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Inverse degree and properties of graphs

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Abstract: Let $G=(V(G), E(G))$ be a simple graph of order n and size m. The inverse degree of a graph G with no isolated vertices is defined by $ID(G)=\sum_{v_i\in V(G)}\frac{1}{d(v_i)}$, where $d(v_i)$ is the degree of the vertex $v_i\in V(G)$. First, in terms of the inverse degree, sufficient conditions for a connected graph to be k-Hamiltonian, k-edge-Hamiltonian, k-path-coverable, Hamilton-connected, k-connected, 2-edge-connected and β-deficient were obtained, respectively. Second, sufficient conditions for the independence number of a connected graph to be less than or equal to the integer k were given. Finally, a sufficient condition for a connected balanced bipartite graph to be Hamiltonian was given.

Key words: inverse degree; degree sequence; graph properties
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逆度和图的性质

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摘要:设 $G=(V(G),E(G))$ 是 n 个顶点 m 条边的简单图. 无孤立点的图 G 的逆度定义为 $ID(G)=\sum_{v_i\in V(G)}\frac{1}{d(v_i)}$,其中,$d(v_i)$表示顶点 v_i 的度. 首先用逆度刻画了连通图分别是 k-哈密尔顿、k-边哈密尔顿、k-路覆盖、哈密尔顿连通、k-连通、2-边连通和 β-亏损的充分条件. 其次用逆度给出了连通图的独立数小于等于整数 k 的充分条件. 最后用逆度给出了连通的平衡二部图是哈密尔顿图的一个充分条件.
关键词:逆度;度序列;图的性质

0 Introduction

Let G be a simple connected graph with vertex set $V(G)=\{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$. For any $v_i\in V(G)$, we denote $d_G(v_i)$ (or simply $d(v_i)$) by the degree of vertex v_i. Denote by K_n the complete graph on n vertices. For two vertex-disjoint graphs G and H, we use $G\vee H$ to denote the join of G and H; $G+H$ to denote their union.

A Hamiltonian path of the graph G of order n is a path of order n contained in G, and a Hamiltonian cycle of the graph G of order n is a cycle of order n contained in G. The graph G is said to be Hamiltonian if it contains a Hamiltonian cycle, is said to be traceable if it contains a Hamiltonian path, and is said to be Hamilton-connected if every two vertices of G are connected by a Hamiltonian path. A graph G is k-Hamiltonian if for all $|X|\leqslant k$, the subgraph induced by $V(G)\backslash X$ is Hamiltonian. A graph G is k-edge-Hamiltonian if any collection of vertex-disjoint paths with at most k edges altogether belong to a Hamiltonian cycle in G. Thus 0-Hamiltonian and 0-edge-Hamiltonian are the same

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as Hamiltonian. More generally, a graph G is k-path-coverable if $V(G)$ can be covered by k or fewer vertex-disjoint paths. In particular, 1-path-coverable is the same as traceable. A connected graph G is said to be k-connected (or k-vertex connected) if it has more than k vertices and remains connected whenever fewer than k vertices are removed. Similarly, G is k-edge-connected if it has at least two vertices and remains connected whenever fewer than k edges are deleted. The deficiency of a graph G, denoted by $\mathrm{def}(G)$, is the number of vertices unmatched under a maximum matching in G. We call G β-deficient if $\mathrm{def}(G)\leqslant\beta$. Thus a β-deficient graph G of order n has a matching number $n-2\beta$. We use $\alpha(G)$ to denote the independence number of a graph G. An integer sequence $\pi=(d_1\leqslant d_2\leqslant\cdots\leqslant d_n)$ is called graphical if there exists a graph G having π as its vertex degree sequence; in that case, G is called a realization of π. If P is a graph property, such as Hamiltonian or k-connected, we call a graphical sequence π is forcibly P if every realization of π has property P.

Topological indices are numbers associated with molecular structures which serve for quantitative relationships between chemical structures and properties. Many of them are based on the distance[1], the vertex degree[2]. Relations between the distance based and degree based topological indices are given in Ref. [3].

The inverse degree of a graph G is also topological index based on the vertex degree of the graph. The inverse degree of a graph G with no isolated vertices is defined[4] as

$$ID(G)=\sum_{v_i\in V(G)}\frac{1}{d(v_i)},$$

where $d(v_i)$ is the degree of the vertex $v_i\in V(G)$.

The inverse degree (also known as the sum of reciprocals of degrees) appeared first through conjectures of the computer program Graffiti[4]. Motivated by a Graffiti conjecture, Zhang et al[5] established upper and lower bounds on $ID(T)+\gamma(T)$ for any tree T, where γ is the number of independent edges. Hu et al.[6] determined the extremal graphs with respect to $ID(G)$ among all connected graphs of order n and with m edges. Dankelmann et al[7] determined a relation between $ID(G)$ and edge-connectivity. In the same paper a bound is established on the diameter in terms of $ID(G)$. Mukwembi[8] further improved this bound. In addition, Li and Shi[9] improved the bound for trees and unicyclic graphs. Chen and Fujita[10] obtained a nice relation between the diameter and inverse degree of a graph, which settled a conjecture in Ref. [8]. Recently Xu et al.[11] determined upper and lower bounds on inverse degree in terms of chromatic number, clique number, independence number, matching number, edge-connectivity, and number of cut edges. Ref. [12] found some lower and upper bounds on $ID(G)$ and characterized the extremal graphs. Moreover, in the same paper, the inverse degree was compared with other degree-based graph invariants. More recent papers on the inverse degree should refer to Refs. [13-14].

Our main goal in this paper is, by utilizing the inverse index and degree conditions, to derive some sufficient conditions for a variety of graph properties including Hamilton-connected, k-Hamiltonian, k-edge-Hamiltonian, k-path-coverable, k-connected, 2-edge-connected and β-deficient. These graph properties are the concerns of plenty of graph theorists.

1 Lemmas

In order to prove the main theorems in this paper, we need the following results as our lemmas.

Lemma 1.1[15] Let $\pi=(d_1\leqslant d_2\leqslant\cdots\leqslant d_n)$ be a graphical degree sequence and $0\leqslant k\leqslant n-3$. If $d_i\leqslant i+k \Rightarrow d_{n-i-k}\geqslant n-i$, for $1\leqslant i<\frac{1}{2}(n-k)$, then π is forcibly k-Hamiltonian.

Lemma 1.2[16] Let $\pi=(d_1\leqslant d_2\leqslant\cdots\leqslant d_n)$ be a graphical degree sequence and $0\leqslant k\leqslant n-3$. If $d_{i-k}\leqslant i \Rightarrow d_{n-i}\geqslant n-i+k$, for $k+1\leqslant i<\frac{1}{2}(n+k)$, then π is forcibly k-edge-Hamiltonian.

Lemma 1.3[17] Let $\pi=(d_1\leqslant d_2\leqslant\cdots\leqslant d_n)$ be a graphical degree sequence and $k\geqslant 1$. If $d_{i+k}\leqslant i\Rightarrow d_{n-i}\geqslant n-i-k$, for $1\leqslant i<\frac{1}{2}(n-k)$, then π is forcibly k-path-coverable.

Lemma 1.4[18] Let G be a graph of order $n\geqslant 3$ with degree sequence $(d_1, d_2, \cdots, d_n)$, where $d_1\leqslant d_2\leqslant\cdots\leqslant d_n$. If $2\leqslant k\leqslant\frac{n}{2}$, $d_{k-1}\leqslant k \Rightarrow d_{n-k}\geqslant n-k+1$. Then G is Hamilton-connected.

Lemma 1.5[19] Let G be a graph of order $n\geqslant 4$ with degree sequence $(d_1, d_2, \cdots, d_n)$, where $d_1\leqslant d_2\leqslant\cdots\leqslant d_n$. If $d_i\leqslant i+k-2 \Rightarrow d_{n-k+1}\geqslant n-i$, for $1\leqslant i\leqslant\frac{1}{2}(n-k+1)$, then G is k-connected.

Lemma 1.6[20] Let $\pi=(d_1\leqslant d_2\leqslant\cdots\leqslant d_n)$ be a graphical degree sequence. Suppose $n\geqslant k+1$, and $d_1\geqslant k\geqslant 1$. If $d_{i-k+1}\leqslant i-1$ and $d_i\leqslant i+k-2$

$\Rightarrow d_n \geqslant n-i+k-1$, for $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$, then π is forcibly k-edge-connected. In particular, if $k \geqslant \lfloor \frac{n}{2} \rfloor$, then π is forcibly k-edge-connected.

Lemma 1.7[21] Let $\pi=(d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n)$ be a graphical degree sequence, and let $0 \leqslant \beta \leqslant n$ with $n=\beta \pmod 2$. If $d_{i+1} \leqslant i-\beta \Rightarrow d_{n+\beta-i} \geqslant n-i-1$, for $1 \leqslant i \leqslant \frac{1}{2}(n+\beta-2)$, then π is forcibly β-deficient.

Lemma 1.8[22] Let $\pi=(d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n)$ be a graphical degree sequence and $k \geqslant 1$. If $d_{k+1} \geqslant n-k$, then π is forcibly $\alpha(G) \leqslant k$.

Lemma 1.9[23] Let $G=(X, Y; E)$ be a bipartite graph such that $X=\{x_1, x_2, \cdots, x_n\}$, $Y=\{y_1, y_2, \cdots, y_n\}$, $n \geqslant 2$, and $d(x_1) \leqslant d(x_2) \leqslant \cdots \leqslant d(x_n)$, $d(y_1) \leqslant d(y_2) \leqslant \cdots \leqslant d(y_n)$. If $d(x_k) \leqslant k < n \Rightarrow d(y_{n-k}) \geqslant n-k+1$, then G is Hamiltonian.

2 Main results

The main results of this paper are as follows.

Firstly, we consider the k-Hamiltonian and k-edge-Hamiltonian properties. When $k=0$, the 0-Hamiltonian and 0-edge-Hamiltonian properties are both equivalent to the Hamiltonian property.

Theorem 2.1 Let G be a connected graph of order $n \geqslant 9$ and $0 \leqslant k \leqslant n-3$.

(ⅰ) For $k=0$ or $k=1$ or $n-5 \leqslant k \leqslant n-3$, if

$$ID(G) \leqslant 1-\frac{2k-2}{n+k-1}+\frac{n+k-1}{2n-2},$$

then G is k-Hamiltonian, unless

$$G \subseteq K_{\frac{n+k-1}{2}} \vee \frac{n-k+1}{2} K_1.$$

(ⅱ) For $2 \leqslant k \leqslant n-6$, if

$$ID(G) \leqslant 1+\frac{1}{1+k}-\frac{k}{n-2}+\frac{1+k}{n-1},$$

then G is k-Hamiltonian, unless

$$G \subseteq (K_1+K_{n-k-2}) \vee K_{k+1}.$$

Proof Let G be a graph satisfying the conditions in the theorem. Suppose that G is not k-Hamiltonian. Then, by Lemma 1.1, there exists an integer $1 \leqslant i < \frac{1}{2}(n-k)$ such that $d_i \leqslant i+k$ and $d_{n-k-i} \leqslant n-i-1$. So we have

$$ID(G)=\sum_{v_i \in V(G)} \frac{1}{d(v_i)}=\frac{1}{d_1}+\frac{1}{d_2}+\cdots+\frac{1}{d_i}+\frac{1}{d_{i+1}}+\cdots+\frac{1}{d_{n-k-i}}+\frac{1}{d_{n-k-i+1}}+\cdots+\frac{1}{d_n} \geqslant$$
$$\frac{i}{d_i}+\frac{n-2i-k}{d_{n-k-i}}+\frac{i+k}{d_n} \geqslant \frac{i}{i+k}+\frac{n-2i-k}{n-i-1}+\frac{i+k}{n-1}=2-\frac{k}{i+k}-\frac{i+k-1}{n-i-1}+\frac{i+k}{n-1}.$$

Let $f(x)=-\frac{k}{x+k}-\frac{x+k-1}{n-x-1}+\frac{x+k}{n-1}$, $1 \leqslant x \leqslant \frac{n-k-1}{2}$, then

$$f'(x)=\frac{k}{(x+k)^2}-\frac{n+k-2}{(n-x-1)^2}+\frac{1}{n-1}=$$
$$\frac{k(n-1)(n-x-1)^2-(n-1)(n+k-2)(x+k)^2+(x+k)^2(n-x-1)^2}{(n-1)(x+k)^2(n-x-1)^2}.$$

Let

$$g(x)=k(n-1)(n-x-1)^2-(n-1)(n+k-2)(x+k)^2+(x+k)^2(n-x-1)^2,$$

then

$$g'(x)=4x^3+6kx^2-6nx^2+6x^2+2k^2x-8knx+8kx+2nx-2x-4k^2n+4k^2-2kn^2+6kn-4k,$$
$$g''(x)=12x^2-12(n-k-1)x+2k^2-8kn+8k+2n-2.$$

For $1 \leqslant x \leqslant \frac{n-k-1}{2}$, $g''(x) \leqslant g''(1)=2k^2-8kn+20k-10n+22$. And when $0 \leqslant k \leqslant n-3$, $g''(1) \leqslant -10n+22 \leqslant 0$. So $g''(x) \leqslant 0$. Therefore,

$$g'(x) \leqslant g'(1)=-4k^2n+6k^2-2kn^2-2kn+10k-4n+8=$$
$$-2(2n-3)k^2-2(n^2+n-5)k-4(n-2) \leqslant 0.$$

So, for $1 \leqslant x \leqslant \frac{n-k-1}{2}$, $f(x) \geqslant \min\{f(1), f(\frac{n-k-1}{2})\}$. By calculation,

$$f(1)=-\frac{k}{1+k}-\frac{k}{n-2}+\frac{1+k}{n-1},\ f(\frac{n-k-1}{2})=-1-\frac{2k-2}{n+k-1}+\frac{n+k-1}{2n-2}.$$

Let $f(1)-f(\frac{n-k-1}{2})=\frac{-k^3n+2k^2n^2-7k^2n+2k^2-kn^3+6kn^2-11kn+4k+n^3-6n^2+11n-6}{(1+k)(n-2)(n+k-1)(2n-2)}=$

0, we get $k=n-3$; or $k=\frac{\sqrt{n^4-12n^3+32n^2-24n+4}-4n+n^2+2}{2n}\in(n-6,n-5)$; or $k=\frac{-\sqrt{n^4-12n^3+32n^2-24n+4}-4n+n^2+2}{2n}\in(1,2)$.

Case 1 $k=0$ or $k=1$ or $n-5\leqslant k\leqslant n-3$.

In this case,

$$f(x)\geqslant\min\{f(1),f(\frac{n-k-1}{2})\}=f(\frac{n-k-1}{2}).$$

Therefore,

$$ID(G)\geqslant 1-\frac{2k-2}{n+k-1}+\frac{n+k-1}{2n-2}.$$

If $ID(G)=1-\frac{2k-2}{n+k-1}+\frac{n+k-1}{2n-2}$, all equalities above should be attained. Thus, we have $i=\frac{n-k-1}{2}$, $d_1=\cdots=d_{n-k-i}=i+k=\frac{n+k-1}{2}$, $d_{n-k-i+1}=\cdots=d_n=n-1$, so $G\cong K_{\frac{n+k-1}{2}}\vee\frac{n-k+1}{2}K_1$. It is easy to check that the graph $K_{\frac{n+k-1}{2}}\vee\frac{n-k+1}{2}K_1$ is not k-Hamiltonian.

Case 2 $2\leqslant k\leqslant n-6$.

In this case, $f(x)\geqslant\min\{f(1),f(\frac{n-k-1}{2})\}=f(1)$. Therefore

$$ID(G)\geqslant 1+\frac{1}{1+k}-\frac{k}{n-2}+\frac{1+k}{n-1}.$$

If $ID(G)=1+\frac{1}{1+k}-\frac{k}{n-2}+\frac{1+k}{n-1}$, all equalities above should be attained. Thus, we have $i=1$, $d_1=1+k$, $d_2=\cdots=d_{n-k-1}=n-2$, $d_{n-k}=\cdots=d_n=n-1$, so $G\cong(K_1+K_{n-k-2})\vee K_{k+1}$. It is easy to check that the graph $(K_1+K_{n-k-2})\vee K_{k+1}$ is not k-Hamiltonian.

This completes the proof.

Theorem 2.2 Let G be a connected graph of order $n\geqslant 9$ and $0\leqslant k\leqslant n-3$.

(ⅰ) For $k=0$ or $k=1$ or $n-5\leqslant k\leqslant n-3$, if

$$ID(G)\leqslant 1-\frac{2k-2}{n+k-1}+\frac{n+k-1}{2n-2},$$

then G is k-edge-Hamiltonian, unless

$$G\cong K_{\frac{n+k-1}{2}}\vee\frac{n-k+1}{2}K_1.$$

(ⅱ) For $2\leqslant k\leqslant n-6$, if

$$ID(G)\leqslant 1+\frac{1}{1+k}-\frac{k}{n-2}+\frac{1+k}{n-1},$$

then G is k-edge-Hamiltonian, unless

$$G\cong(K_1+K_{n-k-2})\vee K_{k+1}.$$

Proof Let G be a graph satisfying the conditions in the theorem. Suppose that G is not k-edge-Hamiltonian. Then, by Lemma 1.2, there exists an integer $k+1\leqslant i\leqslant\frac{1}{2}(n+k-1)$ such that $d_{i-k}\leqslant i$ and $d_{n-i}\leqslant n-i+k-1$. So we have

$$ID(G)=\sum_{v_i\in V(G)}\frac{1}{d(v_i)}=\frac{1}{d_1}+\frac{1}{d_2}+\cdots+\frac{1}{d_{i-k}}+\frac{1}{d_{i-k+1}}+\cdots+\frac{1}{d_{n-i}}+\frac{1}{d_{n-i+1}}+\cdots+\frac{1}{d_n}\geqslant$$
$$\frac{i-k}{d_{i-k}}+\frac{n-2i+k}{d_{n-i}}+\frac{i}{d_n}\geqslant\frac{i-k}{i}+\frac{n-2i+k}{n-i+k-1}+\frac{i}{n-1}=2-\frac{k}{i}-\frac{i-1}{n-i+k-1}+\frac{i}{n-1}.$$

Let $f(x)=-\frac{k}{x}-\frac{x-1}{n-x+k-1}+\frac{x}{n-1}$, $k+1\leqslant x\leqslant\frac{n+k-1}{2}$, then

$$f'(x)=\frac{k(n-1)(n-x+k-1)^2-(n-1)(n+k-2)x^2+x^2(n-x+k-1)^2}{(n-1)x^2(n-x+k-1)^2}.$$

Let $g(x)=k(n-1)(n-x+k-1)^2-(n-1)(n+k-2)x^2+x^2(n-x+k-1)^2$, then

$$g'(x)=4x^3-6kx^2-6nx^2+6x^2+2k^2x+4knx-4kx+2nx-2x+4kn-2k^2n+2k^2-2kn^2-2k,$$
$$g''(x)=12x^2-12kx-12nx+12x+2k^2+4kn-4k+2n-2.$$

For $k+1\leqslant x\leqslant\frac{n+k-1}{2}$, $g''(x)\leqslant g''(k+1)=2k^2-8kn+20k-10n+22$. And when $0\leqslant k\leqslant n-3$, $g''(k+1)\leqslant -10n+22\leqslant 0$.

So, for $k+1\leqslant x\leqslant\frac{n+k-1}{2}$, $f(x)\geqslant\min\{f(k+1),f(\frac{n+k-1}{2})\}$. By calculation,

$$f(k+1)=-\frac{k}{1+k}-\frac{k}{n-2}+\frac{1+k}{n-1},f(\frac{n+k-1}{2})=-1-\frac{2k-2}{n+k-1}+\frac{n+k-1}{2n-2}.$$

Let

$$f(k+1)-f(\frac{n+k-1}{2})=\frac{-k^3n+2k^2n^2-7k^2n+2k^2-kn^3+6kn^2-11kn+4k+n^3-6n^2+11n-6}{(1+k)(n-2)(n+k-1)(2n-2)}=0,$$

we get $k=n-3$; or $k=\dfrac{\sqrt{n^4-12n^3+32n^2-24n+4}-4n+n^2+2}{2n}\in(n-6,n-5)$; or

$$k=\frac{-\sqrt{n^4-12n^3+32n^2-24n+4}-4n+n^2+2}{2n}\in(1,2).$$

Case 1 $k=0$ or $k=1$ or $n-5\leqslant k\leqslant n-3$.

In this case,

$$f(x)\geqslant \min\{f(k+1),f(\frac{n+k-1}{2})\}=f(\frac{n+k-1}{2}).$$

Therefore $ID(G)\geqslant 1-\dfrac{2k-2}{n+k-1}+\dfrac{n+k-1}{2n-2}$.

If $ID(G)=1-\dfrac{2k-2}{n+k-1}+\dfrac{n+k-1}{2n-2}$, all equalities above should be attained. Thus, we have $i=\dfrac{n+k-1}{2}$, $d_1=\cdots=d_{n-i}=i=\dfrac{n+k-1}{2}$, $d_{n-i+1}=\cdots=d_n=n-1$, so $G\cong K_{\frac{n+k-1}{2}}\vee\dfrac{n-k+1}{2}K_1$. It is easy to check that the graph $K_{\frac{n+k-1}{2}}\vee\dfrac{n-k+1}{2}K_1$ is not k-edge-Hamiltonian.

Case 2 $2\leqslant k\leqslant n-6$.

In this case,

$$f(x)\geqslant\min\{f(k+1),f(\frac{n-k-1}{2})\}=f(1).$$

Therefore $ID(G)\geqslant 1+\dfrac{1}{1+k}-\dfrac{k}{n-2}+\dfrac{1+k}{n-1}$.

If $ID(G)=1+\dfrac{1}{1+k}-\dfrac{k}{n-2}+\dfrac{1+k}{n-1}$, all equalities above should be attained. Thus, we have $i=k+1$, $d_1=1+k$, $d_2=\cdots=d_{n-k-1}=n-2$, $d_{n-k}=\cdots=d_n=n-1$, so $G\cong(K_1+K_{n-k-2})\vee K_{k+1}$. It is easy to check that the graph $(K_1+K_{n-k-2})\vee K_{k+1}$ is not k-edge-Hamiltonian.

This completes the proof.

Corollary 2.1 Let G be a connected graph of order $n\geqslant 3$. If

$$ID(G)\leqslant\frac{3}{2}+\frac{2}{n-1},$$

then G is Hamiltonian, unless

$$G\cong K_{\frac{n-1}{2}}\vee\frac{n+1}{2}K_1.$$

Our next task will be to consider k-path-coverable property.

Theorem 2.3 Let G be a connected graph of order $n\geqslant 3$ and $1\leqslant k\leqslant n-3$. If

$$ID(G)\leqslant 1+\frac{2k}{n-k-1}+\frac{2}{n-k-1}+\frac{n-k-1}{2(n-1)},$$

then G is k-path-coverable, unless

$$G\cong(\frac{n+k+1}{2})K_1\vee K_{\frac{n-k-1}{2}}.$$

Proof Let G be a graph satisfying the conditions in the theorem. Suppose that G is not k-path-coverable. Then, by Lemma 1.3, there exists an integer $1\leqslant i\leqslant\frac{1}{2}(n-k-1)$ such that $d_{i+k}\leqslant i$ and $d_{n-i}\leqslant n-i-k-1$. So we have

$$\begin{aligned}ID(G)&=\sum_{v_i\in V(G)}\frac{1}{d(v_i)}=\\&\frac{1}{d_1}+\frac{1}{d_2}+\cdots+\frac{1}{d_{i+k}}+\frac{1}{d_{i+k+1}}+\cdots+\\&\frac{1}{d_{n-i}}+\frac{1}{d_{n-i+1}}+\cdots+\frac{1}{d_n}\geqslant\\&\frac{i+k}{d_{i+k}}+\frac{n-2i-k}{d_{n-i}}+\frac{i}{d_n}\geqslant\\&\frac{i+k}{i}+\frac{n-2i-k}{n-i-k-1}+\frac{i}{n-1}=\\&2+\frac{k}{i}-\frac{i-1}{n-i-k-1}+\frac{i}{n-1}.\end{aligned}$$

Let $f(x)=\dfrac{k}{x}-\dfrac{x-1}{n-x-k-1}+\dfrac{x}{n-1}$, $1\leqslant x\leqslant\dfrac{n-k-1}{2}$, then $f'(x)=-\dfrac{k}{x^2}-\dfrac{n-k-2}{(n-x-k-1)^2}+\dfrac{1}{n-1}<-\dfrac{n-k-2}{(n-x-k-1)^2}+\dfrac{1}{n-1}<-\dfrac{1}{n-k-2}+\dfrac{1}{n-1}<0$.

Therefore, $f(x)\geqslant f(\dfrac{n-k-1}{2})=-1+\dfrac{2k}{n-k-1}+\dfrac{2}{n-k-1}+\dfrac{n-k-1}{2(n-1)}$. Thus,

$$ID(G)\geqslant 1+\frac{2k}{n-k-1}+\frac{2}{n-k-1}+\frac{n-k-1}{2(n-1)}.$$

If $ID(G)=1+\dfrac{2k}{n-k-1}+\dfrac{2}{n-k-1}+\dfrac{n-k-1}{2(n-1)}$, all equalities above should be attained. Thus, we have $i=\dfrac{n-k-1}{2}$, $d_1=\cdots=d_{i+k+1}=i$, $d_{i+k+2}=\cdots=d_n=n-1$, so $G\cong(\dfrac{n+k+1}{2})K_1\vee K_{\frac{n-k-1}{2}}$. It is easy to check that the graph $(\dfrac{n+k+1}{2})K_1\vee K_{\frac{n-k-1}{2}}$ is not k-path-coverable.

This completes the proof.

Corollary 2.2 Let G be a connected graph of order $n\geqslant 4$. If

$$ID(G)\leqslant 1+\frac{4}{n-2}+\frac{n-2}{2(n-1)},$$

then G is traceable, unless $G\cong(\dfrac{n+2}{2})K_1\vee K_{\frac{n-2}{2}}$.

Theorem 2.4 Let G be a connected graph of order $n\geqslant 3$. If

$$ID(G) \leqslant 1 + \frac{n}{2(n-1)},$$

then G is Hamilton-connected, unless

$$G \cong \frac{n}{2}K_1 \vee K_{\frac{n}{2}}.$$

Proof Let G be a graph satisfying the conditions in the theorem. Suppose that G is not Hamilton-connected. Then, by Lemma 1.4, there exists an integer $2 \leqslant k \leqslant \frac{n}{2}$ such that $d_{k-1} \leqslant k$ and $d_{n-k} \leqslant n-k$. So we have

$$ID(G) = \sum_{v_i \in V(G)} \frac{1}{d(v_i)} = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_{k-1}} + \frac{1}{d_k} + \cdots + \frac{1}{d_{n-k}} + \frac{1}{d_{n-k+1}} + \cdots + \frac{1}{d_n} \geqslant \frac{k-1}{d_{k-1}} + \frac{n-2k+1}{d_{n-k}} + \frac{k}{d_n} \geqslant \frac{k-1}{k} + \frac{n-2k+1}{n-k} + \frac{k}{n-1} = 1 - \frac{1}{k} + \frac{n-2k+1}{n-k} + \frac{k}{n-1}.$$

Let

$$f(x) = -\frac{1}{x} + \frac{n-2x+1}{n-x} + \frac{x}{n-1}, \quad 2 \leqslant x \leqslant \frac{n}{2},$$

then

$$f'(x) = \frac{x^4 - 2nx^3 + (3n-2)x^2 - 2n(n-1)x + n^2(n-1)}{(n-1)x^2(n-x)^2}.$$

Let

$$g(x) = x^4 - 2nx^3 + (3n-2)x^2 - 2n(n-1)x + n^2(n-1), \quad 2 \leqslant x \leqslant \frac{n}{2},$$

then

$$g'(x) = 4x^3 - 6nx^2 + 2(3n-2)x - 2n(n-1),$$
$$g''(x) = 2(6x^2 - 6nx + 3n - 2).$$

So $g''(x) \leqslant g''(2) = 2(-9n+22) < 0$. Thus $g'(x) \leqslant g'(2) = -2n^2 - 10n + 24 < 0$.

So, for $2 \leqslant x \leqslant \frac{n}{2}$, $f(x) \geqslant \min\{f(2), f(\frac{n}{2})\}$.

$f(2) = \frac{1}{2} - \frac{1}{n-2} + \frac{2}{n-1}$, $f(\frac{n}{2}) = \frac{n}{2(n-1)} = \frac{1}{2} + \frac{1}{2(n-1)}$. Since $n \geqslant 4$, $f(2) \geqslant f(\frac{n}{2})$, thus $f(x) \geqslant f(\frac{n}{2}) = \frac{n}{2(n-1)}$.

Therefore $ID(G) \geqslant 1 + \frac{n}{2(n-1)}$.

If $ID(G) = 1 + \frac{n}{2(n-1)}$, all equalities above should be attained. Thus, we have $k = \frac{n}{2}$, $d_1 = d_2 = \cdots = d_{k-1} = k$, $d_k = k$, $d_{k+1} = \cdots = d_n = n-1$, so $G \cong \frac{n}{2}K_1 \vee K_{\frac{n}{2}}$. It is easy to check that the graph $\frac{n}{2}K_1 \vee K_{\frac{n}{2}}$ is not Hamilton-connected.

This completes the proof.

Theorem 2.5 Let G be a connected graph of order $n \geqslant k+1 \geqslant 2$.

(i) If $ID(G) \leqslant \frac{2n}{n-2}$, then G is 1-connected.

(ii) If $k \geqslant 2$,

$$ID(G) \leqslant \frac{1}{k-1} + \frac{n-k}{n-2} + \frac{k-1}{n-1},$$

then G is k-connected, unless

$$G \cong (K_1 + K_{n-k}) \vee K_{k-1}.$$

Proof Let G be a graph satisfying the conditions in the theorem. Suppose that G is not k-connected. Then, by Lemma 1.5, there exists an integer $1 \leqslant i \leqslant \frac{1}{2}(n-k+1)$ such that $d_i \leqslant i + k - 2$ and $d_{n-k+1} \leqslant n-i-1$. Obviously, $1 \leqslant k \leqslant n-1$. So we have

$$ID(G) = \sum_{v_i \in V(G)} \frac{1}{d(v_i)} = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_i} + \frac{1}{d_{i+1}} + \cdots + \frac{1}{d_{n-k+1}} + \frac{1}{d_{n-k+2}} + \cdots + \frac{1}{d_n} \geqslant \frac{i}{d_i} + \frac{n-k-i+1}{d_{n-k+1}} + \frac{k-1}{d_n} \geqslant \frac{i}{i+k-2} + \frac{n-k-i+1}{n-i-1} + \frac{k-1}{n-1}.$$

Let $f(x) = \frac{x}{x+k-2} + \frac{n-k-x+1}{n-x-1}$, $1 \leqslant x \leqslant \frac{n-k+1}{2}$, then

$$f'(x) = \frac{(k-2)(n+k-3)(n-k+1-2x)}{(x+k-2)^2(n-x-1)^2}.$$

(i) If $k=1$, $1 \leqslant x \leqslant \frac{n}{2}$, so

$$f'(x) = -\frac{(n-2)(n-2x)}{(x-1)^2(n-x-1)^2} \leqslant 0.$$

Therefore, $f(x) \geqslant f(\frac{n}{2}) = \frac{2n}{n-2}$. Thus, $ID(G) \geqslant \frac{2n}{n-2}$.

If $ID(G) = \frac{2n}{n-2}$. So, all equalities above should be attained. Thus, we have $i = \frac{n}{2}$, $d_1 = d_2 = \cdots = d_n = i-1 = \frac{n}{2} - 1$. So G is $\frac{n}{2}-1$ regular graph. It is easy to check that the graph is 1-connected.

(ii) If $k \geqslant 2$, $f'(x) \geqslant 0$. Therefore, $f(x) \geqslant f(1) = \frac{1}{k-1} + \frac{n-k}{n-2}$. Thus,

$$ID(G)\geqslant\frac{1}{k-1}+\frac{n-k}{n-2}+\frac{k-1}{n-1}.$$

If $ID(G)=\frac{1}{k-1}+\frac{n-k}{n-2}+\frac{k-1}{n-1}$. So, all equalities above should be attained. Thus, we have $i=1$, $d_1=k-1$, $d_2=\cdots=d_{n-k+1}=n-2$, $d_{n-k+2}=\cdots=d_n=n-1$, so $G\subseteq(K_1+K_{n-k})\vee K_{k-1}$. It is stated that the graph $(K_1+K_{n-k})\vee K_{k-1}$ is not k-connected in Ref. [22].

This completes the proof.

Since every k-connected graph is also k-edge-connected, we have also obtained the following sufficient conditions for a graph to be k-edge-connected.

Theorem 2.6 Let G be a connected graph of order $n\geqslant k+1\geqslant 2$.

(ⅰ) If $ID(G)\leqslant\frac{2n}{n-2}$, then G is 1-edge-connected.

(ⅱ) If $k\geqslant 2$,

$$ID(G)\leqslant\frac{1}{k-1}+\frac{n-k}{n-2}+\frac{k-1}{n-1},$$

then G is k-edge-connected, unless

$$G\subseteq(K_1+K_{n-k})\vee K_{k-1}.$$

Especially, when $k=2$, we can get a sufficient condition with a bigger upper bound for $ID(G)$, for large n.

Theorem 2.7 Let G be a connected graph of order $n\geqslant 6$. If

$$ID(G)\leqslant 2+\frac{2}{n},$$

then G is 2-edge-connected.

Proof Let G be a graph satisfying the conditions in the theorem. Suppose that G is not 2-edge-connected. Then, by Lemma 1.6, there exists an integer $3=k+1\leqslant i\leqslant\frac{n}{2}$ such that

$$d_{i-2+1}\leqslant i-1,\ d_i\leqslant i+2-2,\ d_n\leqslant n-i+2-2.$$

We have

$$ID(G)=\sum_{v_i\in V(G)}\frac{1}{d(v_i)}=\frac{1}{d_1}+\frac{1}{d_2}+\cdots+\frac{1}{d_{i-2+1}}+\frac{1}{d_{i-2+2}}+\cdots+\frac{1}{d_i}+\frac{1}{d_{i+1}}+\cdots+\frac{1}{d_n}\geqslant$$
$$\frac{i-2+1}{d_{i-2+1}}+\frac{2-1}{d_i}+\frac{n-i}{d_n}\geqslant$$
$$\frac{i-2+1}{i-1}+\frac{2-1}{i+2-2}+\frac{n-i}{n-i+2-2}=$$
$$2+\frac{1}{i}\geqslant 2+\frac{2}{n}.$$

If $ID(G)=2+\frac{2}{n}$, all equalities above should be attained. Thus, we have $i=k+1=3$, $d_1=d_2=2$, $d_3=3$, $d_4=\cdots=d_n=n-3$. Feng et al. [24] show that if G has this degree sequence, then it must be k-edge-connected.

This completes the proof.

Theorem 2.8 Let G be a connected graph of order $n\geqslant 10$ and $0\leqslant\beta\leqslant n$, $n=\beta(\mathrm{mod}\ 2)$. If

$$ID(G)\leqslant\frac{n+\beta+2}{n-\beta-2}+\frac{n-\beta-2}{2(n-1)},$$

then G is β-deficient, unless

$$G\subseteq\left(\frac{n+\beta+2}{2}\right)K_1\vee K_{\frac{n-\beta-2}{2}}.$$

Proof Let G be a graph satisfying the conditions in the theorem. Suppose that G is not β-deficient. Then, by Lemma 1.7, there exists an integer $1\leqslant i\leqslant\frac{1}{2}(n+\beta-2)$ such that $d_{i+1}\leqslant i-\beta$ and $d_{n+\beta-i}\leqslant n-i-2$. So we have

$$ID(G)=\sum_{v_i\in V(G)}\frac{1}{d(v_i)}=\frac{1}{d_1}+\frac{1}{d_2}+\cdots+\frac{1}{d_{i+1}}+\frac{1}{d_{i+2}}+\cdots+\frac{1}{d_{n+\beta-i}}+\frac{1}{d_{n+\beta-i+1}}+\cdots+\frac{1}{d_n}\geqslant$$
$$\frac{i+1}{d_{i+1}}+\frac{n+\beta-2i-1}{d_{n+\beta-i}}+\frac{i-\beta}{d_n}\geqslant$$
$$\frac{i+1}{i-\beta}+\frac{n+\beta-2i-1}{n-i-2}+\frac{i-\beta}{n-1}.$$

Let $f(x)=\frac{x+1}{x-\beta}+\frac{n+\beta-2x-1}{n-x-2}+\frac{x-\beta}{n-1}$, $1\leqslant x\leqslant\frac{n+\beta-2}{2}$, then

$$f'(x)=-\frac{\beta+1}{(x-\beta)^2}-\frac{n-\beta-3}{(n-x-2)^2}+\frac{1}{n-1}\leqslant$$
$$-\frac{\beta+1}{(n-x-2)^2}-\frac{n-\beta-3}{(n-x-2)^2}+\frac{1}{n-1}=$$
$$-\frac{n-2}{(n-x-2)^2}+\frac{1}{n-1}\leqslant$$
$$-\frac{n-2}{(n-3)^2}+\frac{1}{n-1}=-\frac{3n-7}{(n-1)(n-3)^2}<0.$$

Therefore, $f(x)\geqslant f\left(\frac{1}{2}(n+\beta-2)\right)=\frac{n+\beta+2}{n-\beta-2}+\frac{n-\beta-2}{2(n-1)}$.

Thus, $ID(G)\geqslant\frac{n+\beta+2}{n-\beta-2}+\frac{n-\beta-2}{2(n-1)}$.

If $ID(G)=\frac{n+\beta+2}{n-\beta-2}+\frac{n-\beta-2}{2(n-1)}$, all equalities above should be attained. Thus, we have $i=\frac{n+\beta-2}{2}$. $d_1=\cdots=d_{i+2}=i-\beta, d_{i+3}=\cdots=d_n=n-1$, so $G\subseteq\left(\frac{n+\beta+2}{2}\right)K_1\vee K_{\frac{n-\beta-2}{2}}$. It is easy to check that the graph $\left(\frac{n+\beta+2}{2}\right)K_1\vee K_{\frac{n-\beta-2}{2}}$ is not β-deficient. This completes the proof.

Theorem 2.9 Let G be a connected graph of order n. If

$$ID(G)\leqslant\frac{k+1}{n-k-1}+\frac{n-k-1}{n-1},$$

then $\alpha(G)\leqslant k$, unless $G\cong(\overline{K_{k+1}})\vee K_{n-k-1}$.

Proof Let G be a graph satisfying the conditions in the theorem. Suppose that $\alpha(G)>k$. Then, by Lemma 1.8, $d_{k+1}\leqslant n-k-1$. So we have

$$ID(G)=\sum_{v_i\in V(G)}\frac{1}{d(v_i)}=\frac{1}{d_1}+\frac{1}{d_2}+\cdots+\frac{1}{d_{k+1}}+\frac{1}{d_{k+2}}+\cdots+\frac{1}{d_n}\geqslant\frac{k+1}{d_{k+1}}+\frac{n-k-1}{d_n}\geqslant\frac{k+1}{n-k-1}+\frac{n-k-1}{n-1}.$$

That is to said,

$$ID(G)\geqslant\frac{k+1}{n-k-1}+\frac{n-k-1}{n-1}.$$

Combining this fact with our assumption, we get $ID(G)=\frac{k+1}{n-k-1}+\frac{n-k-1}{n-1}$, all equalities above should be attained. Thus, we have $d_1=d_2=\cdots=d_{k+1}=n-k-1$, $d_{k+2}=\cdots=d_n=n-1$, so $G\cong(\overline{K_{k+1}})\vee K_{n-k-1}$. It is easy to check that the graph $(\overline{K_{k+1}})\vee K_{n-k-1}$ does not satisfy $\alpha(G)\leqslant k$.

This completes the proof.

We define $B_n^k\ (1\leqslant k\leqslant n-1)$ as the graph obtained from $K_{n,n}$ by deleting all edges in its one subgraph $K_{n-k,k}$.

Theorem 2.10 Let $G=(X,Y;E)$ be a bipartite graph such that $X=\{x_1,x_2,\cdots,x_n\}$, $Y=\{y_1,y_2,\cdots,y_n\}$, $n\geqslant2$, and $d(x_1)\leqslant d(x_2)\leqslant\cdots\leqslant d(x_n)$, $d(y_1)\leqslant d(y_2)\leqslant\cdots\leqslant d(y_n)$. If

$$ID(G)\leqslant3,$$

then G is Hamiltonian, unless $G\cong B_n^k$.

Proof Let G be a graph satisfying the conditions in the theorem. Suppose that G is not Hamiltonian. Then, by Lemma 1.9, there exists an integer $k<n$ such that $d(x_k)\leqslant k$ and $d(y_{n-k})\leqslant n-k$. Obviously, $k\geqslant1$. So we have

$$ID(G)=\sum_{v_i\in V(G)}\frac{1}{d(v_i)}=\sum_{i=1}^{k}\frac{1}{d(x_i)}+\sum_{i=k+1}^{n}\frac{1}{d(x_i)}+\sum_{j=1}^{n-k}\frac{1}{d(y_j)}+\sum_{j=n-k+1}^{n}\frac{1}{d(y_j)}\geqslant\frac{k}{d(x_k)}+\frac{n-k}{d(y_{n-k})}+\frac{n}{n}\geqslant\frac{k}{k}+\frac{n-k}{n-k}+\frac{n}{n}=3.$$

If $ID(G)=3$, all equalities above should be attained. Thus, we have $d(x_1)=d(x_2)=\cdots=d(x_k)=k$, $d(x_{k+1})=\cdots=d(x_n)=n$, $d(y_1)=d(y_2)=\cdots=d(y_{n-k})=n-k$, $d(y_{n-k+1})=\cdots=d(y_n)=n$, so $G\cong B_n^k$. It is easy to check that the graph B_n^k is not Hamiltonian.

This completes the proof.

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