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On integrable non-canonical geodesic flow on two-dimensional torus

ZHANG Hai

(School of Mathematical and Computational Sciences, Anqing Normal University, Anqing 246133, China)

Abstract: A non-canonical metric on two-dimensional torus was introduced. It was proved that its geodesic flow is Liouville integrable and has vanishing topological entropy when restricted onto invariant hypersurface.

Key words: geodesic flow; non-canonical metric; topological entropy

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二维环面上的一类可积测地流

章 海

(安庆师范大学数学与计算科学学院,安徽安庆 246133)

摘要:引入了二维环面上的一类非典则的黎曼度量,并且证明了它对应的测地流是 Liouville 可积的.限制到不变超曲面上时,证明了该测地流的拓扑熵为零.

关键词:测地流;非典则度量;拓扑熵

0 Introduction

Let (M^n, g) be a smooth Riemannian manifold with metric $g=(g_{ij})$. Its cotangent bundle T^*M plays an outstanding role in physics. It serves as a natural phase space of particles or systems. There exists a canonical symplectic structure ω on T^*M. In terms of the natural fiberwise coordinates (x^i, p_i) of T^*M, it is given by $\omega=\sum \mathrm{d}x^i \wedge \mathrm{d}p_i$.

For any smooth function f defined on T^*M, one can associate a Hamiltonian vector field X_f defined by $\omega(X_f, \cdot)=\langle \mathrm{d}f, \cdot \rangle$. We call (T^*M, ω, X_H) a Hamiltonian dynamical system with Hamiltonian H. In the canonical coordinates (x^i, p_i), its equations of motion read

$$\dot{x}^i=\frac{\partial H}{\partial p_i},\ \dot{p}_i=-\frac{\partial H}{\partial x^i},\ i=1,\cdots,n.$$

In particular, the system is called the geodesic flow of the metric g, if the Hamiltonian function is

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Biography:ZHANG Hai, male, born in 1982, PhD/lecturer. Research field: Integrable geodesic flows.
E-mail: haizhang@mail.ustc.edu.cn

$$H(x,p)=\frac{1}{2}\mid p\mid_x^2=\frac{1}{2}\sum_{i,j}g^{ij}p_ip_j \quad (1)$$

where $|\cdot|_x^2=\langle\cdot,\cdot\rangle_x$ denotes the norm (scalar product) of the vector space T_x^*M induced by g.

A geodesic flow is called integrable if it is Liouville integrable as a Hamiltonian system. That is, there exist $n=\dim M$ involutive and functionally independent first integrals on T^*M. Integrable geodesic flow has been extensively studied, see e. g. Refs. [1-3].

In this note we will study the dynamic behavior of the geodesic flow on two-dimensional non-canonical torus $\mathbb{T}^2(\varepsilon)$ with the metric

$$g=r^2[\sin^2\theta+(1+\varepsilon)^2\cos^2\theta]d\theta^2+(R+r\cos\theta)^2d\phi^2 \quad (2)$$

where $R>r>0$ are constants and $\varepsilon\in\mathbb{R}$ is a constant. When $\varepsilon=0$, this metric is a canonical metric on torus $\mathbb{T}^2$. From Eq. (1), the corresponding geodesic Hamiltonian reads

$$H=\frac{p_\theta^2}{2r^2[\sin^2\theta+(1+\varepsilon)^2\cos^2\theta]}+\frac{p_\phi^2}{2(R+r\cos\theta)^2} \quad (3)$$

This paper is organized as follows. In section 1 we will prove the complete integrability of the geodesic flow using the method of Killing vector fields. In section 2 we show that the geodesic flow has a vanishing topological entropy.

1 Complete integrability of geodesic flow

In this section, we will show that

Theorem 1.1 The geodesic flow on the non-canonical torus $\mathbb{T}^2(\varepsilon)$ is integrable with an additional linear integral.

Our strategy is to show the existence of such a first integral by working out the Killing tensor of valence one, i. e. the Killing vector field on the base manifold M.

Recall that a Killing tensor T of valence s on (M,g) is a symmetric $(s,0)$-tensor satisfying the equation $[T,G]=0$, where $[\cdot,\cdot]$ is the Schouten bracket and $G=g^{-1}$ is the inverse metric. According to Refs. [6-8], a first integral K polynomial in momenta is in bijective correspondence with Killing tensor T through the relation

$$K=\sum_{i_1\cdots i_s}T^{i_1\cdots i_s}(x)p_{i_1}\cdots p_{i_s} \quad (4)$$

where $T^{i_1\cdots i_s}$ is the component of T.

Let $X=a(\theta,\phi)\partial_\theta+b(\theta,\phi)\partial_\phi$, be a vector field on the torus, where smooth functions a,b are 2π bi-periodic w. r. t. θ and ϕ, that is, $a(\theta+2\pi,\phi)=a(\theta,\phi+2\pi)=a(\theta,\phi)$, and similarly for b.

Lemma 1.1 The Killing tensor equation for X, $[X,G]=0$ is equivalent to the following system of differential equations

$$-(2\varepsilon+\varepsilon^2)\sin\theta\cos\theta\, a+[\sin^2\theta+(1+\varepsilon)^2\cos^2\theta]a_\theta=0 \quad (5)$$

$$r^2[\sin^2\theta+(1+\varepsilon)^2\cos^2\theta]a_\phi+(R+r\cos\theta)^2b_\theta=0 \quad (6)$$

$$-r\sin\theta\, a+(R+r\cos\theta)b_\phi=0 \quad (7)$$

Proof For a vector field, the Schouten bracket coincides with Lie derivative. Let X be a Killing field, then $[X,G]=\mathscr{L}_XG=0$. Since $gG=\mathrm{id}$, the above equation is equivalent to $\mathscr{L}_Xg=0$, the vanishing of Lie derivative of metric g along the direction of X. Observe that

$$\mathscr{L}_Xg=\mathscr{L}_X(r^2[\sin^2\theta+(1+\varepsilon)^2\cos^2\theta]d\theta\odot d\theta)+\mathscr{L}_X((R+r\cos\theta)^2d\phi\odot d\phi)=:A+B \quad (8)$$

where $\odot$ denotes symmetric product of tensors.

The first part is, essentially,

$$A/2r^2=-(2\varepsilon+\varepsilon^2)\sin\theta\cos\theta\, a\, d\theta\odot d\theta+[\sin^2\theta+(1+\varepsilon)^2\cos^2\theta]\cdot(a_\theta d\theta\odot d\theta+a_\phi d\phi\odot d\theta) \quad (9)$$

and the second part is

$$B/2=-(R+r\cos\theta)r\sin\theta\, a\, d\phi\odot d\phi+(R+r\cos\theta)^2(b_\theta d\theta\odot d\phi+b_\phi d\phi\odot d\phi) \quad (10)$$

where we have used the identities $\mathscr{L}_X\theta=a$, $\mathscr{L}_X\phi=b$.

Combining (8) ~ (10) together and simplifying them, one obtains the system of equations (5)~(7).

Lemma 1.2 The system of equations (5)~(7) has only the trivial solution

$$a(\theta,\phi)\equiv 0,\ b(\theta,\phi)\equiv C=\text{const} \quad (11)$$

Proof Note that Eq. (5) implies

$$\frac{\partial \log |a|}{\partial \theta} = \frac{a_\theta}{a} = \kappa(\theta),$$

where $\kappa(\theta)$ is a smooth function of θ only (similarly for functions below). So,

$$\log |a| = \tilde{\kappa}(\theta) + \lambda(\phi).$$

Then we have

$$a(\theta,\phi) = \pm \exp\tilde{\kappa}(\theta) \cdot \exp\lambda(\phi) := \Theta(\theta)\Phi(\phi) \tag{12}$$

Substitute (12) into (6) and (7), one can solve them to get

$$\left.\begin{aligned} b_\theta &= -\frac{r^2\Theta(\theta)\Phi_\phi(\sin^2\theta + \cos^2\theta\varepsilon^2 + 2\cos^2\theta\varepsilon + \cos^2\theta)}{(R + r\cos\theta)^2} \\ b_\phi &= \frac{r\sin\theta\Theta(\theta)\Phi(\phi)}{R + r\cos\theta} \end{aligned}\right\} \tag{13}$$

From Eq. (13), the identity $\partial_\phi b_\theta = \partial_\theta b_\phi$ turns out to be

$$\begin{aligned} &-\Phi_{\phi,\phi} r\Theta(\theta)(\sin^2\theta + \cos^2\theta\varepsilon^2 + 2\cos^2\theta\varepsilon + \cos^2\theta) = \\ &\quad \Phi(\phi)[\cos\theta\sin\theta\Theta_\theta r + \Theta(\theta)\cos\theta R + \\ &\quad \sin\theta\Theta_\theta R + \Theta(\theta) r] \end{aligned}$$

Since both sides are the products of a function of variable θ and that of variable ϕ, it follows that both of them are identical to each other (up to a constant factor C), that is,

$$\begin{aligned} &C\Theta(\theta)(\sin^2\theta + \cos^2\theta\varepsilon^2 + 2\cos^2\theta\varepsilon + \cos^2\theta) = \\ &\quad \cos\theta\sin\theta\Theta_\theta r + \Theta(\theta)\cos\theta R + \sin\theta\Theta_\theta R + \Theta(\theta) r \end{aligned} \tag{14}$$

Also when substituting (12) into (5) and eliminating the functions of ϕ, one has

$$\begin{aligned} &-(2\varepsilon + \varepsilon^2)\sin\theta\cos\theta\Theta(\theta) + \\ &\quad [\sin^2\theta + (1+\varepsilon)^2\cos^2\theta]\Theta_\theta = 0 \end{aligned} \tag{15}$$

Comparing (14) with (15), one concludes that the function Θ must vanish identically, which yields $a(\theta,\phi) \equiv 0$ and $b(\theta,\phi) \equiv C = \text{const}$.

We are now in a position to show the Theorem 1.1.

Proof of Theorem 1.1 Let X be a Killing vector field on the non-canonical torus, then by Lemma 1.2, X must be of the form, $X = C\partial_\phi$, where C is the constant value of function b. Consequently the vector space of Killing vectors is of dimension one with generator ∂_ϕ.

According to the correspondence relation (4), the Killing vector $X = C\,\partial_\phi$ corresponds to linear first integral $K = p_\phi$. It is easy to see p_ϕ is independent of the Hamiltonian (3). This implies that the geodesic system is Liouville integrable.

Remark It is clear that the non-canonical torus admits a one-dimensional algebra of Killing fields. The torus is compact, therefore its isometries group consists of finite connected components with the identity component isomorphic to circle group S^1.

2 Vanishing topological entropy

In this section we investigate the dynamic behavior of the geodesic flow of the non-canonical torus $\mathbb{T}^2(\varepsilon)$. Our main result is the following.

Proposition 2.1 For $c>0$, restricted to each hypersurface $Q = H^{-1}(c)$, the geodesic flow of non-canonical torus (2) has a vanishing topological entropy h_t, i.e. $h_t = 0$.

For the definition of topological entropy in general dynamical system, the reader is referred to, e.g. Refs. [2,4]. Our major method to prove Proposition 2.1 is using a result due to Ref. [9]. Other than for geodesic flow only it can be stated in more general setting as below.

Let $\mathscr{S}$ be a four-dimensional symplectic manifold and H a Hamiltonian function on $\mathscr{S}$. Let $Q = H^{-1}(c)$, $c \in \mathbb{R}$, be a nonsingular compact level set of H, here nonsingularity means the one-form field $\mathrm{d}H$ never vanishes on Q. By implicit function theorem, it implies that $Q \subset \mathscr{S}$ is an embedded submanifold of dimension three, i.e. a smooth hypersurface. Suppose the system X_H is completely integrable with an independent additional integral K on $\mathscr{S}$. We restrict the Hamiltonian flow X_H on full phase space $\mathscr{S}$ to invariant subspace Q. For the restricted flow, we have

Lemma 2.1[9] Under the above general setting, suppose when restricted to Q, function $\widetilde{K} = K|_Q$ verifies either one of the following conditions:

① $\widetilde{K}$ is real analytic,

② the connected components of the set of critical points of $\widetilde{K}$ form submanifolds.

Then the topological entropy of the restricted flow on Q vanishes.

In order to use the above lemma to study geodesic system of the non-canonical torus, we want to know which level set of H is nonsingular.

As a kinematic energy, Hamiltonian (3) satisfies $H\geqslant 0$. Let $Q=H^{-1}(c)\subset T^*\mathbb{T}^2(\varepsilon)$ be a level set, where c is a constant. Then $c\geqslant 0$, and $c=0$ corresponds to the special case

$$Q = H^{-1}(0) = \{(\theta,\phi,0,0) \mid (\theta,\phi) \in \mathbb{T}^2(\varepsilon)\},$$

which is nothing but the smooth zero section of the vector bundle $T^*\mathbb{T}^2(\varepsilon)$. It is diffeomorphic to the base manifold $\mathbb{T}^2(\varepsilon)$, so Q has dimension two. Moreover, we have

Lemma 2.2 The smooth function H is degenerate iff $p_\theta=p_\phi=0$.

Proof In a componentwise form, we calculate the one-form $\mathrm{d}H$ as

$$\mathrm{d}H = \Big[\frac{2p_\theta\sin\theta(W-1)}{r^2W^2\cos\theta}+\frac{2rp_\phi\sin\theta}{(R+r\cos\theta)^3},\ 0,\frac{p_\theta}{r^2W},\frac{p_\phi}{(R+r\cos\theta)^2}\Big]$$

where $W=\sin^2\theta+(1+\varepsilon)^2\cos^2\theta$.

Degeneracy of H corresponds to $\mathrm{d}H=0$, which is exactly $p_\theta=p_\phi=0$ in view of the above explicit expression.

Taking into account the above result, we shall restrict ourselves to the condition $c>0$.

Lemma 2.3 For any $c>0$, $Q\subset T^*\mathbb{T}^2(\varepsilon)$ is a compact hypersurface.

Proof According to the above discussions, when $c>0$, function $H:\mathbb{T}^2(\varepsilon)\to\mathbb{R}$ is smooth and non-singular on $Q=H^{-1}(c)$. Implicit function theorem immediately implies that $Q\subset T^*\mathbb{T}^2(\varepsilon)$ is a regular submanifold. It suffices to show that Q is compact.

We argue by a detailed topological analysis. Note the tangent bundle of a torus is a trivial bundle, which is a general property of any Lie group. Also a vector bundle is always isomorphic to its dual bundle, it follows that $T^*\mathbb{T}^2(\varepsilon)$ is trivial, hence $T^*\mathbb{T}^2(\varepsilon)\simeq\mathbb{T}^2(\varepsilon)\times\mathbb{R}^2$. The coordinates globally split as $(\theta,\phi)\in\mathbb{T}^2(\varepsilon)$ and $(p_\theta, p_\phi)\in\mathbb{R}^2$.

One can see that W is bounded, so let us assume $W\leqslant c_0$, $c_0\in\mathbb{R}^+$. The condition $H=c$ implies that

$$c\geqslant\frac{p_\theta^2}{2r^2W}\geqslant\frac{p_\theta^2}{2r^2c_0}.$$

It follows

$$|p_\theta|\leqslant r\sqrt{2cc_0} \tag{16}$$

Similarly,

$$c\geqslant\frac{p_\phi^2}{2(R+r\cos\theta)^2}\geqslant\frac{p_\phi^2}{2(R+r)^2}$$

implies

$$|p_\phi|\leqslant(R+r)\sqrt{2c} \tag{17}$$

(16) together with (17) give that

$$p_\theta^2+p_\phi^2\leqslant 2c[r^2c_0+(R+r)^2]:=s^2 \tag{18}$$

where s is a constant, $s>0$.

Let us denote by D_s the closed disc, $D_s=\{(p_\theta, p_\phi)\mid p_\theta^2+p_\phi^2\leqslant s^2\}$. The condition (18) means $(p_\theta, p_\phi)\in D_s\subset\mathbb{R}^2$. Therefore, for any $x=(\theta,\phi,p_\theta,p_\phi)\in Q=H^{-1}(c)\subset\mathbb{T}^2(\varepsilon)\times\mathbb{R}^2$, it holds $x\in\mathbb{T}^2(\varepsilon)\times D_s$, namely, $Q\subset\mathbb{T}^2(\varepsilon)\times D_s$. Note $\mathbb{T}^2(\varepsilon)\times D_s$ is compact because of compactness of $\mathbb{T}^2(\varepsilon)$ and D_s. For Q to be compact, it suffices to show $Q\subset\mathbb{T}^2(\varepsilon)\times D_s$ is closed.

It is obvious that $Q\subset T^*\mathbb{T}^2(\varepsilon)$ is closed as it is the preimage of a closed set $\{c\}$ under the continuous mapping H. Also compactness of $\mathbb{T}^2(\varepsilon)\times D_s$ implies $\mathbb{T}^2(\varepsilon)\times D_s\subset T^*\mathbb{T}^2(\varepsilon)$ is closed. The above two results together imply the closeness of Q in $\mathbb{T}^2(\varepsilon)\times D_s$. This completes the proof.

Proof of Proposition 2.1 By Lemmas 2.2 and 2.3, for $c>0$ any level set $Q=H^{-1}(c)$ is a compact and nonsingular hypersurface of cotangent bundle $T^*\mathbb{T}^2(\varepsilon)$. Moreover, according to Proposition 1.1, the geodesic system of (2) admits an integral $K=p_\phi$. Obviously, integral $K:T^*\mathbb{T}^2(\varepsilon)\to\mathbb{R}$ is real analytic. Moreover, the natural inclusion mapping $i: Q\to T^*\mathbb{T}^2(\varepsilon)$ is real analytic as well. The composition of them gives a real analytic function $K\circ i$, which is the restricted function $\widetilde{K}$ of

K to Q. This means that we have verified the condition ① in Lemma 2.1. We thereby complete the proof of Proposition 2.1.

3 Conclusion

In this note we have defined a non-canonical metric on two-dimensional torus $\mathbb{T}^2(\varepsilon)$, and shown that the geodesic flow is Liouville integrable and admits vanishing topological entropy when restricted onto ceratin hypersurface. As remarked earlier, when $\varepsilon=0$, $\mathbb{T}^2(\varepsilon)$ is a canonical torus. On the canonical torus, the corresponding result has been obtained in Ref. [5]. It would be challenging to consider more arbitrary deformation of the canonical metric on torus and study to what extent it can be deformed while still preserving integrability and other good dynamical behaviors.

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