文章编号:0253-2778(2020)03-0328-07

A note on the maximal degree in random k-trees

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Abstract: The random variable Z_n is investigated, the maximal node degree in a random k-tree at step n for $k \geqslant 2$. It is shown that as $n \to \infty$, $Z_n / n^{(k-1)/k}$ has an almost sure limit, which is a positive random variable. The result is also extended to the random k-Apollonian networks model for $k \geqslant 3$.

Key words: random networks; k-tree; Apollonian network; maximal degree

CLC number: O157.4　　**Document code**: A　　doi:10.3969/j.issn.0253-2778.2020.03.010

2010 Mathematics Subject Classification: Primary 05C80, 90B15; secondary 60F15

Citation: FENG Qunqiang, ZHOU Chun, LI Xing. A note on the maximal degree in random k-trees[J]. Journal of University of Science and Technology of China, 2020,50(3):328-334.

冯群强, 邹纯, 李兴. 随机 k-树的最大度数[J]. 中国科学技术大学学报,2020,50(3):328-334.

随机 k-树的最大度数

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摘要:研究了随机 k-树($k \geqslant 2$)在 n 时刻的最大度 Z_n. 当 $n \to \infty$ 时,$Z_n / n^{(k-1)/k}$ 几乎处处收敛到一个正值随机变量. 在此基础上,将类似结果推广到了 $k \geqslant 3$ 的随机阿波罗图上.

关键词:随机网络;k-树;阿波罗网络;最大度

0 Introduction

The random k-trees model, which was first proposed in Ref. [1], is a randomized version of the well-known k-trees in graph theory[2], and plays an important role in graph minor area[3]. There are several equivalent definitions of k-trees, and we employ only one of them, from which a random k-tree can be generated in an iterative manner. Let $k \geqslant 1$ be a fixed integer. Starting with a k-clique of nodes labeled by $0_1, 0_2, \cdots, 0_k$, successively the nodes with labels $1, 2, \cdots, n$ are born, where at each step the new node will be attached to all of the nodes of an already existing k-clique chosen uniformly at random. In particular, for the case $k = 1$ one can get the well studied random tree model—random recursive trees[4-5]. Here, we should emphasize that for $k \geqslant 2$, the random k-trees are no more trees. For instance, the special case $k = 2$ coincides with the scale-free growing network model proposed in the Ref. [6], where triangle is one of the most

Received: 2019-10-29; **Revised**: 2020-01-12

Foundation item: Supported by the National Natural Science Foundation of China(11771418).

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frequently appearing subgraphs (see Fig. 1 for an illustration of this model with $k=2$ at first several steps).

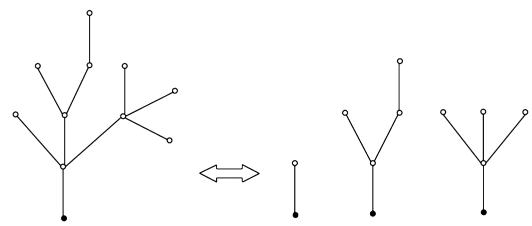

Fig. 1 An evolution of random 2-trees at step $n=0,1$ and 2

One of the most fundamental terminologies in graph theory is the node degree. The degree of a node v in the graph G is the number of edges that are incident to v (i. e., the number of the neighbours of v in G). The degree distribution in a graph is defined to be the fraction of nodes with degree $k\geqslant 0$. In other words, for any $k\geqslant 0$, the degree distribution shows the probability that the degree of a node picked uniformly at random in the graph is k. It has been shown in the literature that the asymptotic degree distribution of random 1-trees (i. e., random recursive trees) is essentially different from that of random k-trees with $k\geqslant 2$, as the tree size goes to infinity. The asymptotic degree distribution for the case $k=1$ is the geometric distribution with parameter 1/2 (see Ref. [7]. That is, the proportion of nodes with degree $d\geqslant 1$ is asymptotically equal to $1/2^d$. While for $k\geqslant 2$, Ref. [1] proved that the proportion of nodes with degree $d\geqslant 1$ follows asymptotically a power law $d^{-\gamma}$ with exponent $\gamma=2+\dfrac{1}{k-1}$.

Another related topic on the node degrees is to consider the maximal degree. The maximal degree in random k-trees with $k=1$ has been well studied by several authors[8-10]. Our main concern here is to study the asymptotic behavior of the maximal degree in a random k-trees at step n for any $k\geqslant 2$, as n goes to infinity. For recent results on random k-trees, we refer readers to Refs. [11-15].

Throughout this work, we fix integer $k\geqslant 2$. To state our main result in the following, we need some necessary notation. In the evolving process of generating a random k-tree, we let $\mathscr{F}_n$ be the σ-algebra generated by the first n steps, and T_n the resulting graph after step n, for any integer $n\geqslant 0$. We denote by $[n]$ the node set of T_n, i. e., $[n]=\{0_1, 0_2, \cdots, 0_k, 1, 2, \cdots, n\}$ with $[0]=\{0_1, 0_2, \cdots, 0_k\}$. For convenience, we say $j=0$ if $j\in[0]$. In a random k-tree T_n, let $D_j(n)$ denote the degree of node $j\in[n]$, and Z_n the maximal degree, i. e.,

$$Z_n=\max_{j\in[n]} D_j(n).$$

It is not hard to see that here the random variable $D_0(n)$ is well defined for each $n\geqslant 0$, since the degrees of all nodes in $[0]$ share a common distribution by symmetry.

Theorem 0.1 In a random k-tree T_n with $k\geqslant 2$, there exists a positive random variable Z such that $Z_n/n^{(k-1)/k}$ converges to Z almost surely and in L^p for all $p\geqslant 1$, as $n\to\infty$.

The rest of the paper is organized as follows. Section 1 is devoted to proving Theorem 0.1, by constructing a sequence of suitable martingales and applying the martingale convergence theorem. In Section 2, we extend our main result to the random k-Apollonian networks model.

1 Proof of Theorem 0.1

To study the maximal degree in a random k-tree, we shall use a martingale method developed in Ref. [16-17]. He investigated the maximal degree in a generalized Barabàsi-Albert random tree by constructing a wide class of martingales[18]. Later, using similar arguments the results are also extended to the preferential attachment graphs model[19]. Our method used here is an adaptation of theirs.

To begin with, we now introduce some useful notation as follows. For real numbers $a, b>-1$ with $a-b>-1$, the generalized binomial coefficient can be written in terms of gamma functions:

$$\binom{a}{b}=\frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)},$$

where a, b are not necessarily integers. For any node $j\in[n]$, we define an operator $\Delta_j(n+1)=D_j(n+1)-D_j(n)$, indicating the increment of the degree of node j from step n to $n+1$. For any real $d>-1$ and $j\in[n]$, we denote

$$M_{j,d}(n)=\frac{\Gamma\left(n+\frac{1}{k}\right)}{\Gamma\left(n+\frac{1+(k-1)d}{k}\right)}\binom{\overline{D}_j(n)+d-1}{d},$$

where

$$\overline{D}_j(n)=D_j(n)-\frac{k(k-2)}{k-1}.$$

It is obvious to see that $M_{j,d}(n)$ is well-defined, since $\overline{D}_j(n)>1$ follows by the simple fact the degree $D_j(n)\geqslant k$ holds for all $j\in[n]$.

At the initial step, there is only one k-clique in the graph T_0. When a node $j\geqslant 1$ is born, it is not hard to see that exactly k new distinct k-cliques are created, containing node j.

As a result, the number of k-cliques in any random k-tree T_n is exactly $kn+1$. If the degree of node j increases by 1 at some step afterwards, however, the number of k-cliques which contains node j only increases by $k-1$. Then, for any given node with degree $D^*\geqslant k$, at any step there are exactly

$$k+(D^*-k)(k-1)=(k-1)D^*-k(k-2)$$

distinct k-cliques containing it. Therefore, conditioning on $\mathscr{F}_n$, we have that for any $j\in[n]$,

$$\mathbb{E}[D_j(n+1)\mid\mathscr{F}_n]=D_j(n)+\frac{(k-1)D_j(n)-k(k-2)}{kn+1},$$

which implies that

$$\mathbb{E}[\Delta_j(n+1)\mid\mathscr{F}_n]=\frac{(k-1)\overline{D}_j(n)}{kn+1},\ j\in[n]\tag{1}$$

Based on the relation (1), the result on the degree of any given node in a random k-tree T_n is given in the next proposition.

Proposition 1.1 Let $D_j(n)$ be the degree of node j in a random k-tree T_n. Then for any node j, as $n\to\infty$, there exists a nonnegative random variable ξ_j such that $D_j(n)/n^{(k-1)/k}$ converges to ξ_j almost surely and in L^p for any $p\geqslant 1$, with moments

$$\mathbb{E}[\xi_j^r]=\frac{\Gamma\left(j+\frac{1}{k}\right)\Gamma\left(r+\frac{k}{k-1}\right)}{\Gamma\left(j+\frac{1+(k-1)r}{k}\right)\Gamma\left(\frac{k}{k-1}\right)},\quad r=1,2,\cdots.\tag{2}$$

Proof In what follows, let node j and real $d>-1$ be fixed. Recall that we set $j=0$ if $j\in[0]$. By considering the two cases $\Delta_j(n)=0$ or $\Delta_j(n)=1$, and using the well-known recursion for gamma functions, i.e., $\Gamma(x)=(x-1)\Gamma(x-1)$ for any $x>1$, it is easy to check that for all $d>-1$,

$$\binom{\overline{D}_j(n+1)+d-1}{d}=\binom{\overline{D}_j(n)+d-1}{d}\left(1+\frac{d\Delta_j(n)}{\overline{D}_j(n)}\right),\ n\geqslant j\tag{3}$$

It follows by (1) that

$$\mathbb{P}(\Delta_j(n)=1\mid\mathscr{F}_n)=\frac{(k-1)\overline{D}_j(n)}{kn+1}=1-\mathbb{P}(\Delta_j(n)=0\mid\mathscr{F}_n),$$

which, together with Eq. (3), implies that

$$\mathbb{E}[M_{j,d}(n+1)\mid\mathscr{F}_n]=\frac{\Gamma\left(n+1+\frac{1}{k}\right)}{\Gamma\left(n+1+\frac{1+(k-1)d}{k}\right)}\mathbb{E}\left[\binom{\overline{D}_j(n+1)+d-1}{d}\mid\mathscr{F}_n\right]=$$

$$\frac{\Gamma\left(n+1+\frac{1}{k}\right)}{\Gamma\left(n+1+\frac{1+(k-1)d}{k}\right)\left(1+\frac{(k-1)d}{kn+1}\right)\binom{\overline{D}_j(n)+d-1}{d}}=$$

$$\frac{\Gamma\left(n+\frac{1}{k}\right)}{\Gamma\left(n+\frac{1+(k-1)d}{k}\right)}\binom{\overline{D}_j(n)+d-1}{d}=M_{j,d}(n).$$

Then, for any node $j\geqslant 0$ and $d>-1$, we have that the sequence $\{M_{j,d}(n)\}_{n=j}^{\infty}$ is a positive martingale with respect to the filtration $\{\mathscr{F}_n\}_{n=j}^{\infty}$. Therefore, by the martingale convergence theorem[20], it follows that $M_{j,d}(n)$ converges almost surely to some nonnegative random variable with finite mean, as $n\to\infty$. Additionally, one can see that the moments $\mathbb{E}[D_j{}^d(n)]$ are finite for all $d>-1$, as d is chosen arbitrarily. More precisely, given the initial value

$$\overline{D}_j(j)=D_j(j)-\frac{k(k-2)}{k-1}=\frac{k}{k-1},$$

we have that for any $d>-1$,

$$\mathbb{E}\left[\binom{\overline{D}_j(n)+d-1}{d}\right]=\frac{\Gamma\left(n+\frac{1+(k-1)d}{k}\right)\Gamma\left(j+\frac{1}{k}\right)}{\Gamma\left(j+\frac{1+(k-1)d}{k}\right)\Gamma\left(n+\frac{1}{k}\right)}\binom{\frac{k}{k-1}+d-1}{d},\ n\geqslant j \tag{4}$$

According to Stirling's formula, it is easy to see that

$$\frac{\Gamma(n+\alpha)}{n!}=n^{\alpha-1}(1+O(n^{-1})) \tag{5}$$

holds for any fixed real number α, as $n\to\infty$. Then, as $n\to\infty$, by the fact that $D_j(n)$ converges almost surely to the infinity, we have

$$\binom{\overline{D}_j(n)+d-1}{d}=\binom{D_j(n)+d-1}{d}\left(1+O\left(\frac{1}{D_j(n)}\right)\right)=\frac{D_j{}^d(n)}{\Gamma(d+1)}\left(1+O\left(\frac{1}{D_j(n)}\right)\right) \tag{6}$$

holds almost surely. Hence, it follows by Eq. (4) that for any $n\geqslant j$,

$$\mathbb{E}[D_j{}^d(n)]=\frac{\Gamma\left(j+\frac{1}{k}\right)\Gamma\left(d+\frac{k}{k-1}\right)}{\Gamma\left(j+\frac{1+(k-1)d}{k}\right)\Gamma\left(\frac{k}{k-1}\right)}n^{\frac{(k-1)d}{k}}(1+O(n^{-1})),\quad d>-1 \tag{7}$$

Indeed, using a similar argument one can show that $M_{j,d}(n)$ also has finite moments of all orders greater than -1. By the L^p martingale convergence theorem[20], it thus follows that $M_{j,d}(n)$ converges to its limit also in L^p for any $p\geqslant 1$ as well, as $n\to\infty$. Consider $d=1$ as a special case. By Eqs. (5) and (6), we have that there exists a nonnegative random variable ξ_j such that

$$\lim_{n\to\infty}M_{j,1}(n)=\lim_{n\to\infty}\frac{D_j(n)}{n^{(k-1)/k}}=\xi_j \tag{8}$$

almost surely and in L^p for any $p\geqslant 1$. In addition, we obtain that

$$\mathbb{E}[\xi_j^r]=\lim_{n\to\infty}\mathbb{E}[M_{j,r}(n)]=\mathbb{E}[M_{j,r}(j)]=\frac{\Gamma\left(j+\frac{1}{k}\right)\Gamma\left(r+\frac{k}{k-1}\right)}{\Gamma\left(j+\frac{1+(k-1)r}{k}\right)\Gamma\left(\frac{k}{k-1}\right)},\ r=1,2,\cdots,$$

and the proof of Proposition 1 is complete.

We remark that by using a connection to two-color triangular Pólya urns[21] obtained a weaker result where their mode of convergence is in distribution. Additionally, we can derive the exact formula for any factorial moment of $D_j(n)$ for any $j\in[n]$ according to Eq. (4). In particular, for any $0\leqslant j\leqslant n$, the exact mean and variance of $D_j(n)$ are given by

$$\mathbb{E}[D_j(n)]=\frac{k}{k-1}\left[\frac{n!\,\Gamma\left(j+\frac{1}{k}\right)}{j!\,\Gamma\left(n+\frac{1}{k}\right)}+k-2\right],$$

and

$$\mathrm{Var}[D_j(n)]=\frac{k\Gamma\left(j+\frac{1}{k}\right)}{(k-1)\Gamma\left(n+\frac{1}{k}\right)}\left[\frac{(2k-1)\Gamma\left(n+\frac{2k-1}{k-1}\right)}{(k-1)\Gamma\left(j+\frac{2k-1}{k-1}\right)}.-\frac{n!}{j!}\left(1+\frac{kn!,\Gamma\left(j+\frac{1}{k}\right)}{(k-1)j!,\Gamma\left(n+\frac{1}{k}\right)}\right)\right].$$

Applying Carleman's condition[22], one could verify that the distribution of ξ_j is uniquely determined by its moments.

In the next lemma, we show that the limit random variable ξ_j given in Proposition 1 has no atom at zero for any $j\geqslant 0$.

Lemma 1.1 For any node $j\geqslant 0$, we have that $\mathbb{P}(\xi_j>0)=1$.

Proof First, it follows by Eqs. (5) and (7) that $D_j(n)/n^{(k-1)/k}$ has finite moments of all $d>-1$. As shown in Proposition 1.1, for any fixed integer $j\geqslant 0$, we have that $D_j(n)/n^{(k-1)/k}$ converges almost surely to ξ_j, suggesting that $D_j(n)/n^{(k-1)/k}$ converges in distribution to ξ_j as well. Applying Markov's inequality, for any $\varepsilon>0$, we thus have

$$\mathbb{P}(\xi_j\leqslant \limsup_{n\to\infty}\mathbb{P}\left(\frac{D_j(n)}{n^{(k-1)/k}}\leqslant\varepsilon\right)\leqslant$$

$$\limsup_{n\to\infty}\sqrt{\varepsilon}\,,\ \mathbb{E}\left[\left(\frac{D_j(n)}{n^{(k-1)/k}}\right)^{-\frac{1}{2}}\right]=O(\sqrt{\varepsilon}).$$

Letting $\varepsilon\downarrow 0$ yields to that $\mathbb{P}(\xi_j=0)=0$ for any integer $j\geqslant 0$. Finally, the nonnegativity of random variable ξ_j completes the proof of this lemma.

We are now ready to give the proof of Theorem 0.1 in the following.

Proof of Theorem 0.1 For $0\leqslant j\leqslant n$, we first write

$$Z_j(n)=\max_{i\in[j]}M_{i,1}(n)=\frac{\Gamma\left(n+\frac{1}{k}\right)}{n!}\max_{i\in[j]}\overline{D}_i(n),$$

from which the simple linear relation between Z_n and $Z_n(n)$ is given by

$$Z_n(n)=\frac{\Gamma\left(n+\frac{1}{k}\right)}{n!}\left(Z_n-\frac{k(k-2)}{k-1}\right).$$

Recall that each sequence $\{M_{j,1}(n)\}_{n=j}^{\infty}$ is a nonnegative martingale. Thus, being the maximum of (finite) martingales, the sequence $\{Z_n(n)\}_{n=0}^{\infty}$ is a nonnegative submartingale.

We next show that $Z_n(n)$ converges almost surely and in L^p to some nonnegative random variable Z for any $p\geqslant 1$, as $n\to\infty$. Since x^r is a convex function on $(0,\infty)$ for any $r\geqslant 1$, it is easy to see that the sequence $\{M_{j,1}^r(n)\}_{n=j}^{\infty}$ is also a submartingale, and that the sequence of the corresponding means $\mathbb{E}[M_{j,1}^r(n)]$ is increasing in n. Hence, for any given $j\geqslant 0$ and $r\geqslant 1$, it follows by Proposition 1.1 and Eq. (8) that $M_{j,1}^r(n)$ converges to ξ_j^r almost surely and in L^p for any $p\geqslant 1$, and for all $n\geqslant j$,

$$\mathbb{E}[M_{j,1}^r(n)]\leqslant\mathbb{E}[\xi_j^r].$$

Noting that the random variables $\{\xi_j, j\in[0]\}$ are identically distributed, we pick the random variable ξ_0 to represent this entire class. Then, we have

$$\mathbb{E}[Z_n^r(n)]\leqslant\sum_{i\in[n]}\mathbb{E}[M_{i,1}^r(n)]\leqslant$$

$$k\,\mathbb{E}[\xi_0^r]+\sum_{j=1}^{\infty}\mathbb{E}[\xi_j^r] \tag{9}$$

which is finite for all n (including $n\to\infty$) according to Eq. (2) provided that $r>k/(k-1)$. Thus, the submartingale $\{Z_n(n)\}_{n=0}^{\infty}$ is bounded in L^p for any $p\geqslant 1$. We conclude that, again by the martingale convergence theorem, $Z_n(n)$ converges not only almost surely but also in L^p to some finite-mean random variable for any $p>1$.

To prove that random variable Z is positive, we shall prove that

$$Z=\lim_{j\to\infty}\max\{\xi_i: i\in[j]\} \tag{10}$$

Note that it is sufficient to show that $Z_n(n)$ converges to the right-hand side of Eq. (10) in L^r for some $r>1$.

Let $r>k/(k-1)$ be fixed. Analogously to Eq. (9), we have that for $1\leqslant j<n$,

$$\mathbb{E}[(Z_n(n)-Z_j(n))^r]\leqslant\sum_{i=j+1}^{n}\mathbb{E}[M_{i,1}^r(n)] \tag{11}$$

Taking the limit as $n\to\infty$ on the both sides of Eq. (11) gives that for any fixed integer $j\geqslant 1$,

$$\mathbb{E}\left[\left(\lim_{n\to\infty}\frac{Z_n}{n^{(k-1)/k}}-\max\{\xi_i:i\in[j]\}\right)^r\right]\leqslant\sum_{i=j+1}^{\infty}\mathbb{E}\left[\xi_i^r\right]$$

which can be arbitrarily small with j sufficiently large, as shown in Eq. (9). Then, letting $j\to\infty$ yields to that the desired result Eq. (10) holds. It is now easy to see that the probability $\mathbb{P}(Z>0)=1$ follows by Lemma 1.1.

2 Random Apollonian networks

A structure closely related to k-trees is the k-Apollonian network when $k\geqslant 3$. The k-Apollonian network is the same as a k-tree in every aspect, except that recruiting cliques are always deactivated. That is, once a clique is chosen to attach the new node at any step $n\geqslant 1$, it will never be chosen again since then. The construction of the simplest case of Apollonian networks with $k=3$ originates from the problem of Apollonian circle packing[23]. The random 3-Apollonian networks model was proposed independently Refs. [24-25] as a model for real-life networks such as the network of internet cables or links, collaboration networks or protein interaction networks. Later, Zhang, et al[26] generalized this model by replacing higher-dimensional curvilinear hyperspheres with triangles (i. e., 3-cliques) to obtain the so-called random k-Apollonian networks. For recent advances on the random k-Apollonian networks model, we refer to Refs. [21, 27-30].

It is clear that the methods applied to k-trees would work for k-Apollonian networks and would produce similar types of result. We summarize these results here, without proof. For random k-Apollonian networks we shall use notation, with tildes. For instance, $\widetilde{Z}_n$ denotes the maximal degree in a k-Apollonian network at step n.

Let I_A be the indicator of an event A. By an argument similar to that for random k-trees, we can construct a positive martingale $\{\widetilde{M}_{j,d}(n)\}_{n=j}^{\infty}$ in order to study the limiting behavior of $\widetilde{D}_j(n)$ for any node j in a random k-Apollonian network, where $d>-1$,

$$\widetilde{M}_{j,d}(n)=\frac{\Gamma\left(n+\frac{1}{k-1}\right)}{\Gamma\left(n+\frac{1+(k-2)d}{k-1}\right)}\binom{\overline{\widetilde{D}}_j(n)+d-1}{d},$$

and

$$\overline{\widetilde{D}}_j(n)=\widetilde{D}_j(n)-\frac{k(k-3)+I_{[j=0]}}{k-2}.$$

An analysis following the steps in the proof of Proposition 1 gives that there exists a nonnegative random variable $\tilde{\xi}_j$ such that $\widetilde{D}_j(n)/n^{(k-2)/(k-1)}$ converges to $\tilde{\xi}_j$ almost surely and in L^p for any $p\geqslant 1$, with moments

$$\mathbb{E}\left[\tilde{\xi}_j^r\right]=\frac{\Gamma\left(j+\frac{1}{k-1}\right)\Gamma\left(r+\frac{k-(k-1)I_{\{j=0\}}}{k-2}\right)}{\Gamma\left(j+\frac{1+(k-2)r}{k-1}\right)\Gamma\left(\frac{k-(k-1)I_{\{j=0\}}}{k-2}\right)},$$

$$r=1,2,\cdots.$$

For any $j\in[n]$, we write

$$\widetilde{Z}_n(n)=\max_{i\in[n]}\widetilde{M}_{i,1}(n)=\frac{\Gamma\left(n+\frac{1}{k-1}\right)}{n!}\left(\widetilde{Z}_n-\frac{k(k-3)+I_{\{j=0\}}}{k-2}\right).$$

Following the proof of Theorem 1, then one can obtain that the sequence $\{\widetilde{Z}_n(n)\}_{n=0}^{\infty}$ is a submartingale bounded in L^p for any $p\geqslant 1$. Finally, by the martingale convergence theorem, we arrive at the corresponding results for the random k-Apollonian networks model: In a random k-Apollonian network with $k\geqslant 3$, there exists a positive random variable $\widetilde{Z}$ such that $\widetilde{Z}_n/n^{(k-2)/(k-1)}$ converges to $\widetilde{Z}$ almost surely and in L^p for all $p\geqslant 1$, as $n\to\infty$.

3 Conclusion

In this work, we show the maximal degree in a random k-tree, as well as in an Apollonian network, has an almost sure limit as the tree size grows to infinity. Although this limit is shown to be a positive random variable, the more basic information on its distribution is still absent. The positivity of random variable indicates the non-normality. We put the derivation of the asymptotic distribution of the maximal degree in random k-

trees into our further work.

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