\frac{n^2-4n+3}{r^3}\bar{u}'(r)=\overline{\Delta^2u}(r). \tag{4}$$

Note that $(-\Delta)^3u=0$, so Δ^2u is harmonic. By the mean value formula, we have $\Delta^2u(x)=\overline{\Delta^2u}(r)$ for any $0<r<d_x$. Then, it is easy to check that $\frac{r^4}{8n(n+2)}\Delta^2u(x)$ is a special solution to Eq. (4). Therefore, the general solutions to Eq. (4) can be given by

$$\bar{u}(r)=\begin{cases}c_{11}+c_{12}r^{4-n}+c_{13}r^{2-n}+c_{14}r^2+\dfrac{r^4}{8n(n+2)}\Delta^2u(x), & n\geqslant 5;\\ c_{21}+c_{22}r^{-2}+c_{23}\ln r+c_{24}r^2+\dfrac{r^4}{192}\Delta^2u(x), & n=4;\\ c_{31}+c_{32}r^{-1}+c_{33}r+c_{34}r^2+\dfrac{r^4}{120}\Delta^2u(x), & n=3;\\ c_{41}+c_{42}\ln r+c_{43}r^2\ln r+c_{44}r^2+\dfrac{r^4}{64}\Delta^2u(x), & n=2;\end{cases}$$

where c_{ij} $(i,j=1,\cdots,4)$ are all constant. Next, we calculate these constants.

Consider the case $n\geqslant 5$ first. Since $\bar{u}$ is continuous in $[0,d_x]$ and $\bar{u}(0)=u(x)=c_{11}$, which implies that $c_{12}=c_{13}=0$,

$$u(x)+c_{14}r^2+\frac{r^4}{8n(n+2)}\Delta^2u(x)=\bar{u}(r). \tag{5}$$

Then, taking the Laplacian operator $\Delta=\partial_{rr}^2+\frac{n-1}{r}\partial_r$ on both sides of Eq. (5), we obtain

$$2c_{14}+\frac{3r^2}{2n(n+2)}\Delta^2u(x)+\frac{n-1}{r}\left(2c_{14}r+\frac{r^3}{2n(n+2)}\Delta^2u(x)\right)=\Delta\bar{u}(r),$$

which yields

$$2nc_{14}+\frac{r^2}{2n}\Delta^2u(x)=\Delta\bar{u}(r).$$

Since $\Delta\bar{u}(r)=\overline{\Delta u}(r)$ by Lemma 2.1, letting $r\to 0$, we get $c_{14}=\frac{1}{2n}\Delta u(x)$.

For the cases $n=2,3,4$, we can use the same strategy as the cases $n\geqslant 5$. Therefore, for $n\geqslant 2$, we have the following uniform formula

$$u(x)+\frac{r^2}{2n}\Delta u(x)+\frac{r^4}{8n(n+2)}\Delta^2u(x)=\bar{u}(r). \tag{6}$$

For any $R\in[0,d_x]$, multiplying $\omega_n r^{n-1}$ on both sides of Eq. (6) and integrating with respect to $r\in[0,R]$, we get

$$u(x)+\frac{R^2}{2(n+2)}\Delta u(x)+\frac{R^4}{8(n+2)(n+4)}\Delta^2u(x)=\frac{1}{|B_R(0)|}\int_{B_R(0)}u(x+y)\mathrm{d}y. \tag{7}$$

Fixing $R=r$ in Eq. (7) and combining it with Eq. (6), we obtain

$$u(x)-\frac{r^4}{8(n+2)(n+4)}\Delta^2u(x)=\frac{n+2}{2}\frac{1}{|B_r(0)|}\int_{B_r(0)}u(x+y)\mathrm{d}y-\frac{n}{2\omega_n}\int_{\partial B_1(0)}u(x+r\xi)\mathrm{d}\sigma_\xi. \tag{8}$$

Again, for any $R\in[0,d_x]$, multiplying r^{n+k} $(k\geqslant 0)$ on both sides of Eq. (8) and integrating with respect to $r\in[0,R]$, we obtain

$$\frac{R^{n+k+1}}{n+k+1}u(x)-\frac{R^{n+k+5}}{8(n+2)(n+4)(n+k+5)}\Delta^2u(x)=$$
$$\frac{n(n+2)}{2\omega_n}\int_0^R r^k\int_{B_r(0)}u(x+y)\mathrm{d}y\mathrm{d}r-$$
$$\frac{n}{2\omega_n}\int_0^R r^{n+k}\int_{\partial B_1(0)}u(x+r\xi)\mathrm{d}\sigma_\xi\mathrm{d}r=$$
$$\frac{n(n+2)}{2\omega_n}\int_0^R r^k\int_0^r\rho^{n-1}\int_{\partial B_1(0)}u(x+\rho\xi)\mathrm{d}\sigma_\xi\mathrm{d}\rho\mathrm{d}r-\frac{n}{2}\int_0^R r^{n+k}\bar{u}(r)\mathrm{d}r=$$
$$\frac{n(n+2)}{2}\int_0^R r^k\int_0^r\rho^{n-1}\bar{u}(\rho)\mathrm{d}\rho\mathrm{d}r-\frac{n}{2}\int_0^R r^{n+k}\bar{u}(r)\mathrm{d}r.$$

Let $f(r):=\int_0^r\rho^{n-1}\bar{u}(\rho)\mathrm{d}\rho$, then integration by parts implies that

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$$l\int_0^R r^k \int_0^r \rho^{n-1}\bar{u}(\rho)\mathrm{d}\rho\mathrm{d}r = \frac{R^{k+1}}{k+1}f(R) - \int_0^R \frac{r^{k+1}}{k+1}f'(r)\mathrm{d}r = \frac{R^{k+1}}{k+1}\int_0^R r^{n-1}\bar{u}(r)\mathrm{d}r - \int_0^R \frac{r^{n+k}}{k+1}\bar{u}(r)\mathrm{d}r.$$

It follows that

$$\frac{R^{n+k+1}}{n+k+1}u(x) - \frac{R^{n+k+5}}{8(n+2)(n+4)(n+k+5)}\Delta^2 u(x) = \frac{n(n+2)}{2(k+1)}R^{k+1}\int_0^R r^{n-1}\bar{u}(r)\mathrm{d}r - \frac{n(n+k+3)}{2(k+1)}\int_0^R r^{n+k}\bar{u}(r)\mathrm{d}r.$$

Therefore,

$$u(x) - \frac{(n+k+1)R^4}{8(n+2)(n+4)(n+k+5)}\Delta^2 u(x) = \frac{n+k+1}{2(k+1)}\frac{1}{|B_R(x)|}\int_{B_R(x)} u(y)\left[(n+2)-(n+k+3)\frac{|y-x|^{k+1}}{R^{k+1}}\right]\mathrm{d}y. \tag{9}$$

Plugging $k=0$ and $k=1$ into Eq. (9), we have two special equalities as follows:

$$u(x) - \frac{(n+1)R^4}{8(n+2)(n+4)(n+5)}\Delta^2 u(x) = \frac{n+1}{2}\frac{1}{|B_R(x)|}\int_{B_R(x)} u(y)\left[(n+2)-(n+3)\frac{|y-x|}{R}\right]\mathrm{d}y$$

and

$$u(x) - \frac{R^4}{8(n+4)(n+6)}\Delta^2 u(x) = \frac{n+2}{4}\frac{1}{|B_R(x)|}\int_{B_R(x)} u(y)\left[(n+2)-(n+4)\frac{|y-x|^2}{R^2}\right]\mathrm{d}y,$$

which implies that

$$u(x) = \frac{(n+1)(n+2)}{16}\frac{1}{|B_R(x)|}\int_{B_R(x)} u(y)\cdot \left[(n+2)(n+4) - 2(n+3)(n+5)\frac{|y-x|}{R} + (n+4)(n+6)\frac{|y-x|^2}{R^2}\right]\mathrm{d}y.$$

By the similar arguments, generally, we can obtain the mean value formulas for k-harmonic functions.

Lemma 2.3. Assume that u is a weak solution to $(-\Delta)^k u = 0$ in Ω, $k \geqslant 3$. For any $x \in \Omega$, denoted by $d_x = \mathrm{dist}(x, \partial\Omega)$, then for any $0 < R < d_x$, the following mean value equality holds

$$u(x) = a(n,k)\frac{1}{|B_R(x)|}\int_{B_R(x)} u(y)\sum_{i=0}^{k-1} b_i(k)\frac{|y-x|^i}{R^i}\mathrm{d}y,$$

where $a(n,k) = \dfrac{\prod_{m=1}^{k-1}(n+m)}{2^{k-1}[(k-1)!]^2}$ and $b_i(k) = (-1)^i C_{k-1}^i \cdot \prod_{m=1}^{k-1}(n+2m+i)$, $i = 0, \cdots, k-1$.

Proof. If we denote by

$$f(R,x) = a(n,k)\frac{1}{|B_R(x)|}\int_{B_R(x)} u(y)\sum_{i=0}^{k-1} b_i(k)\frac{|y-x|^i}{R^i}\mathrm{d}y,$$

where $a(n,k) = \dfrac{\prod_{m=1}^{k-1}(n+m)}{2^{k-1}[(k-1)!]^2}$ and $b_i(k) = (-1)^i C_{k-1}^i \cdot \prod_{m=1}^{k-1}(n+2m+i)$, $i = 0, \cdots, k-1$. Then by direct calculation, we get

$$\frac{\partial}{\partial R}f(R,x) = 0$$

and

$$\lim_{R\to 0} f(R,x) = u(x),$$

which finishes the proof.

3 Proof of Theorem 1.1

Now, we can give the proof of Theorem 1.1.

Proof. On the one hand, for any $x \in \Omega$, $0 < 2R < \mathrm{dist}(x, \partial\Omega)$, by Lemma 2.3 we have

$$u(x) \leqslant C(n,k)\frac{1}{|B_R(x)|}\int_{B_R(x)} u(y)\mathrm{d}y.$$

If $z \in B_{R/2}(x)$, we have $B_{R/2}(z) \subset B_R(x)$ and

$$u(z) \leqslant C(n,k)\frac{1}{|B_{R/2}(z)|}\int_{B_{R/2}(z)} u(y)\mathrm{d}y \leqslant C(n,k)\frac{2^n}{|B_R(x)|}\int_{B_R(x)} u(y)\,\mathrm{d}y.$$

Therefore,

$$\sup_{B_{R/2}(x)} u \leqslant \frac{C(n,k)}{|B_R(x)|}\int_{B_R(x)} u(y)\mathrm{d}y. \tag{10}$$

The value of the above positive constants $C(n,k)$ may vary in different places.

On the other hand, since $u \geqslant 0$ and $-\Delta u \geqslant 0$ in Ω, by the mean value inequality, for any $x \in \Omega$, $t > 0$, if $B_t(x) \subset\subset \Omega$, then

$$u(x) \geqslant \frac{1}{|B_t(x)|}\int_{B_t(x)} u(y)\mathrm{d}y.$$

By Hopf's lemma[1, 2], we have $\inf_{B_{R/2}(x)} u = \inf_{\partial B_{R/2}(x)} u$. Without loss of generality, we assume the minimum point $x_0 \in \partial B_{R/2}(x)$, then $u(x) \geqslant u(x_0)$. Obviously, $B_{R/2}(x) \subset B_R(x) \subset B_{3R/2}(x_0) \subset\subset \Omega$, so we have

$$\inf_{B_{R/2}(x)} u = u(x_0) \geqslant \frac{1}{|B_{3R/2}(x_0)|}\int_{B_{3R/2}(x_0)} u(y)\mathrm{d}y \geqslant \left(\frac{2}{3}\right)^n \frac{1}{|B_R(x)|}\int_{B_R(x)} u(y)\,\mathrm{d}y. \tag{11}$$

By (10) and (11), the proof is completed.

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Conflict of interest

The authors declare that they have no conflict of interest.

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