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# Bowley reinsurance with asymmetric information under reinsurer's default risk

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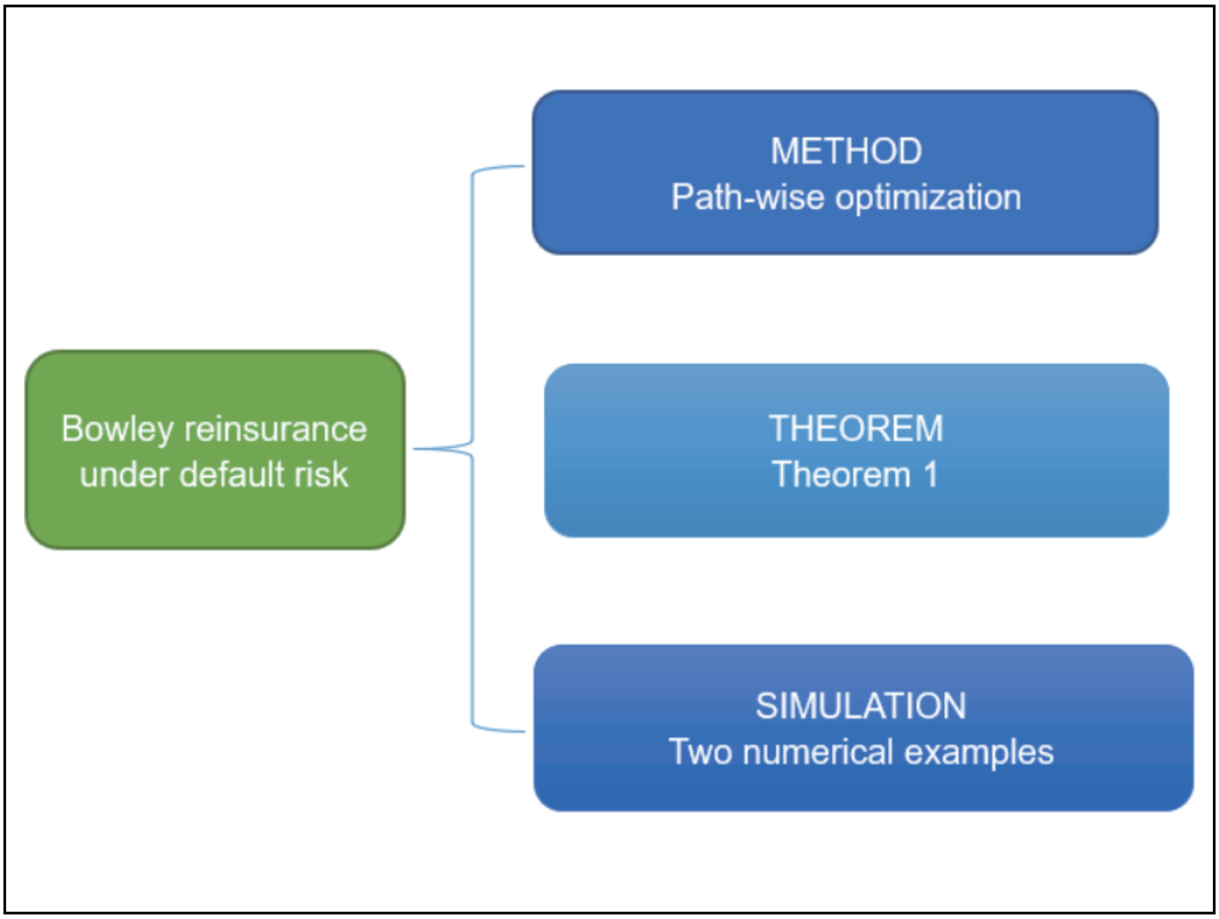
## Graphical abstract




*The method, theorem, and simulation study for Bowley reinsurance under default risk.*

## Public summary

■ We study the problem of Bowley reinsurance with asymmetric information under the reinsurer's default risk.

■ We adopt the path-wise optimization to solve the problem of Bowley reinsurance under default risk.

■ We give two numerical examples to illustrate Theorem 1.

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**Abstract:** Bowley reinsurance with asymmetric information means that the insurer and reinsurer are both presented with distortion risk measures but there is asymmetric information on the distortion risk measure of the insurer. Motivated by predecessors research, we study Bowley reinsurance with asymmetric information under the reinsurer's default risk. We call this solution the Bowley solution under default risk. We provide Bowley solutions under default risk in a closed form under general assumptions. Finally, some numerical examples are provided to illustrate our main conclusions.

**Keywords:** Bowley reinsurance; asymmetric information; distortion-deviation premium principle; distortion risk measure; default risk

**CLC number:** F224.7 **Document code:** A

**2020 Mathematics Subject Classification:** C61; G22; G32

## 1 Introduction

Reinsurance is an effective risk management tool in today's complex business environment and has been an important research field. Since the seminal paper of Borch[1], optimal (re)insurance design problems have been widely studied in the literature. Borch[1] showed that stop-loss reinsurance is optimal by minimizing the variance of an insurer's total risk exposure under the expected value premium principle. Borch's result has constantly been extended in many directions. One direction is to consider the expected utility maximization[2–4]. Another direction is to consider the same optimal problem under the risk management framework. For example, Cai et al.[5] and Cheung[6] minimized the value-at-risk (VaR) and tail value-at-risk (TVaR) of the insurer's total risk exposure. Cui et al.[7] and Cheung et al.[8] considered the problem under general distortion risk measures including VaR, TVaR.

As noted by Borch[1], an insurer and a reinsurer may have conflicting interests under a reinsurance contract. If they cooperate, in the same work, Borch gained the optimal retention of the quota-share and stop-loss reinsurance contracts to maximize the product of individual expected utility at the end. Following this way, this research line leads to Pareto-optimal[9–11]. Later, Borch[12] considered that a reinsurance contract could be attractive to one party, but may not be acceptable to another. One of the best ways to solve this noncooperative conflict between reinsurer and insurer is the theory of Nash equilibrium[13, 14]. Under different assumptions, Aase[13] and Boonen et al.[14] adopted the Nash bargaining framework to price reinsurance contracts. Except for Nash equilibrium, some other important equilibrium concepts, such as the Bowley solution, the Stackelberg game, and the principal-agent problem, are also used in optimal reinsurance theory. For instance, in the Stackelberg game, one player (the 'leader') chooses first, and all other players (the 'followers') move after the leader. In other words, there is an order in the game; once the buyer chooses her indemnity, then the seller cannot modify its premium[15–17]. And under the principal-agent problem, the principal (monopoly) has the right to determine the optimal insurance contracts and the corresponding premiums charged to each type of agents. Meanwhile, the principal can only rely on the prior knowledge of the proportion of each type, not the hidden characteristics of any single agent[18, 19].

Recently, there has been increasing interest in studying Bowley reinsurance since Ref. [16]. Cheung et al.[16] focused on the preferences given by distortion risk measures. As far as we know, Chan et al.[20] first used the nature of the reinsurer's monopolistic and built a Bowley solution (Stackelberg equilibria) of equilibrium reinsurance arrangements to maximize the expected utility of both the insurer and reinsurer in order, and then back. Motivated by Boonen et al.[21] and Boonen et al.[22], we consider Bowley reinsurance solutions with asymmetric information under the reinsurer's default risk with a general pricing principle. Asymmetric information refers to that the insurer is given a distortion risk measure but the reinsurer does not have any idea about the preferences of the insurer. The reinsurer sets the premium principle first to the insurer, and the insurer decides his own optimal reinsurance indemnity based on the premium principle. For the proceedings of Bowley solutions, we refer to Ref. [22].

However, in the above mentioned papers involving Bowley solutions, it is assumed that the reinsurer will be able to compensate for the losses incurred. In reality, the reinsurer could fail to pay the promised part of loss if the loss is huge, which


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is denoted as default risk. There is an extensive literature on default risk[24–27]. Therefore, the aim of the paper is to fill the gap between default risk and the Bowley solutions. We follow the framework of Asimit et al.[24] and Lo[27]. Note that there is a difference between this paper and Ref. [24] in the calculation of the premium principle. In this paper, the premium principle bases on the promised part of loss whereas Asimit et al.[24] considered the default risk. The reason for this is that even if there is a default in a (re)insurance contract, the insurer does not know this in advance. Our results indicate that the optimal reinsurance indemnity depends on the default rate, i.e., with the increase in the default rate, the insurer cedes less risk to the reinsurer and retains more risk. Finally, examples are also given to illustrate the main results, where the explicit expressions for the optimal reinsurance treaties are provided.

The rest of this paper is organized as follows. Section 2 states the asymmetric information and default risk problem studied in this paper. Section 3 solves this problem. In section 4 we give two examples to illustrate the main result when the two distortion functions of the insurer are ordered. Section 5 concludes the paper.

## 2 Preliminaries

Throughout, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. We assume that the total loss faced by the insurer is denoted by a bounded, nonnegative random loss variable $X$, and its cumulative distribution function and survival function are denoted by $F_X$ and $S_X$, respectively. We assume that both the insurer and the reinsurer are known about $F_X$ and $S_X$. The quantile of $X$ at level $p\in[0,1]$ is denoted by $F_X^{-1}(p):=\inf\{x\in\mathbb{R}_+|F_X(x)\geqslant p\}$. Denote $\mathbb{I}_A(s)$ as the indicator function such that $\mathbb{I}_A(s)=1$ for $s\in A$ and $\mathbb{I}_A(s)=0$ for $s\notin A$. We also denote the following two sets:

$$\mathcal{G}_d:=\{g:[0,1]\mapsto[0,1]|g(0)=0,g(1)=1,\ g \text{ is a nondecreasing and left-continuous}\},$$

and

$$\mathcal{G}=\{g:[0,1]\mapsto\mathbb{R}_+|g \text{ is continuous and concave},\ g(0)=0\}.$$

### 2.1 Indemnities, distortion risk measures and premium principles

To manage the risk explosion, the insurer cedes the risk $f(X)$ to the reinsurer, where $f(x)$ is called a ceded function. Avoiding moral hazard or insurance swindles[28], we impose an incentive compatibility constraint on $f$, i.e., we assume that $f\in\mathcal{F}$ with

$$\mathcal{F}=\{f:\mathbb{R}_+\mapsto\mathbb{R}_+|f(0)=0,\ 0\leqslant f(x)-f(y)\leqslant x-y \text{ for any } 0\leqslant y\leqslant x\}.$$

Under the incentive compatibility constraint, it is easy to see that $f$ is almost everywhere differentiable on $\mathbb{R}_+$. Moreover, there exists a Lebesgue integrable function $h:\mathbb{R}_+\mapsto[0,1]$ such that

$$f(x)=\int_0^x h(t)\mathrm{d}t,\ x\geqslant 0, \tag{1}$$

where $h$ is the slope of the ceded loss function $f$. Zhuang et al.[29] termed $h$ as a marginal indemnification function. Next, we discuss the preferences of the insurer. Following most of the literature, we assume that the insurer wants to minimize its distortion risk measure. The formal definition is given below.

**Definition 1.** For $g\in\mathcal{G}_d$ or $g\in\mathcal{G}$, a functional $\rho_g$ of a nonnegative random variable $Z$ is given by

$$\rho_g(Z)=\int_0^\infty g(S_Z(z))\mathrm{d}z, \tag{2}$$

whenever the integral exists. When $g\in\mathcal{G}_d$, we call $\rho_g$ as distortion risk measure.

Note that, the class of distortion risk measure is a big umbrella under which many common risk measures fall, for example the most notably VaR, whose definition is recalled below.

**Definition 2.** The VaR of a random variable $Z$ at the probability level of $p\in(0,1]$ is the left-continuous inverse of its distribution function $F_Z$ at $p$:

$$\mathrm{VaR}_p(Z):=F_Z^{-1}(p)=\inf\{x\in\mathbb{R}|F_Z(x)\geqslant p\},$$

with the convention $\inf\emptyset=+\infty$. The VaR is a distortion risk measure with distortion function $g(t)=\mathbb{I}_{(1-\alpha,1]}(t)$.

When the insurer cedes loss $f(X)$ to a reinsurer, the reinsurer charges premium in turn. In this paper, we assume that the premium principle is comonotonically additive and law invariant, but not necessarily monotone. In particular, for any ceded loss function $f\in\mathcal{F}$ and $g_r\in\mathcal{G}$, we define the reinsurance premium principle

$$\Pi_{g_r}(f(X))=(1+\theta)\mathbb{E}f(X)+\rho_{g_r}(f(X))=(1+\theta)\mathbb{E}f(X)+\int_0^\infty g_r(S_X(t))h(t)\mathrm{d}t,\ \theta>-1, \tag{3}$$

where $h$ satisfies Eq. (1) and the second equality follows from Lemma 1 below. Note that we do not require $\Pi_{g_r}$ to be monotone because $g_r\in\mathcal{G}$ need not be monotone. The above distortion-deviation premium principle $\Pi_{g_r}$ was recently introduced by Liang et al.[23] who derived the form by analogy with the mean-standard deviation premium principle. The distortion-deviation premium principle encompasses a large class of premium principles typically used in the literature, such as the expected premium principle, Wang's premium principle[30], Gini premium principle, and the absolute deviation premium principle[31]. Liang et al.[23] termed (3) as canonical representation. They showed that there exist two functions $h_1,h_2\in\mathcal{G}$ such that

$$\Pi_{g_r}(f(X))=\rho_{h_1}[f(X)]+\rho_{h_2}[f(X)],$$

where $h_1$ is nondecreasing and $h_2$ is a deviation distortion, i.e., $h_2\in\mathcal{G}$ and $h_2(0)=h_2(1)=0$.

Finally, we end this subsection with a lemma, which is a useful tool for our following analysis.

**Lemma 1.** Let $f:\mathbb{R}_+\mapsto\mathbb{R}_+$ be a nondecreasing, absolutely continuous function with $f(0)=0$ and $Z$ be a nonnegative random variable. Then, for $g\in\mathcal{G}_d$ or $g\in\mathcal{G}$, we have


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$$\rho_g[f(Z)] = \int_0^\infty g(S_Z(t))h(t)\mathrm{d}t,$$

where $h$ is the derivative of $f$, which exists almost everywhere.

### 2.2 Bowley reinsurance solutions with asymmetric information under default risk

The main goal in this paper is to incorporate default risk under the setting of Bowley reinsurance with asymmetric information. We follow the framework of Asimit et al.[24], who were motivated by the recent implementation of the Solvency Ⅱ regulatory framework in the countries of the European Union. Assume that the reinsurer operates in a VaR-regulated environment and thus prescribes its regulatory capital in accordance with the $\beta$-level VaR of the reinsured loss for some large probability level $\beta$ (e.g., $\beta = 99.5\%$ in Solvency Ⅱ). If $f:\mathbb{R}_+ \mapsto \mathbb{R}_+$ represents the reinsurance indemnity function and $\delta \in [0,1]$ is the recovery rate of the loss given default, then the insurer is in effect compensated for $f(X)\wedge \mathrm{VaR}_\beta[f(X)] + \delta(f(X) - \mathrm{VaR}_\beta[f(X)])_+$. Obviously, the no-default scenario is recovered if we set $\delta = 1$. It is easy to see that the greater $\delta$, the less likely the default. For more on default risk, we refer to Refs. [26, 27].

Under the setting of Bowley reinsurance, we assume that the reinsurer does not have any idea about the distortion risk measures used by the insurer, while the reinsurer only knows that the insurer may have finitely many possible distortion risk measures. For brevity of our result, we consider the case in which there are only two possible distortion risk measures of the insurer. To be precise, the reinsurer holds the opinion that the insurer minimizes either $\rho_{g_{i1}}$ or $\rho_{g_{i2}}$ with probability $p$ and $1-p$, respectively, where $p\in[0,1]$ and $\{g_{i1}, g_{i2}\}\subset \mathcal{G}_d$ are the two possible distortion functions adopted by the insurer.

Recall that there exists default risk on the reinsurer. Denote the reinsurance indemnity function $f\in\mathcal{F}$. Then the total retained loss for the insurer is equal to $X - f(X)\wedge \mathrm{VaR}_\beta[f(X)] - \delta(f(X)-\mathrm{VaR}_\beta[f(X)])_+ + \Pi_{g_r}(f(X))$, rather than $X - f(X) + \Pi_{g_r}(f(X))$, where $\Pi_{g_r}$ is the distortion-deviation premium principle given by (3). Note, we assume that even if default risk exists, the premium principle is calculated based on $f(X)$, rather than $f(X)\wedge \mathrm{VaR}_\beta[f(X)] + \delta(f(X)-\mathrm{VaR}_\beta[f(X)])_+$. This is different from Ref. [24]. Because we believe that even if there is a default risk in the future, the reinsurer charges the most desirable premium at first. In conclusion, the two-step game played by the insurer and the reinsurer is formalized as follows:

**Decision problem faced by the insurer.** For any given $g_r\in\mathcal{G}$ provided by the reinsurer, the insurer chooses the optimal ceded loss function $f\in\mathcal{F}$ by solving

$$\min_{f\in\mathcal{F}} \rho_{g_i}(X - f(X)\wedge \mathrm{VaR}_\beta[f(X)] + \delta(f(X)-\mathrm{VaR}_\beta[f(X)])_+ + \Pi_{g_r}(f(X))), \tag{4}$$

where $g_i = g_{i1}$ or $g_i = g_{i2}$, depending on the type of the insurer.

**Decision problem faced by the reinsurer.** The reinsurer is uncertain about the type of the insurer, but knows the distortion functions $g_{i1}$ and $g_{i2}$ and probability $p$. Thus for the reinsurer, the goal is to select the optimal reinsurance premium generating function $g_r^*$ by maximizing the expected net profit. Then, the optimization problem of interest is

$$\begin{aligned}&\max_{g_r\in\mathcal{G}} W(g_r; f_{g_r,g_{i1}}, f_{g_r,g_{i2}}) := \\ &\max_{g_r\in\mathcal{G}}\{p\{\mathbb{E}[\Pi_{g_r}(f_{g_r,g_{i1}}(X)) - f_{g_r,g_{i1}}(X)] - C(f_{g_r,g_{i1}}(X))\} + \\ &(1-p)\{\mathbb{E}[\Pi_{g_r}(f_{g_r,g_{i2}}(X)) - f_{g_r,g_{i2}}(X)] - C(f_{g_r,g_{i2}}(X))\}\},\end{aligned} \tag{5}$$

where $C(f_{g_r,g_{ij}})$ denotes the aggregate administrative cost paid by the reinsurer if the insurer purchases the policy $f_{g_r,g_{ij}}$, for $j=1,2$.

In the absence of default, i.e., $\delta = 1$, Boonen and Zhang[22] also considered the above two-step game. When the insurer chooses the indemnity function that is optimal for him/her, the reinsurer will know the type of the insurer that is revealed by indemnity selection. Thus, the problem of this paper is summarized as follows:

$$\begin{aligned}&\max_{g_r\in\mathcal{G}} W(g_r; f_{g_r,g_{i1}}, f_{g_r,g_{i2}}),\\ &\text{s.t. } f_{g_r,g_{ij}} \in \arg\min_{f\in\mathcal{F}} \rho_{g_{ij}}(X - f(X)\wedge \mathrm{VaR}_\beta[f(X)] - \\ &\qquad \delta(f(X)-\mathrm{VaR}_\beta[f(X)])_+ + \Pi_{g_r}(f(X))),\end{aligned}$$

where $W(g_r; f_{g_r,g_{i1}}, f_{g_r,g_{i2}})$ is the expected net profit of the reinsurer in Eq. (5) for given indemnity functions $f_{g_r,g_{i1}}$ and $f_{g_r,g_{i2}}$. Solutions are called Bowley solutions under default risk.

For problem (4), similar problem has been solved in the literature[16, 24, 27, 29]. We state it in the following proposition and provide a self-contained proof.

**Proposition 1.** For any $g_r\in\mathcal{G}$, the optimal ceded loss function $f_{g_r,g_{ij}}$ that solves problem (4) is given by

$$f^*_{g_r,g_{ij}}(x) = \mu(\{z\in[0,x] | \psi_j(F_X(z))>0\}) + \int_0^x \xi_j(z)\mathbb{I}_{\{\psi_j(F_X(z))=0\}}\mu(\mathrm{d}z),\ x\geqslant 0, \tag{6}$$

where the function $\psi_j$ is defined as

$$\psi_j(z) := \delta g_{ij}(1-z) + (1-\delta)g_{ij}(1-z)\mathbb{I}_{\{F_X^{-1}(z)<VaR_\beta(X)\}} - ((1+\theta)(1-z) + g_r(1-z)),$$

and $\xi_j$ is any measurable function with $0\leqslant \xi_j(z)\mathbb{I}_{\{\psi_j(F_X(z))=0\}}\leqslant 1$, for $j=1,2$.

**Proof.** Due to translation invariance, we can rewrite (4) as

$$\rho_{g_{ij}}(X - f(X)\wedge \mathrm{VaR}_\beta[f(X)] - \delta(f(X)-\mathrm{VaR}_\beta[f(X)])_+) + \Pi_{g_r}(f(X)).$$

Being nondecreasing, 1-Lipschitz functions of the ground-up loss $X$, the three random variables,

$$\begin{aligned}&X - f(X)\wedge \mathrm{VaR}_\beta[f(X)] - \delta(f(X)-\mathrm{VaR}_\beta[f(X)])_+,\\ &f(X)\wedge \mathrm{VaR}_\beta[f(X)],\\ &\delta(f(X)-\mathrm{VaR}_\beta[f(X)])_+,\end{aligned}$$

are all comonotonic. By virtue of the comonotonic additivity and positive homogeneity of $\rho_{g_{ij}}$, we further have

$$\begin{aligned}&\rho_{g_{ij}}[X - f(X)\wedge \mathrm{VaR}_\beta[f(X)] - \delta(f(X)-\mathrm{VaR}_\beta[f(X)])_+] = \\ &\rho_{g_{ij}}[X] - \rho_{g_{ij}}[f(X)\wedge \mathrm{VaR}_\beta[f(X)]] - \\ &\delta\rho_{g_{ij}}[(f(X)-\mathrm{VaR}_\beta[f(X)])_+].\end{aligned}$$


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Apply Lemma 1 to the two absolutely continuous functions (noting that $\mathrm{VaR}_\beta[f(X)]$ is merely a constant):

$$x \mapsto f(x) \wedge \mathrm{VaR}_\beta[f(X)] \text{ and } x \mapsto (f(x) - \mathrm{VaR}_\beta[f(X)])_+,$$

whose derivatives are almost everywhere equal to

$$h(x)\mathbb{I}_{\{x<\mathrm{VaR}_\beta(X)\}} \text{ and } x \mapsto h(x)\mathbb{I}_{\{x\geqslant\mathrm{VaR}_\beta(X)\}} = h(x)(1-\mathbb{I}_{\{x<\mathrm{VaR}_\beta(X)\}}),$$

respectively. Then,

$$\begin{aligned}
&\rho_{g_{ij}}[X - f(X)\wedge \mathrm{VaR}_\beta[f(X)] - \\
&\quad \delta(f(X)-\mathrm{VaR}_\beta[f(X)])_+ + \Pi_{g_r}(f(X))] = \\
&\rho_{g_{ij}}[X] - \rho_{g_{ij}}[f(X)\wedge \mathrm{VaR}_\beta[f(X)]] - \\
&\delta\rho_{g_{ij}}[(f(X)-\mathrm{VaR}_\beta[f(X)])_+] + \Pi_{g_r}(f(X)) = \\
&\rho_{g_i}[X] - \int_0^\infty g_{ij}(S_X(z))\mathbb{I}_{\{z<\mathrm{VaR}_\beta(X)\}}h(z)\mathrm{d}z - \\
&\delta\int_0^\infty g_{ij}(S_X(z))(1-\mathbb{I}_{\{z<\mathrm{VaR}_\beta(X)\}})h(z)\mathrm{d}z + \\
&\int_0^\infty ((1+\theta)S_X(z) + g_r(S_X(z)))h(z)\mathrm{d}z = \\
&\rho_{g_i}[X] - \delta\int_0^\infty g_{ij}(S_X(z))h(z)\mathrm{d}z - \\
&(1-\delta)\int_0^\infty g_{ij}(S_X(z))\mathbb{I}_{\{z<\mathrm{VaR}_\beta(X)\}}h(z)\mathrm{d}z + \\
&\int_0^\infty ((1+\theta)S_X(z) + g_r(S_X(z)))h(z)\mathrm{d}z = \\
&\rho_{g_i}[X] - \int_0^\infty \psi_j(z)h(z)\mathrm{d}z.
\end{aligned}$$

To minimize the objective function in the form of the above equation over all $f\in\mathcal{F}$, we use the method that minimizes the integrand. To this end, we can easily see that

$$h(t) = \begin{cases} 0, & \text{if } \psi_j(t) < 0; \\ \xi_j(t), & \text{if } \psi_j(t) = 0; \\ 1, & \text{if } \psi_j(t) > 0. \end{cases}$$

Thus, the solution is given by (6), and we complete the proof.

**Remark 1.** We consider only two parties, namely an insurer and a default-prone reinsurer. It is interesting to consider the insurance-reinsurance model in which three agents, namely a policyholder, an insurer, and a default-prone reinsurer, coexist[27]. This will be our future work to consider the insurance-reinsurance model under the Bowley question.

Refs. [21, 22] considered Bowley reinsurance without reinsure's default risk. Ref. [21] considered the administrative cost of offering the compensation is proportional to the expectation of the ceded loss, i.e., $C(f) =: \gamma\mathbb{E}(f(X))$ for any $f\in\mathcal{F}$, where $\gamma\geqslant 0$ is a fixed constant representing the cost coefficient. Ref. [22] considered the administrative cost of offering the compensation to be proportional to a distortion risk measure. To make our model generality, we follow the assumption of Ref. [22], i.e., let $C(f) =: \gamma\rho_{\hat{g}_R}(f(X))$ for any $f\in\mathcal{F}$, where $\gamma\geqslant 0$ is a fixed constant and $\hat{g}_R\in\mathcal{G}_d$. Then, for $j\in\{1,2\}$, we have

$$\begin{aligned}
&\mathbb{E}[f_{g_r,g_{ij}}(X)] + \gamma\rho_{\hat{g}_R}(f_{g_r,g_{ij}}(X)) = \\
&\int_0^\infty g_R(S_{f_{g_r,g_{ij}}(X)}(z))\mathrm{d}z =: \Pi_{g_R}(f_{g_r,g_{ij}}(X)),
\end{aligned}$$

where $g_R(t) := t + \gamma\hat{g}_R(t), t\in[0,1]$. Thus, we rewrite the objective in Eq. (5) as

$$\begin{aligned}
&W(g_r; f_{g_r,g_{i1}}, f_{g_r,g_{i2}}) := \\
&p\{\mathbb{E}[\Pi_{g_r}(f_{g_r,g_{i1}}(X)) - f_{g_r,g_{i1}}(X)] - C(f_{g_r,g_{i1}}(X))\} + \\
&(1-p)\{\mathbb{E}[\Pi_{g_r}(f_{g_r,g_{i2}}(X)) - f_{g_r,g_{i2}}(X)] - C(f_{g_r,g_{i2}}(X))\} = \\
&p\{\Pi_{g_r}(f_{g_r,g_{i1}}(X)) - \Pi_{g_R}(f_{g_r,g_{i1}}(X))\} + \\
&(1-p)\{\Pi_{g_r}(f_{g_r,g_{i2}}(X)) - \Pi_{g_R}(f_{g_r,g_{i2}}(X))\}.
\end{aligned}$$

Under such indifference circumstances, it is common in the literature to achieve definiteness assuming that $h_j(z) = 1$, which means that the insurer is "willing to" act in favor of the reinsurer[16, 21, 22, 32]. In this way, we shall set $h_j(z) = 1$ in the sequel. Under this setup, problem (5), by Lemma 1, boils down to solving

$$\begin{aligned}
&\max_{g_r\in\mathcal{G}} W(g_r; f_{g_r,g_{i1}}, f_{g_r,g_{i2}}) = \\
&\max_{g_r\in\mathcal{G}} \int_0^1 [(1+\theta)t + g_r(t) - g_R(t)]\times \\
&\left(\left[p\mathbb{I}_{\{(1+\theta)t+g_r(t)\leqslant\delta g_{i1}(t)+(1-\delta)g_{i1}(t)\mathbb{I}_{\{F_X^{-1}(1-t)<\mathrm{VaR}_\beta(X)\}}\}}\right] + \right. \\
&\left.\left[(1-p)\mathbb{I}_{\{(1+\theta)t+g_r(t)\leqslant\delta g_{i2}(t)+(1-\delta)g_{i2}(t)\mathbb{I}_{\{F_X^{-1}(1-t)<\mathrm{VaR}_\beta(X)\}}\}}\right]\right)\nu_X(\mathrm{d}t) = \\
&\max_{g_r\in\mathcal{G}} \left\{p\int_0^1 [(1+\theta)t + g_r(t) - g_R(t)] \times \right. \\
&\left[\mathbb{I}_{\{(1+\theta)t+g_r(t)\leqslant\delta g_{i1}(t)+(1-\delta)g_{i1}(t)\mathbb{I}_{\{F_X^{-1}(1-t)<\mathrm{VaR}_\beta(X)\}}\}}\right]\nu_X(\mathrm{d}t) + \\
&(1-p)\int_0^1 [(1+\theta)t + g_r(t) - g_R(t)] \times \\
&\left.\left[\mathbb{I}_{\{(1+\theta)t+g_r(t)\leqslant\delta g_{i2}(t)+(1-\delta)g_{i2}(t)\mathbb{I}_{\{F_X^{-1}(1-t)<\mathrm{VaR}_\beta(X)\}}\}}\right]\nu_X(\mathrm{d}t)\right\},
\end{aligned} \tag{7}$$

where $\nu_X$ is the Radon measure on $[0,1)$ such that $\nu_X([a,b)) = (-F_X^{-1}(1-b)) - (-F_X^{-1}(1-a))$ for $0\leqslant a<b<1$.

## 3 Main result

In this section, we provide our main result for problem (7). We assume $\{g_{i1}, g_{i2}\}\subset\mathcal{G}_d$. It will be helpful to define, for $t\in[0,1]$,

$$\begin{aligned}
&G_r(t) := (1+\theta)t + g_r(t), \\
&G_{i1}(t) := \delta g_{i1}(t) + (1-\delta)g_{i1}(t)\mathbb{I}_{\{F_X^{-1}(1-t)<\mathrm{VaR}_\beta(X)\}}, \\
&G_{i2}(t) := \delta g_{i2}(t) + (1-\delta)g_{i2}(t)\mathbb{I}_{\{F_X^{-1}(1-t)<\mathrm{VaR}_\beta(X)\}},
\end{aligned}$$

and

$$\begin{aligned}
\phi(t) :=& \mathbb{I}_{\{G_{i1}(t)>G_{i2}(t)\}}(G_{i2}(t) - (pG_{i1}(t) + (1-p)g_R(t))) + \\
& \mathbb{I}_{\{G_{i1}(t)<G_{i2}(t)\}}((1-p)G_{i2}(t) + pg_R(t) - G_{i1}(t)).
\end{aligned}$$

Moreover, we also define

$$\begin{aligned}
&\mathcal{A} = \{t\in[0,1] : \phi(t) < 0, G_{i1}(t)\geqslant g_R(t)\}, \\
&\mathcal{B} = \{t\in[0,1] : \phi(t) = 0, G_{i2}(t)\geqslant g_R(t)\}, \\
&\mathcal{C} = \{t\in[0,1] : \phi(t) > 0, G_{i2}(t)\geqslant g_R(t)\}.
\end{aligned}$$

These sets allow us to state Theorem 1, which provides the Bowley solutions under asymmetric information and the reinsurer's default risk. Note that Theorem 1 is provided with respect to $G_r$ since there exists a one-to-one correspondence between $G_r$ and $g_r$. In Remark 2, we express the solution with



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respect to $g_r$.

**Theorem 1.** The solution set to problem (7) contains those $G_r^* \in \mathcal{G}$ such that

$$G_r^*(t) = \begin{cases} G_{i1}(t), & \text{if } t \in \mathcal{A}; \\ \in \{G_{i1}(t), G_{i2}(t)\}, & \text{if } t \in \mathcal{B}; \\ G_{i2}(t), & \text{if } t \in \mathcal{C}; \end{cases}$$

and

$$\nu_X\{t \in [0,1] \setminus (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}) : G_r^*(t) \leqslant \max\{G_{i1}(t), G_{i2}(t)\}\} = 0.$$

Moreover, for any of these $G_r^*$, we have, for $x \in \mathbb{R}_+$,

$$\begin{aligned} f^*_{g_r^*, g_{i1}}(x) =& \mu(\{t \in [0,x] | S_X(t) \in \mathcal{A}\}) + \\ & \mu(\{t \in [0,x] | S_X(t) \in \mathcal{B}, G_r^*(S_X(t)) \leq G_{i1}(S_X(t))\}) + \\ & \mu(\{t \in [0,x] | S_X(t) \in \mathcal{C}, G_{i1}(S_X(t)) > G_{i2}(S_X(t))\}), \end{aligned}$$

and

$$\begin{aligned} f^*_{g_r^*, g_{i2}}(x) =& \mu(\{t \in [0,x] | S_X(t) \in \mathcal{A}, G_{i1}(S_X(t)) < G_{i2}(S_X(t))\}) + \\ & \mu(\{t \in [0,x] | S_X(t) \in \mathcal{B}, G_r^*(S_X(t)) \leq G_{i2}(S_X(t))\}) + \\ & \mu(\{t \in [0,x] | S_X(t) \in \mathcal{C}\}). \end{aligned}$$

**Proof.** The proof is similar to that of Theorem 3.1 in Ref. [22].

We use the technique of path-wise optimization. Eq. (7) is written as

$$\begin{aligned} & \max_{g_r \in \mathcal{G}} \int_0^1 [(1+\theta)t + g_r(t) - g_R(t)] \times \\ & \quad \left\{ \left[ p \mathbb{I}_{\{(1+\theta)t + g_r(t) \leqslant \delta g_{i1}(t) + (1-\delta) g_{i1}(t) \mathbb{I}_{\{F_X^{-1}(1-t) < \mathrm{VaR}_\beta(X)\}}\}} \right] + \right. \\ & \quad \left. \left[ (1-p) \mathbb{I}_{\{(1+\theta)t + g_r(t) \leqslant \delta g_{i2}(t) + (1-\delta) g_{i2}(t) \mathbb{I}_{\{F_X^{-1}(1-t) < \mathrm{VaR}_\beta(X)\}}\}} \right] \right\} \nu_X(\mathrm{d}t) = \\ & \max_{G_r \in \mathcal{G}} \int_0^1 [G_r(t) - g_R(t)] \left[ p \mathbb{I}_{\{G_r(t) \leqslant G_{i1}(t)\}} + (1-p) \mathbb{I}_{\{G_r(t) \leqslant G_{i2}(t)\}} \right] \nu_X(\mathrm{d}t) = \\ & \max_{G_r \in \mathcal{G}} \int_0^1 [G_r(t) - g_R(t)] \left[ p \mathbb{I}_{\{G_{i2}(t) < G_r(t) \leqslant G_{i1}(t)\}} + \right. \\ & \quad \left. (1-p) \mathbb{I}_{\{G_{i1}(t) < G_r(t) \leqslant G_{i2}(t)\}} + \mathbb{I}_{\{G_r(t) \leqslant \min\{G_{i1}(t), G_{i2}(t)\}\}} \right] \nu_X(\mathrm{d}t) = \\ & \max_{G_r \in \mathcal{G}} \int_0^1 \varphi(G_r(t), t) \nu_X(\mathrm{d}t) \leqslant \int_0^1 \max_{G_r(t) \geqslant 0} \varphi(G_r(t), t) \nu_X(\mathrm{d}t), \end{aligned}$$

where

$$\varphi(G_r(t), t) := \begin{cases} [G_r(t) - g_R(t)] \left[ p \mathbb{I}_{\{G_{i2}(t) < G_r(t) \leqslant G_{i1}(t)\}} + \mathbb{I}_{\{G_r(t) \leqslant G_{i2}(t)\}} \right] \\ \qquad \text{if } G_{i2}(t) < G_{i1}(t); \\ [G_r(t) - g_R(t)] \mathbb{I}_{\{G_r(t) \leqslant G_{i1}(t)\}} \\ \qquad \text{if } G_{i2}(t) = G_{i1}(t); \\ [G_r(t) - g_R(t)] \left[ (1-p) \mathbb{I}_{\{G_{i1}(t) < G_r(t) \leqslant G_{i2}(t)\}} + \mathbb{I}_{\{G_r(t) \leqslant G_{i1}(t)\}} \right] \\ \qquad \text{if } G_{i2}(t) > G_{i1}(t). \end{cases}$$

Now, we solve the maximization problem path wise, and therefore, we first fix $t \in [0,1]$. Next, we construct solutions of $\max_{G_r(t) \geqslant 0} \varphi(G_r(t), t)$. We consider three different cases.

(i) Suppose that $G_{i2}(t) < G_{i1}(t)$: Then,

$$\varphi(G_r(t), t) = [G_r(t) - g_R(t)][p \mathbb{I}_{\{G_{i2}(t) < G_r(t) \leqslant G_{i1}(t)\}} + \mathbb{I}_{\{G_r(t) \leqslant G_{i2}(t)\}}].$$

For all $t \in [0,1]$, it holds that $\varphi(\cdot, t)$ is strictly increasing on $[0, G_{i2}(t)]$ and on $(G_{i2}(t), G_{i1}(t)]$, and $\varphi(\cdot, t) = 0$ on $(G_{i1}(t), \infty)$. Thus, the maximum value of $\varphi(\cdot, t)$ is either located at the possible discontinuities, $G_{i2}(t)$ and $G_{i1}(t)$, or it is zero. Hence,

$$\begin{aligned} \max_{G_r(t) \geqslant 0} \varphi(G_r(t), t) =& \max\{p[G_{i1}(t) - g_R(t)], [G_{i2}(t) - g_R(t)], 0\} = \\ & \max\{p[G_{i1}(t) - g_R(t)] + \\ & \max\{0, G_{i2}(t) - [pG_{i1}(t) + (1-p)g_R(t)]\}, 0\} = \\ & \max\{p[G_{i1}(t) - g_R(t)] + \max\{0, \phi(t)\}, 0\}. \end{aligned}$$

If $\phi(t) < 0$ and $G_{i1}(t) \geqslant g_R(t)$, then $\max_{G_r(t) \geqslant 0} \varphi(G_r(t), t)$ is solved by $G_r(t) = G_{i1}(t)$. Likewise, if $\phi(t) > 0$ and $G_{i2}(t) \geqslant g_R(t)$, then $\max_{G_r(t) \geqslant 0} \varphi(G_r(t), t)$ is solved by $G_r(t) = G_{i2}(t)$. If $\phi(t) = 0$ and $G_{i2}(t) \geqslant g_R(t)$, then $\max_{G_r(t) \geqslant 0} \varphi(G_r(t), t)$ is solved by either $G_r(t) = G_{i1}(t)$ or $G_r(t) = G_{i2}(t)$. Finally, if $G_{i1}(t) < g_R(t)$, then $\max_{G_r(t) \geqslant 0} \varphi(G_r(t), t) = 0$, and it is thus solved by any $G_r(t) > G_{i1}(t)$.

(ii) Suppose that $G_{i2}(t) = G_{i1}(t)$: Then,

$$\varphi(G_r(t), t) = [G_r(t) - g_R(t)] \mathbb{I}_{\{G_r(t) \leqslant G_{i1}(t)\}}.$$

For all $t \in [0,1]$, it holds that $\varphi(\cdot, t)$ is strictly increasing on $[0, G_{i1}(t)]$, and $\varphi(\cdot, t) = 0$ on $(G_{i1}(t), \infty)$. Thus, the maximum value of $\varphi(\cdot, t)$ is either located at, $G_{i1}(t)$, or it is zero. Hence,

$$\max_{G_r(t) \geq 0} \varphi(G_r(t), t) = \max\{[G_{i1}(t) - g_R(t)], 0\}.$$

Recall that $\phi(t) = 0$. If $G_{i1}(t) \geqslant g_R(t)$, then $\max_{G_r(t) \geqslant 0} \varphi(G_r(t), t)$ is solved by $G_r(t) = G_{i1}(t) = G_{i2}(t)$. If $G_{i1}(t) < g_R(t)$, then $\max_{G_r(t) \geqslant 0} \varphi(G_r(t), t) = 0$, and it is thus solved by any $G_r(t) > G_{i1}(t)$.

(iii) Suppose that $G_{i2}(t) > G_{i1}(t)$: Then,

$$\varphi(G_r(t), t) = [G_r(t) - g_R(t)][(1-p) \mathbb{I}_{\{G_{i1}(t) < G_r(t) \leqslant G_{i2}(t)\}} + \mathbb{I}_{\{G_r(t) \leqslant G_{i1}(t)\}}].$$

For all $t \in [0,1]$, it holds that $\varphi(\cdot, t)$ strictly increases on $[0, G_{i1}(t)]$ and on $(G_{i1}(t), G_{i2}(t)]$, and $\varphi(\cdot, t) = 0$ on $(G_{i2}(t), \infty)$. Thus, the maximum value of $\varphi(\cdot, t)$ is either located at the possible discontinuities, $G_{i2}(t)$ and $G_{i1}(t)$, or it is zero. Hence,

$$\begin{aligned} \max_{G_r(t) \geqslant 0} \varphi(G_r(t), t) =& \max\{(1-p)[G_{i2}(t) - g_R(t)], [G_{i1}(t) - g_R(t)], 0\} = \\ & \max\{(1-p)[G_{i2}(t) - g_R(t)] + \\ & \max\{0, G_{i1}(t) - [(1-p)G_{i2}(t) + p g_R(t)]\}, 0\} = \\ & \max\{(1-p)[G_{i2}(t) - g_R(t)] + \max\{0, -\phi(t)\}, 0\}. \end{aligned}$$

If $\phi(t) < 0$ and $G_{i1}(t) \geqslant g_R(t)$, then $\max_{G_r(t) \geqslant 0} \varphi(G_r(t), t)$ is solved by $G_r(t) = G_{i1}(t)$. Likewise, if $\phi(t) > 0$ and $G_{i2}(t) \geqslant g_R(t)$, then $\max_{G_r(t) \geqslant 0} \varphi(G_r(t), t)$ is solved by $G_r(t) = G_{i2}(t)$. If $\phi(t) = 0$ and $G_{i2}(t) \geqslant g_R(t)$, then $\max_{G_r(t) \geqslant 0} \varphi(G_r(t), t)$ is solved by either $G_r(t) = G_{i1}(t)$ or $G_r(t) = G_{i2}(t)$. Finally, if $G_{i1}(t) < g_R(t)$, then $\max_{G_r(t) \geq 0} \varphi(G_r(t), t) = 0$, and it is thus solved by any $G_r(t) > G_{i1}(t)$.

Now we constructed the solutions of $\max_{G_r(t) \geqslant 0} \varphi(G_r(t), t)$ for all $t \in [0,1]$. Let $G_r^* \in \mathcal{G}$ such that it solves $\max_{G_r(t) \geqslant 0} \varphi(G_r(t), t)$ for all $t \in (0,1]$. Note that $G_r^*(0) = 0$ is a solution to $\max_{G_r(0) \geqslant 0} \varphi(G_r(0), 0) = 0$. Thus,


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$$\int_0^1 \max_{G_r(t)\geqslant 0}\varphi(G_r(t),t)\nu_X(\mathrm{d}t)=\int_0^1\varphi(G_r^*(t),t)\nu_X(\mathrm{d}t)\leqslant \max_{G_r\in\mathcal{G}}\int_0^1\varphi(G_r(t),t)\nu_X(\mathrm{d}t)\leqslant \int_0^1\max_{G_r(t)\geqslant 0}\varphi(G_r(t),t)\nu_X(\mathrm{d}t).$$

Thus, the inequalities can be replaced by equalities, and we conclude the proof of the premium generating functions $G_r^*$ in Bowley solutions under the reinsurer's default risk.

For a fixed premium generating function $G_r^*$, the optimal indemnity functions $f^*_{g_r^*,g_{i1}}$ and $f^*_{g_r^*,g_{i1}}$ follow from Proposition 1. This concludes the proof.

**Remark 2.** ① From Theorem 1 and $G_r^*(t)=(1+\theta)t+g_r^*(t)$, we obtain $g_r^*(t)=G_r^*(t)-(1+\theta)t$, i.e.,

$$g_r^*(t)=\begin{cases} G_{i1}(t)-(1+\theta)t, & \text{if } t\in\mathcal{A};\\ \in\{G_{i1}(t)-(1+\theta)t, G_{i2}(t)-(1+\theta)t\}, & \text{if } t\in\mathcal{B};\\ G_{i2}(t)-(1+\theta)t, & \text{if } t\in\mathcal{C};\end{cases}$$

and

$$\nu_X\{t\in[0,1]\setminus(\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}): g_r^*(t)\leqslant\max\{G_{i1}(t)-(1+\theta)t, G_{i2}(t)-(1+\theta)t\}\}=0.$$

② When setting $\delta=1$, we recover Theorem 3.1[22] under the same premium principle.

While the function $\phi$ is merely used as an ancillary function to construct the Bowley solutions under default risk, it has an interpretation as follows[22]. At a given value $t\in[0,1]$, $\phi(t)$ is the marginal profit that the reinsurer makes by choosing $G_r^*(t)=g_{i2}(t)$ instead of $G_r^*(t)=g_{i1}(t)$. Therefore, if $\phi(t)$ is positive (negative), then it is profitable for the reinsurer to select the premium generating function $G^*(t)$ that makes the type 2 (or 1) insurer indifferent between buying or not buying marginal reinsurance. While, for the marginal profit, reinsurance prices often make one type of insurer "indifferent", this does not imply that the insurer will be indifferent between insuring or not insuring. In fact, since it may hold that $G_r^*(t)<g_{ij}(t)$ for some $t\in[0,1]$, the insurer can strictly profit from buying reinsurance.

## 4 Two examples with ordered distortion functions of the insurer

In this section, two examples are provided for illustrating Theorem 1 in Section 3. These two examples have order distortion functions, i.e., $g_{i1}(t)\geqslant g_{i2}(t)$ for all $t\in[0,1]$. Then $G_{i1}(t)\geqslant G_{i2}(t)$. For ease of implementing the calculation, we first set $\hat{g}_R(t)=t$ so that $g_R(t)=(1+\gamma)t$ and $\Pi_{g_R}(f_{g_r;g_{ij}}(X))=(1+\gamma)\mathbb{E}[f_{g_r;g_{ij}}(X)], j=1,2$. Under these circumstances, the function $\phi$ simplifies as

$$\phi(t)=G_{i2}(t)-(pG_{i1}(t)+(1-p)(1+\gamma)t),$$

where $G_{i1}(t)$ and $G_{i2}(t)$ are defined in Section 3.

From Theorem 1, we obtain that, for any optimal $G_r^*$ and $x\in[0,\infty)$,

$$f^*_{g_r^*,g_{i1}}(x)=\mu(\{t\in[0,x]|S_X(t)\in\mathcal{A}\cup\mathcal{B}\})+\mu(\{t\in[0,x]|S_X(t)\in\mathcal{C},G_{i1}(S_X(t))>G_{i2}(S_X(t))\}),\quad(8)$$

and

$$f^*_{g_r^*,g_{i2}}(x)=\mu(\{t\in[0,x]|S_X(t)\in\mathcal{B},G_r^*(S_X(t))=G_{i2}(S_X(t))\})+\mu(\{t\in[0,x]|S_X(t)\in\mathcal{C}\}).\quad(9)$$

When $\delta=1$, i.e., there is no reinsurer's default risk, then we recover the results obtained by Boonen and Zhang[22]. Additionally, we remark that there exists an error in Section 4 of Ref. [22] for $f^*_{g_r^*,g_{i1}}(x)$ and $f^*_{g_r^*,g_{i2}}(x)$. The correct forms are given in Eqs. (8) and (9).

For the first example below, as we will observe, the value of probability $p$ plays a key role in determining the optimal premium generating function and the corresponding ceded loss functions.

**Example 1.** Suppose that the risk $X$ has an exponential distribution with mean 1, the recovery rate $\delta=0.5$ and the reinsurer sets its regulatory capital at the level of $\beta=0.95$. Then $\mathrm{VaR}_{0.95}(X)=2.9957$. Assume that the distortion functions of the insurer are given by $g_{i1}(t)=t^{\alpha_1}$ and $g_{i2}(t)=t^{\alpha_2}$, for $t\in[0,1]$, where $\alpha_1=0.2$ and $\alpha_2=0.4$. Clearly, $g_{i1}(t)\geqslant g_{i2}(t)$, for all $t\in[0,1]$, which implies $G_{i1}(t)\geqslant G_{i2}(t)$. For $\gamma=0.1$, the solutions of the equations $G_{i1}(t)=g_R(t)$ and $G_{i2}(t)=g_R(t)$ on $t\in(0,1)$ are $t_1=0.8877$ and $t_2=0.8531$, respectively. According to the definitions of sets $\mathcal{A},\mathcal{B}$, and $\mathcal{C}$, we first need to determine the signs of the function $\phi(t)$ for $t\in[0,t_1]$, and then obtain the explicit expressions of these three sets. Consider the following three values of the probability $p$:

(i) Suppose that $p=0.1$. In this case, we calculate that

$$\begin{aligned}\phi(t)=&G_{i2}(t)-(pG_{i1}(t)+(1-p)(1+\gamma)t)=\\ &0.5t^{0.4}+0.5(t^{0.4}\mathbb{I}_{\{t>0.05\}})-\\ &0.1\times(0.5t^{0.2}+0.5(t^{0.2}\mathbb{I}_{\{t>0.05\}}))-0.9\times1.1t=\\ &0.5t^{0.4}-0.05t^{0.2}+(0.5t^{0.4}-0.05t^{0.2})\mathbb{I}_{\{t>0.05\}}-0.99t.\end{aligned}$$

From Fig. 1a, we can easily see that $\phi(t)=0$ has a unique solution on $(0,t_1]$, which is given by $t_3=0.8478$. Meanwhile, $\phi(t)>0$ when $t\in(0,t_3)$ and $\phi(t)<0$ when $t\in(t_3,t_1)$. Hence, we have $\mathcal{A}=(t_3,t_1)$, $\mathcal{B}=\{0,t_3\}$, and $\mathcal{C}=(0,t_3)$. Premium generating functions $G_r^*$ in Bowley solutions with default risk are then given by

$$G_r^*(t)=\begin{cases}0.5t^{0.4}+0.5(t^{0.4}\mathbb{I}_{\{t>0.05\}}), & \text{if } t\in[0,0.8478];\\ t^{0.2}, & \text{if } t\in(0.8478,0.8877];\\ \tilde{G}_r(t), & \text{if } \nu_X\{t\in(0.8877,1]:\tilde{G}_r(t)\leqslant G_{i1}(t)\}=0.\end{cases}$$

Furthermore, the optimal ceded loss functions for $x\in\mathbb{R}_+$ are given by

$$\begin{aligned}f^*_{g_r^*,g_{i1}}(x)=&\mu(\{t\in[0,x]|\mathrm{e}^{-t}\in(t_3,t_1]\})+0+\\ &\mu(\{t\in[0,x]|\mathrm{e}^{-t}\in(0,t_3]\})=\\ &\mu(\{t\in[0,x]|\mathrm{e}^{-t}\in(0,t_1]\})=\\ &\mu(\{t\in[0,x]|t\in(0.1191,\infty)\})=\\ &(x-0.1191)_+,\end{aligned}$$

where $x_+:=\max\{x,0\}$, and



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$$f^*_{g_r^*,g_{i2}}(x) = 0 + \mu(\{t\in[0,x] | \mathrm{e}^{-t}\in(0,t_3]) = \mu(\{t\in[0,x] | t\in[0.1651,\infty)\}) = (x-0.1651)_+.$$

This means that the type of our optimal ceded loss function is a traditional stop-loss policy. Moreover the expected net profit can be calculated as $W(g_r^*) = 1.0306$.

(ii) Suppose that $p = 0.4$. In this case, we have

$$\begin{aligned}\phi(t) =& G_{i2}(t) - (pG_{i1}(t) + (1-p)(1+\gamma)t) = \\ & 0.5t^{0.4} + 0.5(t^{0.4}\mathbb{I}_{\{t>0.05\}}) - \\ & 0.4\times(0.5t^{0.2} + 0.5(t^{0.2}\mathbb{I}_{\{t>0.05\}})) - 0.6\times 1.1t = \\ & 0.5t^{0.4} - 0.2t^{0.2} + (0.5t^{0.4} - 0.2t^{0.2})\mathbb{I}_{\{t>0.05\}} - 0.66t.\end{aligned}$$

From Fig. 1b, we can easily see that $\phi(t)=0$ has two solutions on $[0,t_1]$, which are given by $t_3 = 0.0202$ and $t_4 = 0.8136$. Meanwhile, $\phi(t)>0$ when $t\in(t_3,t_4)$ and $\phi(t)<0$ when $t\in(0,t_3)\cup(t_4,t_1)$. Hence, we have $\mathcal{A} = (0,t_3)\cup(t_4,t_1)$, $\mathcal{B} = \{0,t_3,t_4\}$, and $C = (t_3,t_4)$. Premium generating functions $G_r^*$ in Bowley solutions with default risk are then given by

$$G_r^*(t) = \begin{cases} 0.5t^{0.2} + 0.5(t^{0.2}\mathbb{I}_{\{t>0.05\}}), \\ \qquad \text{if } t\in(0,0.0202)\cup(0.8136,0.8877); \\ 0.5t^{0.4} + 0.5(t^{0.4}\mathbb{I}_{\{t>0.05\}}), \\ \qquad \text{if } t\in(0.0202,0.8136]; \\ \tilde{G}_r(t), \quad \text{if } \nu_X\{t\in(0.8877,1]: \tilde{G}_r(t)\leqslant G_{i1}(t)\} = 0. \end{cases}$$

Furthermore, the optimal ceded loss functions for $x\in\mathbb{R}_+$ are given by

$$\begin{aligned} f^*_{g_r^*,g_{i1}}(x) =& \mu(\{t\in[0,x] | \mathrm{e}^{-t}\in[0,t_3]\cup[t_4,t_1]\}) + 0 + \\ & \mu(\{t\in[0,x] | \mathrm{e}^{-t}\in(t_3,t_4)\}) = \\ & \mu(\{t\in[0,x] | \mathrm{e}^{-t}\in[0,t_1]\}) = \\ & (x-0.1191)_+, \end{aligned}$$

and

$$f^*_{g_r^*,g_{i2}}(x) = 0 + \mu(\{t\in[0,x] | \mathrm{e}^{-t}\in(t_3,t_4)\}) = \min\{(x-0.2063)_+, 3.9658\}.$$

Hence, the stop-loss contract is signed between the reinsurer and the type 1 insurer, while a two layer stop-loss policy is provided for the type 2 insurer. Moreover the net gain can be calculated as $W(g_r^*) = 1.2401$.

(iii) Suppose that $p = 0.9$. In this case, we have

$$\begin{aligned}\phi(t) =& G_{i2}(t) - (pG_{i1}(t) + (1-p)(1+\gamma)t) = \\ & 0.5t^{0.4} + 0.5(t^{0.4}\mathbb{I}_{\{t>0.05\}}) - \\ & 0.9\times(0.5t^{0.2} + 0.5(t^{0.2}\mathbb{I}_{\{t>0.05\}})) - 0.1\times 1.1t = \\ & 0.5t^{0.4} - 0.45t^{0.2} + (0.5t^{0.4} - 0.45t^{0.2})\mathbb{I}_{\{t>0.05\}} - 0.11t.\end{aligned}$$

From Fig. 1c, we know that $\mathcal{A} = (0,t_1]$, $\mathcal{B} = \{0\}$, and $C = \emptyset$. Premium generating functions $G_r^*$ in Bowley solutions with default risk are then given by

$$G_r^*(t) = \begin{cases} 0.5t^{0.2} + 0.5(t^{0.2}\mathbb{I}_{\{t>0.05\}}), & \text{if } t\in(0,0.8877]; \\ \tilde{G}_r(t), & \text{if } \nu_X\{t\in(0.8877,1]: \tilde{G}_r(t)\leqslant G_{i1}(t)\} = 0. \end{cases}$$

Furthermore, the optimal ceded loss functions for $x\in[0,\infty)$

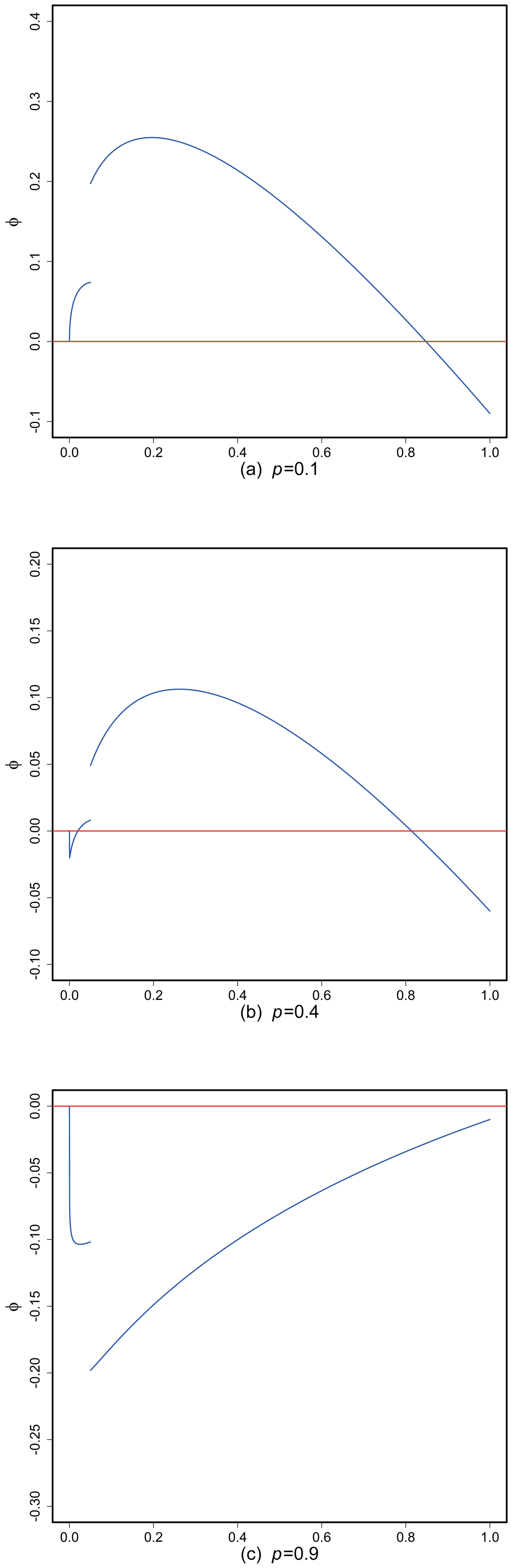


**Fig. 1.** Plots of the function $\phi(t)$ on $t\in[0,1]$ for three different values of $p$. Corresponding to Example 1, (a) $p=0.1$, (b) $p=0.4$, and (c) $p=0.9$.



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are given by

$$f^*_{g^*_r,g_{i1}}(x)=(x-0.1191)_+$$

and

$$f^*_{g^*_r,g_{i2}}(x)=0.$$

Hence, the stop-loss contract is signed between the reinsurer and the type 1 insurer. When the probability of the type 1 insurer is sufficiently high, for example $p=0.9$, the type 2 insurer will not cede any function. Furthermore, we do not know exactly the critical point $p$ to the insurers that cede no loss. This will be our future study. Moreover the profit acquired by the reinsurer can be computed as $W(g^*_r)=2.2794$.

The following example illustrates how the default rate $\delta$ affects the optimal contract form. We will see that $\delta$ has the greatest impact on the type 1 insurer, and that the optimal contract forms depend on the value of $\delta$.

**Example 2.** Suppose that the risk $X$ is a nonnegative continuous random variable possibly with a jump point at 0. The reinsurer sets its regulatory capital at $\beta\in(0,1)$ and the recovery rate $\delta\in(0,1)$. The distortion functions of the insurer are given by $g_{i1}(t)=(d_1t)\wedge 1$ and $g_{i2}(t)=(d_2t)\wedge 1$ for $t\in[0,1]$, respectively. We assume that $d_{i1}\geqslant d_{i2}$. Clearly, under this setting, we have $g_{i1}(t)\geqslant g_{i2}(t)$, that is, $G_{i1}(t)\geqslant G_{i2}(t)$ and $\{G_{i1}(t)>G_{i2}(t)\}=(0,1/d_2)$. Let $\gamma=a-1$ with $a\geqslant 1$. If $1-\beta\leqslant 1/d_j$, then $G_{ij}(t)=\delta d_jt\mathbb{I}_{\{t\leqslant 1-\beta\}}+d_jt\mathbb{I}_{\{1-\beta<t\leqslant 1/d_j\}}+\mathbb{I}_{\{1/d_j<t\leqslant 1\}}$, and

$$\begin{aligned}\phi(t)=&G_{i2}(t)-(pG_{i1}(t)+(1-p)(1+\gamma)t)=\\&(g_{i2}(t)-pg_{i1}(t))(\delta\mathbb{I}_{\{t\leqslant 1-\beta\}}+\mathbb{I}_{\{1-\beta<t\leqslant 1\}})-(1-p)at.\end{aligned}$$

We have $\phi(t)<0$ for $g_{i2}(t)-pg_{i1}(t)<0$ and $\phi(t)>0$ for $g_{i2}(t)-pg_{i1}(t)>0$. Thus the following relationship between $\phi$ and 0 holds:

$$\phi>(<)0\quad\Longleftrightarrow\quad \delta\mathbb{I}_{\{t\leqslant 1-\beta\}}+\mathbb{I}_{\{1-\beta<t\leqslant 1\}}>(<)\frac{(1-p)at}{g_{i2}(t)-pg_{i1}(t)}.$$

The right hand side can be written as

$$\frac{(1-p)at}{g_{i2}(t)-pg_{i1}(t)}=\begin{cases}\dfrac{(1-p)a}{d_2-d_1p}, & \text{if } t\in(0,1/d_1];\\ \dfrac{(1-p)a}{d_2-p/t}, & \text{if } t\in(1/d_1,1/d_2];\\ at, & \text{if } t\in(1/d_2,1].\end{cases}$$

To analysis how $\delta$ influences $f^*$, we consider the following cases:

(i) Suppose that $d_2<d_1p$, $1-\beta<p/d_2$. If $a/d_2<1$, let $t_1=p/(d_2-(1-p)a)\leqslant 1/d_2$ and $t_2=1/a$. We have $\{\phi(t)<0\}=(0,t_1)\cup(t_2,1)$, $\{\phi(t)>0\}=(t_1,t_2)$. Then we consider different values of $\delta$. If $\delta$ is relatively large, that is $d_2\delta\geqslant a$, then $\{G_{i1}(t)\geqslant g_R(t)\}=\{G_{i2}(t)\geqslant g_R(t)\}=(0,1/a)$. Hence, $\mathcal{A}_1=(0,t_1)$, $\mathcal{B}_1=\{t_1,t_2\}$, and $C_1=(t_1,t_2)$. If $\delta\in(a/d_1,a/d_2)$, we have $\{G_{i1}(t)\geqslant g_R(t)\}=(0,1/a)$ but $\{G_{i2}(t)\geqslant g_R(t)\}=(1-\beta,1/a)$. Thus $\mathcal{A}_2=(0,t_1)$, $\mathcal{B}_2=\{t_1,t_2\}$, and $C_2=(t_1,t_2)\cap(1-\beta,1/a)=(1-\beta,1/a)=(t_1,t_2)$. The result is the same as the first condition. However if $\delta$ becomes smaller, i.e., $\delta<a/d_1$, we have $\{G_{i1}(t)\geqslant g_R(t)\}=\{G_{i2}(t)\geqslant g_R(t)\}=(1-\beta,1/a)$. Thus $\mathcal{A}_3=(1-\beta,t_1)$, $\mathcal{B}_3=\{t_1,t_2\}$, and $C_3=(t_1,t_2)$.

In conclusion, if $\delta>a/d_1$, the premium generating functions $G^*_r$ in Bowley solutions with default risk are then given by

$$G^*_r(t)=\begin{cases}G_{i1}(t), & \text{if } t\in[0,t_1];\\ G_{i2}(t), & \text{if } t\in(t_1,t_2];\\ \tilde{G}_r(t), & \text{if } \nu_X\{t\in(t_2,1]:\tilde{G}_r(t)\leqslant G_{i1}(t)\}=0;\end{cases}$$

and the cede functions are given by

$$\begin{aligned}f^*_{g^*_r,g_{i1}}(x)=&\mu(\{t\in[0,x]|S_X(t)\in(0,1/d_2)\}),\\ f^*_{g^*_r,g_{i2}}(x)=&\mu(\{t\in[0,x]|S_X(t)\in(t_1,t_2)\}).\end{aligned}$$

If $\delta<a/d_1$, the premium generating functions $G^*_r$ are

$$G^*_r(t)=\begin{cases}G_{i1}(t), & \text{if } t\in(1-\beta,t_1];\\ G_{i2}(t), & \text{if } t\in(t_1,t_2];\\ \tilde{G}_r(t), & \text{if } \nu_X\{t\in(0,1-\beta]\cup(t_2,1]:\tilde{G}_r(t)\leqslant G_{i1}(t)\}=0;\end{cases}$$

and the cede functions are

$$\begin{aligned}f^*_{g^*_r,g_{i1}}(x)=&\mu(\{t\in[0,x]|S_X(t)\in(1-\beta,1/d_2)\}),\\ f^*_{g^*_r,g_{i2}}(x)=&\mu(\{t\in[0,x]|S_X(t)\in(t_1,t_2)\}).\end{aligned}$$

(ii) Suppose that $d_2<d_1p$, $1-\beta<p/d_2$. If $a/d_2>1$, then $\phi(t)<0$ for $t\in(0,1)$ and $C=\emptyset$. By a similar argument in part (i), we have $\mathcal{A}=(0,t_1)$ for $\delta>a/d_1$ and $\mathcal{A}=(1-\beta,t_1)$ for $\delta<a/d_1$.

In this condition, if $\delta>a/d_1$, the premium generating functions $G^*_r$ are

$$G^*_r(t)=\begin{cases}G_{i1}(t), & \text{if } t\in(0,t_1];\\ \tilde{G}_r(t), & \text{if } \nu_X\{t\in(t_1,1]:\tilde{G}_r(t)\leqslant G_{i1}(t)\}=0;\end{cases}$$

and the cede functions are

$$f^*_{g^*_r,g_{i1}}(x)=\mu(\{t\in[0,x]|S_X(t)\in(0,t_1)\}),\quad f^*_{g^*_r,g_{i2}}(x)=0.$$

If $\delta<a/d_1$, the premium generating functions $G^*_r$ are

$$G^*_r(t)=\begin{cases}G_{i1}(t), & \text{if } t\in(1-\beta,t_1];\\ \tilde{G}_r(t), & \text{if } \nu_X\{t\in(0,1-\beta]\cup(t_2,1]:\tilde{G}_r(t)\leqslant G_{i1}(t)\}=0;\end{cases}$$

and the cede functions are

$$f^*_{g^*_r,g_{i1}}(x)=\mu(\{t\in[0,x]|S_X(t)\in(1-\beta,t_1)\}),\quad f^*_{g^*_r,g_{i2}}(x)=0.$$

(iii) Suppose that $d_2>d_1p$, $1-\beta<1/d_1$, and $1<\dfrac{(1-p)a}{d_2-d_1p}$.

For $a/d_2>1$, $\phi(t)<0$ for $t\in(0,1)$, the result is the same as condition (ii).

If $\delta>a/d_1$, the premium generating functions $G^*_r$ are

$$G^*_r(t)=\begin{cases}G_{i1}(t), & \text{if } t\in(0,t_1];\\ \tilde{G}_r(t), & \text{if } \nu_X\{t\in(t_1,1]:\tilde{G}_r(t)\leqslant G_{i1}(t)\}=0;\end{cases}$$

and the cede functions are

$$f^*_{g^*_r,g_{i1}}(x)=\mu(\{t\in[0,x]|S_X(t)\in(0,t_1)\}),\quad f^*_{g^*_r,g_{i2}}(x)=0.$$

If $\delta<a/d_1$, the premium generating functions $G^*_r$ are

$$G^*_r(t)=\begin{cases}G_{i1}(t), & \text{if } t\in(1-\beta,t_1];\\ \tilde{G}_r(t), & \text{such that } \nu_X\{t\in(0,1-\beta]\cup(t_1,1]:\\ & \qquad\tilde{G}_r(t)\leqslant G_{i1}(t)\}=0;\end{cases}$$


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and the cede functions are

$$f^*_{g^*_r,g_{i1}}(x)=\mu(\{t\in[0,x]|S_X(t)\in(1-\beta,t_1)\}),\quad f^*_{g^*_r,g_{i2}}(x)=0.$$

For $a/d_2<1$, there exist $t_1=\dfrac{p}{d_2-(1-p)a}\leqslant 1/d_2$, $t_2=1/a$ such that $\{\phi(t)<0\}=(0,t_1)\cup(t_2,1)$ and $\{\phi(t)>0\}=(t_1,t_2)$.

If $\delta>a/d_1$, the result is the same as condition (i). In this case, the premium generating functions $G^*_r$ in Bowley solutions with default risk are then given by

$$G^*_r(t)=\begin{cases}G_{i1}(t), & \text{if } t\in[0,t_1];\\ G_{i2}(t), & \text{if } t\in(t_1,t_2];\\ \tilde{G}_r(t), & \text{if } \nu_X\{t\in(t_2,1]:\tilde{G}_r(t)\leqslant G_{i1}(t)\}=0;\end{cases}$$

and the cede functions are

$$f^*_{g^*_r,g_{i1}}(x)=\mu(\{t\in[0,x]|S_X(t)\in(0,1/d_2)\}),$$
$$f^*_{g^*_r,g_{i2}}(x)=\mu(\{t\in[0,x]|S_X(t)\in(t_1,t_2)\}).$$

If $\delta<a/d_1$, the premium generating functions $G^*_r$ are

$$G^*_r(t)=\begin{cases}G_{i1}(t), & \text{if } t\in(1-\beta,t_1];\\ G_{i2}(t), & \text{if } t\in(t_1,t_2];\\ \tilde{G}_r(t), & \text{if } \nu_X\{t\in(0,1-\beta]\cup(t_2,1]:\tilde{G}_r(t)\leqslant G_{i1}(t)\}=0;\end{cases}$$

and the cede functions are

$$f^*_{g^*_r,g_{i1}}(x)=\mu(\{t\in[0,x]|S_X(t)\in(\beta,1/d_2)\}),$$
$$f^*_{g^*_r,g_{i2}}(x)=\mu(\{t\in[0,x]|S_X(t)\in(t_1,t_2)\}).$$

(iv) Suppose that $d_2>d_1p$, $1-\beta<1/d_1$, and $1>\dfrac{(1-p)a}{d_2-d_1p}$, which implies $d_2>a$.

If $\delta>\dfrac{(1-p)a}{d_2-d_1p}$, then $\delta d_2>a$, $\{\phi(t)>0\}=(0,1/a)$, and $\{\phi(t)<0\}=(1/a,1)$. Thus $\mathcal{A}=\emptyset$, $C=(0,1/a)$, and the premium generating functions $G^*_r$ are

$$G^*_r(t)=\begin{cases}G_{i2}(t), & \text{if } t\in(0,1/a];\\ \tilde{G}_r(t), & \text{such that } \nu_X\{t\in(1/a,1]:\tilde{G}_r(t)\leqslant G_{i1}(t)\}=0.\end{cases}$$

The ceded loss functions are

$$f^*_{g^*_r,g_{i1}}(x)=\mu(\{t\in[0,x]|S_X(t)\in(0,1/d_2)\}),$$
$$f^*_{g^*_r,g_{i2}}(x)=\mu(\{t\in[0,x]|S_X(t)\in(0,1/a)\}).$$

If $\delta<\dfrac{(1-p)a}{d_2-d_1p}$, then $\{\phi(t)>0\}=(1-\beta,1/a)$, $\{\phi(t)<0\}=(0,1-\beta)\cup(1/a,1)$, and $C=(1-\beta,1/a)$. Moreover if $\delta>a/d_1$, then $\mathcal{A}=(0,1-\beta)$, the premium generating functions $G^*_r$ and the ceded loss functions are

$$G^*_r(t)=\begin{cases}G_{i1}(t), & \text{if } t\in(0,1-\beta];\\ G_{i2}(t), & \text{if } t\in(1-\beta,1/a];\\ \tilde{G}_r(t), & \text{such that } \nu_X\{t\in(1/a,1]:\tilde{G}_r(t)\leqslant G_{i1}(t)\}=0;\end{cases}$$

$$f^*_{g^*_r,g_{i1}}(x)=\mu(\{t\in[0,x]|S_X(t)\in(0,1/d_2)\}),$$
$$f^*_{g^*_r,g_{i2}}(x)=\mu(\{t\in[0,x]|S_X(t)\in(1-\beta,1/a)\}).$$

If $\delta<a/d_1$, then $\mathcal{A}=\emptyset$, the premium generating functions $G^*_r$ and the ceded loss functions are

$$G^*_r(t)=\begin{cases}G_{i2}(t), & \text{if } t\in(1-\beta,1/a];\\ \tilde{G}_r(t), & \text{such that } \nu_X\{t\in(0,1-\beta]\cup(1/a,1]:\\ & \tilde{G}_r(t)\leqslant G_{i1}(t)\}=0;\end{cases}$$

$$f^*_{g^*_r,g_{i1}}(x)=\mu(\{t\in[0,x]|S_X(t)\in(1-\beta,1/d_2)\}),$$
$$f^*_{g^*_r,g_{i2}}(x)=\mu(\{t\in[0,x]|S_X(t)\in(1-\beta,1/a)\}).$$

From the above, in the first three cases, we find that the optimal reinsurance contract for the type 1 insurer is from a stop-loss contract to a stop-loss with an upper limit contract, and the optimal reinsurance remains unchanged even if $\delta$ becomes smaller for the type 2 insurer. For the fourth case, the optimal contract for the type 1 is the same as the first case, and the optimal contract for the type 2 will be from stop-loss contract to stop-loss with upper limit contract. Thus, we know that the insurer is not willing to cede more risk to the reinsurer when $\delta$ is smaller, i.e., the higher default risk.

## 5 Conclusions

Under the framework of distortion risk measures, we revisit the Bowley reinsurance problem with the reinsurer's default risk when the type of insurer is unknown to the reinsurer. By assuming that the reinsurer adopts a general premium principle[23] and distortion risk measures, the Bowley solutions under the reinsurer's default risk are derived in full generality with the help of marginal profit functions. Our results generalize Ref. [22, Theorem 3.1] to the case where the reinsurer defaults his risk. The optimal ceded loss functions depend on the underlying risk distribution, the distortion functions used by the insurer, the cost function, recovery rate, default risk level, and the probabilities that the reinsurer assigns to the insurer of being a certain type. By implementing some numerical examples, we find that the shut-down policy, the pooling stop-loss policies, the layer or limited stop-loss contracts are possible candidates of the optimal indemnity functions for the insurer.

As a future study, we are interested in extending the current study to the case where there is a multiplier type of background risk[33]. In addition, since different types of insurers may have different distributions of losses, we wish to design optimal Bowley reinsurance contracts with such asymmetric information as well.

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## Conflict of interest

The authors declare that they have no conflict of interest.

## Biography

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