Intersection complex via residue

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Graphical abstract

A normal crossing divisor gives rise to a stratification of a smooth scheme, and a logarithmic connection of a vector bundle along the divisor induces residue maps along each stratum.

Public summary

- We provide an intrinsic definition of intersection subcomplex via these residues.
- We present an explicit geometric description of it.

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Abstract: We provide an intrinsic algebraic definition of the intersection complex for a variety.

Keywords: algebraic geometry; intersection complex; weight filtration

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1 Introduction

Intersection homology theory is a generalization of singular homology for singular algebraic varieties.

In Ref. [1], Sheng and Zhang established a positive characteristic analog of an intersection cohomology theory for polarized variations of Hodge structures and proposed an algebraic definition of the intersection complex, but with the help of coordinate systems. Here, we provide an intrinsic definition of the intersection complex via residues and provide a geometric description of it.

The remainder of this paper is organized as follows. Section 2 establishes notations and presents key definitions. Section 3 provides the main theorem and its proof. Finally, in Section 4, an explicit computation following the spirit of proof in surface case is made, and a counterexample is discussed.

2 Intersection complex

Let $(X, D)$ be a smooth scheme over a regular locally Noetherian scheme $S$ with a reduced smooth normal crossing divisor $D = \sum_{i} D_{i}$, where $I$ is a finite index set, and $\varepsilon$ be a locally free coherent sheaf with an integrable logarithmic divisor, where $I$ is a finite index set, and $\lambda$ be a $S$-scheme.

We introduce some natural morphisms of log-differential sheaves before providing our definitions.

Suppose $X$ is of relative dimension $n$ over $S$. Owing to smoothness of $X$ and the definition of simple normal crossing divisors, for any $x \in X$, there exists a neighborhood $U$ of $x$ such that we can find a coordinate system $(t_{1}, \ldots, t_{n}, \ldots, t_{n})$ such that $D \cap U$ is defined by the equation $t_{1} \times \cdots \times t_{n} = 0$. As an immediate result, $\Omega_{\mathcal{E}^{\alpha}}(\log D)$ admits an $O_{U}$ basis

$$\{\omega_{\varepsilon} = \text{dlog} t_{1}, \ldots, \omega_{\varepsilon} = \text{dlog} t_{n}\}.$$ 

Moreover, it induces a free system of generators for $\Omega_{\mathcal{E}^{\alpha}}(\log D)$.

For $1 \leq i, j \leq r$ and $a \geq 1$, we define

$$\beta^{i}_{j} : \Omega_{\mathcal{E}^{\alpha}}^{a}(\log D) \rightarrow \Omega_{\mathcal{E}^{\alpha}}^{a}(\log (D - D_{i})|_{D_{i}}),$$

$$\phi' + \phi \wedge \text{dlog} t_{i} \rightarrow \phi_{ij}D_{i},$$

where $\phi'$ lies in span $\omega_{\varepsilon}$ with $i \notin I$.

One can consider $\beta^{i}_{j}$ as taking the residual part of a log differential form along $D_{i}$, and $\gamma^{j}_{i}$ is the restriction of the $D_{i}$-regular log differential forms to $D_{i}$. Obviously, $\beta^{i}_{j}$ and $\gamma^{j}_{i}$ are surjective and independent of the coordinate system, respectively. For simplicity, we omit the upper symbol $a$.

Clearly, for any log connection $\nabla$, the composite map $(\beta \otimes I_{D}) \circ \nabla$ factors through $\gamma_{i}$.

We call the second map the residue map of $\nabla$ along $D_{i}$, and denote it as $\text{Res}_{i}(\nabla)$.

We can generalize morphisms above to the multi-indices case as follows. For a subset $I = \{j_{1}, \ldots, j_{r}\} \subseteq \{1, 2, \ldots, r\}$ with $j_{1} < j_{2} < \cdots < j_{r}$, set $D_{I} = \cap_{j \in I} D_{j}$, and define the residue $\text{Res}_{I}$ of the connection $\nabla$ along $D_{I}$ as follows:

$$\text{Res}_{I}(\nabla) = \text{Res}_{I_{1}}(\nabla) \circ \cdots \circ \text{Res}_{I_{r}}(\nabla).$$

We define $\beta$ and $\gamma_{I}$ in a similar manner.

The following diagram naturally commutes.

$$\begin{array}{ccc}
\Omega^{a}_{\mathcal{E}^{\alpha}}(\log (D_{I} - D_{I})) & \rightarrow & \Omega^{a}_{\mathcal{E}^{\alpha}}(\log (D)) \\
\downarrow & & \downarrow \\
\Omega^{a}_{\mathcal{E}^{\alpha}}(\log (D_{I} - D_{I})|_{D_{I}}) & \rightarrow & \Omega^{a}_{\mathcal{E}^{\alpha}}(\log (D - D_{I})|_{D_{I}})
\end{array}$$

where $l : \varepsilon \rightarrow \varepsilon|_{D_{I}}$ is the canonical restriction map.

Now we can define the intersection complex. Set
Intersection complex via residue

Let \(X := \bigcup_{j \in \mathbb{J}} \bigcap_{s \in [I]} D_s, 1 \leq s \leq |I|\), then the following descending chain gives rise to a stratification of \(X\):

\[X := X \supset X_{s-1} \supset \cdots \supset X_{|\mathbb{J}|-1} \supset X_{|\mathbb{J}|} = \emptyset.
\]

And let \(j_s : U_s := X - X_{s-1} = X - X_{s} \) be the natural inclusion for \(n - |I| \leq j \leq \infty\).

**Definition 2.1.** Notations as above. We inductively define res-intersection complex \(IC\) as follows:

- \(IC_j(e) = IC_j(e)_{|I_j} \)
- Assume \(IC_j(H, \nabla)(U_s)\) is defined. A section \(\beta \in j_s\) belongs to \(IC_j(e, \nabla)(U_{s-1})\) when the following two conditions are satisfied:
  1. \(\beta\) has log poles along \(D_{s-1}\), and
  2. \(Res_{j_s} \beta \in \text{Im}(Res_{j_s} e : \mathcal{E}|_{D_s} \otimes \Omega^{\geq s})\)

Then we provide a geometric description of res-intersection complex in the sequel of this section. For any subset \(j\) of \(I\), let \(D_j := \cap_{s \in [I]} D_s\), \(D^j = D_j - \cup_{s \in I^j} (D^j \cap D_s)\) and let \(D^j = X - D\).

Set theoretically, we have the equation \(X = \cup_{j \in I} D^j\). Each \(D^j\) is a locally closed subspace of \(X\), and thus we can endow \(D^j\) with reduced subscheme structure.

**Proposition 2.1.** If \(Res(e, \nabla)\) are bundle morphisms for all \(i \in \{1, 2, \cdots, r\}\), then the res-intersection complex is a complex of locally free sheaves if it is restricted to each stratum \(D^j\), where \(I\) is an index subset of \(\{1, 2, \cdots, r\}\) and \(D^j\) is endowed with a reduced subscheme structure.

We employ the following lemma to prove Proposition 2.1.

**Lemma 2.1.** Let \(X\) be a reduced Noetherian scheme, and let \(F\) be a coherent sheaf on \(X\). Consider the function

\[\phi_\theta(x) = \dim_{k(x)} F \otimes \theta,\]

where \(k(x) = O/I_m\) is the residue field at point \(x\). If \(\phi\) is constant, then \(F\) is locally free.

**Proof of Proposition 2.1.** Consider the reduced scheme \(D^j\) and its associated coherent sheaf \(IC_j = IC_j(X, e)_{|I_j}\). Because of the assumption the divisors are reduced, Lemma 2.1, the proposition is proven if we can show that the dimension of the fibre of sheaf, which is \(\phi_{IC_j}\), is constant over \(D^j\).

For each \(x \in D^j\), \(IC_j(X, e)(x)\) is an \(\Omega^\bullet_{x, I}\) module spanned by basis

\[\{ \text{Res}_s(e) \otimes \text{dlog}\theta_s \}\]

where \(\text{Res}_s(e)\) represents the restriction of \(\text{Res}_s(e)\) on \(e\). Due to that \(\text{Res}_s(e)\) are bundle morphisms, \(\phi_{IC_j}\) is constant if we restrict it to each degree and stratum \(D^j\). Therefore, \(IC_j(X, e)_{|I_j}\) is a complex of locally free sheaves.

**Proof of Proposition 2.1.** It is a local problem, we may assume \(X = \text{Spec} A\) and \(F = M\), where \(A\) is a reduced commutative local ring with maximal ideal \(m\) and \(M\) is a finite \(A\)-module.

We only have to show \(M\) is a free \(A\)-module. Assume that \(k(m)\) vector space \(M/mM\) has dimension \(n\). We use Nakayama’s lemma to lift the basis for \(M/mM\) into a set of generators \(m_1, m_2, \cdots, m_n\). It is sufficient to demonstrate that \(m_i\) is linearly independent. Suppose that \(\sum a_i m_i = 0\), where \(a_i \in A\).

In addition, \(a_i\) must lie in \(m\) for all \(i\), because the generators \(m_i\) form the basis of the fibre \(M/mM\). Choose \(q \in \text{Spec} A\) arbitrarily; then, the images of \(m_i\) in \(M/qM\) generate vector space. In addition, \(\phi\) is constant, implying that they are, in fact, a basis, similarly to \(a_i \in q\) for all \(i\).

Therefore, \(a_i\) lies in the intersection of the prime ideals of \(A\), which is the nilradical of \(A\), and thus \(a_i = 0\) because \(A\) is assumed to be reduced. This completes this proof.

It is interesting to investigate the case where \((e, \nabla)\) comes from the polarized variation of Hodge structures. Let us consider a quick recall of this (cf. Ref. [3]). If \(X\) is a complex variety, and \(E\) is a local system over \(X - D\) underlies a polarized variation of Hodge structures, then we obtain a vector bundle \((e, \nabla)\) equipped with a flat connection via a Riemann-Hilbert correspondence over \(X - D\). There is a canonical extension of \(e\) to a vector bundle with a logarithmic flat connection over \(X\), with the residue of the connection along divisor \(D\) being the log of the monodromy of the divisor (up to a scalar), which we denote as \(N\). It can be observed that \(N\) is topologically defined.

In Refs. [4, 5], the intermediate extension complex can be fibre-wisely expressed as follows: for \(x \in D^j\) and a set of coordinates \(z\), the fibre of intermediate extension complex at \(x\) is an \(\Omega^\bullet_{x, E}\) sub-module generated by the sections \(\bigwedge_{r \leq j} d_{z_r}^{(j)}\) for \(r \in N, E\) and \(j \in I\). The differential map of the complex at fibre is defined as

\[d(\bar{v} \otimes \text{dlog} t_s) = \sum(N(v) \otimes \text{dlog} t_s) \wedge \text{dlog} t_s\]

Note that the residue of the connection \(\nabla_i\) is exactly (up to a scalar) the endomorphism \(N_i\) if it is restricted to the stratum \(D^j\). It can be easily seen that the res-intersection complex coincides with the intermediate extension complex. From this perspective, we provide an algebraic definition of the intermediate extension complex.

**3 Main theorem**

In the following, we show that the res-intersection subcomplex above coincides with the intersection subcomplex defined in Ref. [1].

Let \(X, D, e\) be as in the previous section. Given a coordinate system

\[\{t_1, t_2, \cdots, t_r, t_{r+1}, \cdots, t_L\}\]

of \(U\), locally we can write

\[\nabla = \sum_{s \in I} \nabla_s \text{dlog} t_s + \sum_{s \in J} \nabla_s \text{dlog} t_s.\]

Due to that the set \(\{t_1, t_2, \cdots, t_l\}\) forms a basis of the log sheaf. For subset \(I = \{j_1, j_2, \cdots, j_t\} \subseteq \{1\}\) with \(j_1 < j_2 < \cdots < j_t\), let \(\nabla = \nabla_{j_1} \circ \nabla_{j_2} \circ \cdots \circ \nabla_{j_t}\). We can generalize diagram (1) as follows:

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\[ e \otimes \Omega^{\geq 0}_g \left( \log (D - D_i) \right) \xrightarrow{\gamma_{x,i}} e \otimes \Omega^{g-\geq}_g \left( \log (D - D_i) \right)|_{D_i} \]

\[ e \otimes \Omega^{g\leq 0}_g \left( \log (D - D_i) \right) \xrightarrow{\beta_i} e \otimes \Omega^{d-\leq 0}_g \left( \log (D - D_i) \right)...(2) \]

In Ref. [1], the intersection complex is defined as follows:

**Definition 3.1.** \( IC(X, e) \) is an \( \Omega \) graded submodule of \( \Omega_g (\log D) \otimes e \) generated by the abelian substructure

\[ \sum_{i \in I} \nabla e_i \otimes \omega_i, \]

where \( U \) is a open subset of \( X \), and \( M = \{1, 2, \ldots, r \} \).

Our main theorem is as follows:

**Theorem 3.2.** If \( \text{Res}(\nabla): e|_{D_i} \rightarrow e|_{D_i} \) are bundle morphisms for all \( i \in I \), then \( IC(X, e) = IC(X, e) \).

This proof makes essential use of the weight filtration of the log complex.

**Definition 3.3.** Weight filtration \( W \) of the logarithmic complex is defined as follows:

\[ e \otimes \Omega^g \left( \log (D - D_i) \right) \xrightarrow{\sum_{i \in I}} e \otimes \Omega^g \left( \log (D - D_i) \right) \]

where \( l: e \rightarrow e|_{D_i} \) is the canonical restriction map. And if \( \text{Res} \) is a bundle morphism, then

\[ (l_i \otimes \beta_i)(\nabla, \omega_i)(e \otimes \Omega^{g\geq 0}_g) = (\text{Res}(\nabla))(e|_{D_i}) \otimes \Omega^g_{D_i} \]

**(6)**

**Proof.** It is a basic fact of weight filtration, for a rigorous proof of this lemma the reader is referred to [6] and [7].

Firstly, one have to verify that the upper arrow is well defined. That is, one have to show \( \nabla, \omega_i(e \otimes \Omega^{g\geq 0}_g) \) is contained in \( W^g\Omega^g (\log D) \otimes e \). It is straightforward because the source the map \( e \otimes \Omega^g \) is weight zero and the map \( \nabla, \omega_i \) is of weight \( [I] \).

Note that we have \( \beta_i(W_g \Omega^g (\log D)) = \Omega^{g\geq 0}_g \), hence the vertical arrow on the right is well-defined. The commutativity of the diagram follows from restricting the diagram 1 on subbundle \( e \otimes \Omega^{g\geq 0}_g \subset e \otimes \Omega^g_{D_i} (\log (D - \sum_{j \neq i} D_j)|_{D_i}) \). It remains to show the equation 6. It is easy to see the sheaf on right side is contained in left side. By the commutativity of the diagram 1 again, one has the left side of the equation 6 is contained in

\[ (\text{Res}(\nabla))(e|_{D_i}) \otimes \Omega^g_{D_i} (\log (D - \sum_{j \neq i} D_j)|_{D_i}). \]

Therefore, the equation 6 follows from the following claim.

**Claim:** If \( \text{Res} \) is a bundle morphism, then we have the equation

\[ (\text{Res}(\nabla))(e|_{D_i}) \otimes \Omega^{g\geq 0}_g (\log (D - \sum_{j \neq i} D_j)|_{D_i}) \cap (e|_{D_i}) \otimes \Omega^{\leq 0}_g = (\text{Res}(\nabla))(e|_{D_i}) \otimes \Omega^{\leq 0}_g. \]

For the \( \supseteq \) direction, it is obvious. For the other direction, the sheaf \( \text{Res}(\nabla)(e|_{D_i}) \) is locally free due to the assumption that \( \text{Res} \) is a bundle morphism, thus it has no torsion along \( D_i \), where \( j \in I - I \). This completes the proof of the claim.

The second provides a local description of the weight filtration along divisor in terms of the coordinates.

We now return to the proof of the main theorem.

**Proof of Theorem 3.1.** Without a loss of generality, we asume that \( r = n \).

Clearly, \( IC(X, e) \subset IC(X, e) \), since the restriction \( \nabla \) on \( D \) is exactly the residue map \( \text{Res} \).

Conversely, we consider any \( s \in IC(X, e) \subset IC(X, e) \), such that \( \ker(\beta_i) = e \otimes \ker(\beta_i) \), and we denote it as \( s \). Taking \( m = a \) in the exact sequence (4), we know that \( \ker(\beta_i) \) is exactly the residue map \( \text{Res} \).

Replacing \( s \) with \( s \), by the definition of res-intersection complex, one has \( \beta_i(W_g \Omega^g (\log D)) = e \otimes \Omega^{g\geq 0}_g \), and one has \( s \in e \otimes \Omega^{g\geq 0}_g (\log D) \) by the construction. Then we can chase in the diagram (5) for index set \( I \) with \( |I| = a \), by the equation 6, we obtain a section \( e_i \in e \otimes \Omega^{g\leq 0}_g \) such that \( s_i = \nabla(e_i) \omega_i = e \otimes \ker(\beta_i) \), therefore, we have

\[ s := s_i = \sum_i \nabla(e_i) \omega_i = e \otimes \ker(\beta_i), \]

where \( I \) is a of cardinal \((a - 1)\). Therefore, we have \( s_i \in W^{l}(\Omega^g (\log D)) \otimes e \) by (4).

Repeat the processes above \( a \) times, we obtain \( s \in e \otimes W^a = \Omega^a (\log D) \otimes e \) and \( e_i \in e \otimes \Omega^{g\leq 0}_g \) with \( I \subseteq I \), satisfying

\[ s = \sum_i \nabla(e_i) \omega_i + s, \]

which implies \( s \in IC(X, e) \cap \Omega^a (\log D) \otimes e \). This completes our proof.

Repeat the processes \( a \) times, finally we obtain \( s \in W^a = \Omega^a (\log D) \otimes e \), which implies

**4 Surface case**

In this section, we provide an explicit calculation for the surface case and present an example to the main theorem without bundle morphism condition.
Let \( X = \text{Spec}(k[t_1, t_2]) \) be a surface and let \( D = D_1 + D_2 \) defined by the equation \( t_1 t_2 = 0 \). The divisor gives rise to a stratification of the surface as \( X = D_1 D_1' D_1 D_2' \). With the help of the coordinates \( t_i \), we can write \( \nabla = \nabla_i d \log t_i + \nabla_d \log t_d \).

1. \( IC^i(X, e) = IC^i(X, e) \), because both are equal to \( E \).

2. In the one-degree term, note that the sections of the sheaf \( IC^i(X, e) \) of the form:

\[
\tilde{s} = \nabla_1(e_1) d \log t_1 + \nabla_2(e_2) d \log t_2 + f_1 dt_1 + f_2 dt_2.
\]

Let \( s \in IC^i(X, e) \) (X). To verify \( s \in IC^i_1(X, e) \), we aim to find \( e_1, e_2 \). By definition, \( s \) satisfies \( \beta(s) \in \text{Im}(\text{Res}_1) \). Consider a commutative diagram

\[
\begin{array}{ccc}
\varepsilon & \xrightarrow{\nabla} & \varepsilon \otimes \Omega^1 \text{log}(D) \\
\downarrow l_1 & & \downarrow l_1 \otimes \beta_1 \\
\varepsilon|_{l_1} & \xrightarrow{\text{Res}_1} & \varepsilon|_{l_1}.
\end{array}
\]

The first vertical arrow is surjective, so we can find a section \( e_i \in \varepsilon \) such that \( s - \nabla(e_i) d \log t_1 \) is in the kernel of the second vertical, which is \( \varepsilon \otimes \Omega^1 \text{log}(D_1) \), replacing \( 1 \) with \( g \), and we obtain a section \( e_i \) of \( \varepsilon \), by the exact sequence (4), \( s_1 = s - (\nabla_1(e_1) d \log t_1 + \nabla_2(e_2) d \log t_2) \) is of weight zero. In other words, it is regular, which allows us to write:

\[
s = \nabla_1(e_1) d \log t_1 + \nabla_2(e_2) d \log t_2 + f_1 dt_1 + f_2 dt_2,
\]

where \( e_1, e_2, f_1, f_2 \in \varepsilon(U) \).

3. Using the same pattern, a section \( \omega \) in \( IC^i(X, e) \) (X) is of the form

\[
t = (\nabla_2(e_2)) d \log t_2 \wedge d \log t_1 + \nabla_1(e_1) d \log t_1 \wedge d t_2 + \nabla_2(e_2) d \log t_2 \wedge d t_2 + \nabla_1(e_1) d \log t_1 \wedge d t_1 + f_1 d t_1 + f_2 d t_2.
\]

by definition. For any section \( s \in IC^i(X, e) \) (X), we aim to get section \( e_{12}, e_1, e_2, f \). By (5), we have the following commutative diagram:

\[
\begin{array}{ccc}
\varepsilon & \xrightarrow{\nabla_{12 \otimes 12}} & \varepsilon \otimes \Omega^2 \text{log}(D) \\
\downarrow l_{12} & & \downarrow l_{12} \otimes \beta_{12} \\
\varepsilon|_{l_{12}} & \xrightarrow{\text{Res}_{12}} & \varepsilon|_{l_{12}}.
\end{array}
\]

where both vertical arrows are surjective, we obtain \( e_{12} \in \varepsilon \) such that \( s_1 := s - (\nabla(e_{12}) \omega_{12}) \in \ker \varepsilon \otimes \beta_{12} \), which is of weight one by short exact sequence (4).

Then consider diagram (5) and set \( a = 2, I = [1, 2] \). Using the same argument as above, we obtain \( \tilde{e}_2 \) in \( \varepsilon \otimes \Omega^2 \), such that \( s_1 - \nabla_1(e_1) d \log t_1 \wedge d t_2 + \nabla_2(e_2) d \log t_2 \wedge d t_2 \) in \( \ker(\beta_2 \otimes \beta_1) \), which is exactly \( \Omega^2 \otimes \varepsilon \) by (4). Therefore, \( e_{12}, e_1, e_2, f \) are the section we want. This completes our proof.

In the sequel of the section we will present an example, which is provided by Ref. [1], to show that the main theorem will be wrong if the residue morphisms are not assumed to be bundle morphisms. Let \( k \) be a perfect field of character \( p, (X, D) \) is as above. Define logarithmic connection over

\[
\nabla : O_X \rightarrow \Omega^1 \text{log}(D),
\]

\[
f \mapsto df + (f t^p)^* \cdot \frac{dt}{t^n}.
\]

One can verify that \( \nabla \) is integral, and the residue morphisms are as follows:

\[
\text{Res}_1(\nabla) = \tau^2, \quad \text{Res}_2(\nabla) = 0.
\]

One can see that \( \text{Res}_1(\nabla) \) has torsion at \( t_1 \), hence it is not a bundle morphism. By definition, we have \( IC_1 = (t^p, dt) \wedge \Omega_2 \). Consider the section \( \tau^2 dt \), it is a section in \( IC_1 \) but not in \( IC^i \).

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Conflict of interest

The author declares that he has no conflict of interest.

Biographies

Xiaojin Lin is currently a graduate student under the tutelage of Prof. Mao Sheng at the University of Science and Technology of China. His research interests focus on Hodge theory and vector bundle.

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