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Cycle lengths in graphs of chromatic number five and six

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Abstract: A problem was proposed by Moore and West to determine whether every $(k+1)$-critical non-complete graph has a cycle of length 2 modulo k. We prove a stronger result that for $k=4, 5$, every $(k+1)$-critical non-complete graph contains cycles of all lengths modulo k.

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1 Introduction

The problem of deciding whether a given graph contains cycles of all lengths modulo a positive integer k shows up in many literature (see Refs. [1−8]). Recently, Moore and West[9] asked whether every $(k+1)$-critical non-complete graph has a cycle of length 2 modulo k. Here, a graph is k-critical if it has chromatic number k but deleting any edge will decrease the chromatic number. Very recently, Gao et al.[10] partially answered this question by showing the following theorem.

Theorem 1.1 For $k \geqslant 6$, every $(k+1)$-critical non-complete graph contains cycles of all lengths modulo k.

However, methods in Ref. [10] do not work for $k<6$. In this note, we give a new method and prove that the conclusion of Theorem 1.1 also holds for $k=4, 5$.

Theorem 1.2 For $k=4, 5$, every $(k+1)$-critical non-complete graph contains cycles of all lengths modulo k.

Thus, combined with the Theorems 1.1 and 1.2, we completely give an affirmative answer to the question of Moore and West.

The rest of the paper is organized as follows. In Section 2, we introduce the notation. In Section 3, we give a key lemma. In Section 4, we consider graphs of chromatic number five and prove Theorem 1.2 for the case $k = 4$. In Section 5, we consider graphs of chromatic number six and prove Theorem 1.2 for the case $k=5$.

2 Notation

All graphs considered are finite, undirected, and simple. Let G be a graph and let H be a subgraph of a graph G. We say that H and a vertex $v \in V(G)-V(H)$ are adjacent in G if v is adjacent in G to some vertex in $V(H)$. Let $N_G(H) := \bigcup_{v \in V(H)} N_G(v) - V(H)$ and $N_G[H] := N_G(H) \cup V(H)$. For a subset S of $V(G)$, $G[S]$ denotes the subgraph induced by S in G, and $G-S$ denotes the subgraph $G[V(G)-S]$. A vertex is a leaf in G if it has degree one in G. We say that a path P is internally disjoint from H if no vertex of P other than its endpoints is in $V(H)$. For two vertex-disjoint subgraphs H, H' of G, let $N_H(H')$ be the set of vertices in H which are adjacent to some vertex in H'.

A cycle or a path is said to be odd (resp. even) if its length is odd (resp. even). Given a cycle C and an orientation of C, for two vertices x and y in C, let $C[x, y]$ denote the path on C from x to y in the direction, including x and y. Let $C[x, y) := C[x, y]-y$, $C(x, y] := C[x, y]-x$, and $C(x, y) := C[x, y]-\{x, y\}$. We use the similar notation to a path P.

Let u and v be two vertices of a graph. If there are three internally disjoint paths between u and v, then we call such a graph as the theta graph. Note that any theta graph contains an even cycle.

A vertex v of a graph G is a cut-vertex of G if $G-v$ contains more components than G. A block B in G is a maximal connected subgraph of G such that there exists no cut-vertex of B. So a block is an isolated vertex, an edge or a 2-connected graph. An end-block in G is a block in G containing at most one cut-vertex of G. If D is an end-block of G and a vertex x is the only cut-vertex of G with $x \in V(D)$, then we say that D is an end-block with cut-vertex x.

Let T be a tree, and fix a vertex r as its root. Let v be a vertex of T. The parent of v is the vertex adjacent to v on the path from v to r. An ascendant of v is any

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vertex which is either the parent of v or is recursively the ascendant of the parent of v. A child of v is a vertex of which v is the parent. A descendant of v is any vertex which is either the child of v or is recursively the descendant of any of the children of v. Let Y be a subset of $V(T)$. We say a vertex x is the descendant of Y if x is the descendant of some vertex in Y. Let a, b be two vertices of T. Denote $T_{a,b}$ the unique path between a and b in T.

3 Key lemma

Let G be a 2-connected graph and let C and D be two cycles in G. We say that (C, D) is an opposite pair in G, if C is odd and D is even satisfying that C and D are edge-disjoint and share at most one common vertex.

Lemma 3.1 Let G be a 2-connected graph of minimum degree at least 4. Let (C, D) be an opposite pair in G. Then G contains cycles of all lengths modulo 4.

Proof Suppose to the contrary that G does not contain cycles of all lengths modulo 4. Since G is 2-connected and $|V(C)\cap V(D)|\leqslant 1$, there exist two vertex-disjoint paths P, Q between C and D satisfying $(V(C)\cap V(D))-V(Q)=\emptyset$①. We take such an opposite pair (C, D), paths P and Q as the following manner:

① $|E(P)|$ is as large as possible;

② $|E(Q)|$ is as large as possible subject to ①.

Let p and q be the endpoints of P and Q in D, respectively.

Claim 1 Every even cycle in the block of $G-(V(C\cup P\cup Q)-\{p, q\})$ including D contains both p and q. In particular, every theta graph in the block includes both p and q.

Proof of Claim 1 Let H be the block of

$$G-(V(C\cup P\cup Q)-\{p, q\})$$

including D. Let D' be an even cycle in H other than D. Suppose that $p\notin V(D')$. Since H is 2-connected, there are two vertex-disjoint paths L_1, L_2 from $\{p, q\}$ to D' in H. We may assume that L_1 links p and D'. Note that L_1 has a length at least 1 and (C, D') is an opposite pair. Then $P\cup L_1$ and $Q\cup L_2$ are two internally disjoint paths between C and D' such that $P\cup L_1$ is longer than P, a contradiction. Therefore, $p\in V(D')$.

Suppose that $q\notin V(D')$. Since H is 2-connected, there is a path L_3 from q to D' internally disjoint from $V(D')$ in H. Note that L_3 has a length at least 1 and (C, D') is an opposite pair. Then P and $Q\cup L_3$ are two internally disjoint paths between C and D' such that $Q\cup L_3$ is longer than Q, a contradiction. Therefore, $q\in V(D')$. Since every theta graph contains an even cycle, every theta graph in H includes both p and q. This completes the proof of Claim 1.

Since D is an even cycle, we partition $V(D)$ into the sets A and B alternatively along D. By symmetry between A and B, we may assume that $p\in A$.

Claim 2 For any $b\in B-\{q\}$, there is no path from b to $C\cup P\cup Q-\{p, q\}$ internally disjoint from $C\cup D\cup P\cup Q$.

Proof of Claim 2 Suppose to the contrary that there is a path R from b to $x\in V(C\cup P\cup Q)-\{p, q\}$ internally disjoint from $C\cup D\cup P\cup Q$. By symmetry, we may assume that $b\in D(p, q)$.

Assume that $|E(D)|\equiv 0 \bmod 4$. As C is an odd cycle, there is an even path X_1 and an odd path Y_1 between p and q in $C\cup P\cup Q$. If $q\in B$, then both $|E(D[p, q])|$ and $|E(D[q, p])|$ are odd, and furthermore, since their sum is 0 modulo 4, they differ by 2 modulo 4. Then $X_1\cup D[p, q]$, $X_1\cup D[q, p]$, $Y_1\cup D[p, q]$ and $Y_1\cup D[q, p]$ are 4 cycles of different lengths modulo 4, a contradiction. Therefore, we have that $q\in A$.

Suppose that $x\in V(P)-\{p\}$. Since C is an odd cycle, there is an even path X_2 and an odd path Y_2 between b and q in $C\cup P\cup Q\cup R$. However, since both $|E(D[b, q])|$ and $|E(D[q, b])|$ are odd and differ by 2 modulo 4, $X_2\cup D[b, q]$, $X_2\cup D[q, b]$, $Y_2\cup D[b, q]$ and $Y_2\cup D[q, b]$ are 4 cycles of different lengths modulo 4, a contradiction. Thus, x is not contained in $V(P)-\{p\}$.

Suppose that $x\in V(C\cup Q)-(V(P)\cup\{q\})$. Then there is an even path X_3 and an odd path Y_3 between b and p in $C\cup P\cup Q\cup R$. However, since both $|E(D[b, p])|$ and $|E(D[p, b])|$ are odd and differ by 2 modulo 4, $X_3\cup D[b, p]$, $X_3\cup D[p, b]$, $Y_3\cup D[b, p]$ and $Y_3\cup D[p, b]$ are 4 cycles of different lengths modulo 4, a contradiction. Thus, x is not contained in $V(C\cup Q)-(V(P)\cup\{q\})$.

Therefore, $|E(D)|\equiv 2 \bmod 4$. As C is an odd cycle, there is an even path X_4 and an odd path Y_4 between p and q in $C\cup P\cup Q$. If $q\in A$, then both $|E(D[p, q])|$ and $|E(D[q, p])|$ are even, and furthermore, since their sum is 2 modulo 4, they differ by 2 modulo 4. Then $X_4\cup D[p, q]$, $X_4\cup D[q, p]$, $Y_4\cup D[p, q]$ and $Y_4\cup D[q, p]$ are 4 cycles of different lengths modulo 4, a contradiction. Therefore, we have that $q\in B$.

Suppose that $x\in V(C\cup P)-(V(Q)\cup\{p\})$. Since C is an odd cycle, there is an even path X_5 between b and q and an odd path Y_5 between b and q in $C\cup P\cup Q\cup R$. However, since both $|E(D[b, q])|$ and

① We remark that (i) if $V(C)\cap V(D)=\emptyset$, then P and Q are vertex-disjoint, (ii) if C and D share one common vertex, then $V(Q)=V(C)\cap V(D)$.

$|E(D[q,b])|$ are even and differ by 2 modulo 4, $X_5 \cup D[b,q]$, $X_5 \cup D[q,b]$, $Y_5 \cup D[b,q]$ and $Y_5 \cup D[q,b]$ are 4 cycles of different lengths modulo 4, a contradiction. Thus, x is not contained in $V(C \cup P) - (V(Q) \cup \{p\})$.

Suppose that $x \in V(Q) - \{q\}$. Since G is a 2-connected graph of minimum degree at least 4, there exists a path T from b to $y \in V(C \cup D \cup P \cup Q \cup R) - \{b\}$ internally disjoint from $C \cup D \cup P \cup Q \cup R$. As the same reason for x, y is not contained in $V(C \cup P) - (V(Q) \cup \{p\})$. Therefore $y \in V(Q \cup D \cup R) - \{b\}$.

If $y \in V(R \cup Q \cup D(b,p)) - \{b\}$, then $D[b,p) \cup R \cup T \cup Q$ contains a theta graph. It follows that there is an even D_1 cycle in $G-(C \cup P-Q)$. Note that (C, D_1) is an opposite pair in G. It is easy to see that there are two internally disjoint paths P' and Q' between C and D' satisfying that P' contains P and is longer than P and $Q' \subseteq Q \cup D(b,q] \cup R$, a contradiction. Thus, y is not contained in $V(R \cup Q \cup D(b,p)) - \{b\}$.

Suppose that $y \in V(D[p,b))$. Since $T \cup D[y,b]$ does not contain q and $D[b,q] \cup R \cup Q[x,q]$ does not contain p, by the choice of opposite pairs, we have that $T \cup D[y,b]$ and $D[b,q] \cup R \cup Q[x,q]$ are both odd cycles. Since C is an odd cycle, there is an odd path X' and an even path Y' between y and x in $C \cup P \cup Q \cup D[p,y]$. Note that the lengths of X' and Y' differ by 1 modulo 4, the lengths of T and $D[y,b]$ differ by 1 modulo 4 and the lengths of $D[b,q] \cup Q[x,q]$ and R differ by 1 modulo 4. Then the set

$$\{L_1 \cup L_2 \cup L_3 \mid L_1 \in \{X', Y'\},\ L_2 \in \{T, D[y,b]\},\ L_3 \in \{D[b,q] \cup Q[x,q], R\}\}$$

contains cycles of all lengths modulo 4, a contradiction. Thus, y is not contained in $V(D[p,b))$.

This completes the proof of Claim 2.

Let z be a vertex in $B-\{q\}$. By symmetry between two orientations of C, we may assume that $z \in V(D(p,q))$. Since the degree of z is at least 4 in G and G is 2-connected, there is a path Z from z to $C \cup D \cup P \cup Q - \{z\}$ internally disjoint from $C \cup D \cup P \cup Q$. By Claim 2, the endpoint of Z other than z is contained in $D-\{z\}$. Let r be the endpoint of Z other than z. Since the degree of z is at least 4 in G and G is 2-connected, there is a path S from z to $s \in V(C \cup D \cup P \cup Q \cup Z) - \{z\}$ internally disjoint from $C \cup D \cup P \cup Q \cup Z$. By Claim 2, s is contained in $V(D \cup Z) - \{z\}$.

Suppose that $s \in V(Z) - \{z\}$.

If $r \in V(D(z,p))$, then $D[z,r] \cup Z \cup S$ is a theta graph not containing p, contradicting Claim 1.

If $r \in V(D[p,z))$, then $D[r,z] \cup Z \cup S$ is a theta graph not containing q, contradicting Claim 1.

Thus, s is not contained in $V(Z) - \{z\}$.

Suppose that $s \in V(D) - \{z, r\}$. By symmetry between r and s, we may assume that $s \in V(D(r,z))$.

If $r \in V(D(q,z))$, then $D[r,z] \cup Z \cup S$ is a theta graph not containing q, contradicting Claim 1.

If $r \in V(D(z,q])$ and $s \in V(D(r,p))$, then $D[z,s] \cup Z \cup S$ is a theta graph not containing p, contradicting Claim 1.

Therefore $r \in V(D(z,q])$ and $s \in V(D[p,z))$. Since $S \cup D[s,z]$ does not contain q and $D[z,r] \cup Z$ does not contain p, by Claim 1, we have that $S \cup D[s,z]$ and $D[z,r] \cup Z$ are both odd cycles. Since C is an odd cycle, there is an odd path X'' and an even path Y'' between s and r in $C \cup P \cup Q \cup D[p,s] \cup D[r,q]$. Note that the lengths of X'' and Y'' differ by 1 modulo 4, the lengths of S and $D[s,z]$ differ by 1 modulo 4 and the lengths of $D[z,r]$ and Z differ by 1 modulo 4. Then the set $\{L_1 \cup L_2 \cup L_3 \mid L_1 \in \{X'', Y''\},\ L_2 \in \{S, D[s,z]\},\ L_3 \in \{D[z,r], Z\}\}$ contains cycles of all lengths modulo 4, a contradiction.

This completes the proof of Lemma 3.1.

4 Graphs of chromatic number five

In this section, we prove the following theorem on 2-connected graphs of the minimum degree at least four, from which Theorem 1.2 can be inferred as a corollary for the case $k=4$.

Theorem 4.1 Every 2-connected non-bipartite graph of the minimum degree at least 4 contains cycles of all lengths modulo 4, except that it is the complete graph of five vertices.

Proof Let G be a 2-connected non-bipartite graph of the minimum degree at least 4. Assume that G is not a K_5 and does not contain cycles of all lengths modulo 4. Let $C := v_0 v_1 \cdots v_{2l} v_0$ be an odd cycle in G such that $|V(C)|$ is minimum, where the indices are taken under the additive group $\mathbb{Z}_{2l+1}$. Note that C is induced. Let $H := G - V(C)$. By Lemma 3.1, there is no opposite pairs in G, hence H does not contain an even cycle. It follows that every block of H is either an odd cycle, an edge or an isolated vertex.

Claim G does not contain a triangle.

Proof of Claim Suppose that G contains a triangle. Then C is a triangle. Let H_1 be a component of H. Since the minimum degree of G is at least 4, H_1 has at least two vertices. Suppose that H_1 contains an odd cycle C_1.

If H_1 is not 2-connected, then there exists an end-block B_1 of H_1 with cut-vertex b_1 such that

$$(V(B_1) - \{b_1\}) \cap V(C_1) = \emptyset.$$

As B_1 is either an odd cycle or an edge, there exists $w \in V(B_1) - \{b_1\}$ such that w has at least two neighbors on C. Since C is an odd cycle, $G[C \cup \{w\}]$ contains an even cycle D_1. Then C_1 and D_1 form an opposite pair in G, a contradiction.

Therefore, H_1 is 2-connected, that is H_1 is an induced odd cycle, we denote $H_1 := u_0u_1\cdots u_{2h}u_0$, where the indices are taken under the additive group $\mathbb{Z}_{2h+1}$. Since the minimum degree of G is at least 4, u_0 and u_2 have at least two neighbors on C. Without loss of generality, we may assume that u_0 is adjacent to v_0 and v_1 and u_2 is adjacent to v_0. Then C, $u_0v_0v_2v_1u_0$, $u_0u_1u_2v_0v_1u_0$ and $u_0u_1u_2v_0v_2v_1u_0$ are cycles of lengths 3, 4, 5 and 6, respectively, a contradiction.

Therefore, every component of H does not contain an odd cycle, that is, every component of H is a tree.

If $|V(H_1)|=2$, then $G[C\cup H_1]$ is a K_5. Suppose that there is another component $H_2\neq H_1$ of H. Since G is 2-connected, there are two disjoint paths L_1 and L_2 from H_2 to C internally disjoint from C in $G[H_2\cup C]$. Without loss of generality, we may assume that $V(L_i)\cap V(C)=\{v_i\}$ for $i=1, 2$. Concatenating L_1, L_2 and a path in H_2, there exists a path L from v_1 to v_2 internally disjoint from C in $G[H_2\cup C]$. As there are paths of lengths 1, 2, 3 and 4 from v_1 to v_2 in $G[H_1\cup C]$, we could easily obtain 4 cycles of consecutive lengths, a contradiction. Therefore, $H=H_1$. It follows that $G=G[C\cup H_1]$, a contradiction.

Therefore $|V(H_1)|\geqslant 3$. For any two leaves x, y of H_1, let T be the fixed path between x and y in H_1. Since the minimum degree of G is at least 4, x and y have at least three neighbors on C. Without loss of generality, we may assume that x is adjacent to v_0 and v_1 and y is adjacent to v_0. If T is even, then C and v_0yTxv_0 form an opposite pair, a contradiction. Therefore T is odd. Suppose that there exist three leaves x, y and z in H_1. Let $T_{x,y}$, $T_{y,z}$ and $T_{z,x}$ be the fixed paths between x and y, y and z and z and x in H_1, respectively. Note that all of them are odd. However, their sum is even, a contradiction. Therefore, H_1 is a path. Let $H_1 := z_0z_1z_2\cdots z_n$ for some $n\geqslant 2$. Since the minimum degree of G is at least 4, z_0 is adjacent to all vertices of C and z_2 is adjacent to at least 2 vertices of C. Without loss of generality, we may assume that z_2 is adjacent to v_0 and v_1. Then C, $z_0v_0v_2v_1z_0$, $z_0z_1z_2v_0v_1z_0$ and $z_0z_1z_2v_0v_2v_1z_0$ are cycles of lengths 3, 4, 5 and 6, respectively, a contradiction.

This completes the proof of Claim.

By Claim, G does not contain a triangle. Suppose that there is a vertex u of degree at most one in H. Since the minimum degree of G is at least 4, u has at least three neighbors on C. Since C is odd, there exist two distinct neighbors v_i, v_j of u on C such that the odd path between v_i and v_j on C has no internal vertices which are the neighbors of u in G. Let Q_o, Q_e be the odd and even paths between v_i and v_j in C respectively. Let $C' := uv_i\cup Q_o\cup v_ju$. Note that C' is an odd cycle. By the choice of C, we have that $|E(C')|\geqslant|E(C)|$. This forces that $|E(Q_e)|=2$ and u is adjacent to all vertices of $V(Q_e)$. It follows that there is a triangle in G, a contradiction. Therefore, the minimum degree of H is at least 2.

Suppose that H has more than one component. Let W_1 and W_2 be two components of H. Since the degree of any vertex in W_1 is at least 2, we have that W_1 contains an odd cycle C_2. Since G is 2-connected and C is an odd cycle, there is an even cycle D_2 in $G[V(C)\cup W_2]$. Thus, C_2 and D_2 form an opposite pair, a contradiction. Therefore, H is connected.

Note that the minimum degree of H is at least 2 and every block of H is either an odd cycle, an edge or an isolated vertex. There is a vertex t of H which has at least two neighbors on C. Since C is odd, there exist two distinct neighbors v_i, v_j of t on C such that the odd path between v_i and v_j on C has no internal vertices which are the neighbors of t in G . Let Q_o', Q_e' be the odd and even paths between v_i and v_j in C respectively. Let $C'' := tv_i\cup Q_o'\cup v_jt$. Note that C'' is an odd cycle. By the choice of C, we have that $|E(C'')|\geqslant|E(C)|$. This fores that $|E(Q_e')|=2$. Without loss of generality, we may assume that $i=j+2$. Let s be the neighbor of v_{j+l+1} in H. Note that C has length at least five. If follows that $v_{j+l+1}\neq v_i$, v_j. Since H is connected, there is a path L between t and s in H. Then $C[v_{j+2}, v_{j+l+1}]\cup v_{j+l+1}s\cup L\cup tv_{j+2}$, $C[v_{j+l+1}, v_j]\cup v_jt\cup L\cup sv_{j+l+1}$, $C[v_j, v_{j+l+1}]\cup v_{j+l+1}s\cup L\cup tv_j$, $C[v_{j+l+1}, v_{j+2}]\cup v_{j+2}t\cup L\cup sv_{j+l+1}$ are 4 cycles of consecutive lengths, a contradiction. This completes the proof of Theorem 4. 1.

We remark that Theorem 4. 1 is best possible by the following examples. For any positive integer t, let $P_t := v_0v_1\cdots v_{2t+1}$ and $Q_t := u_0u_1\cdots u_{2t+1}$ be two vertex-disjoint paths. Let H_t be the graph obtained from $P_t\cup Q_t$ by adding edges in $\{v_{2i}u_{2i+1}, u_{2i}v_{2i+1}, u_0v_0, u_{2t+1}v_{2t+1} \mid i=0, 1, \cdots, t\}$. We see that H_t is a 2-connected non-bipartite graph of the minimum degree 3 without cycles of length 1 modulo 4.

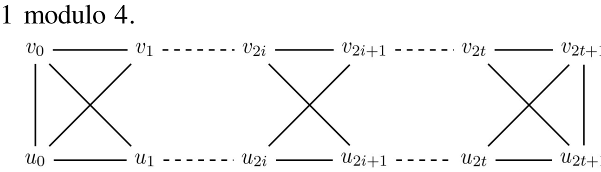

Figure 1. Graphs without cycles of length 1 modulo 4.

5 Graphs of chromatic number six

In this section, we consider graphs of chromatic number six and prove Theorem 1. 2 for the case $k=5$. Very recently, Gao et al. [10] proved following theorems on cycles lengths in graphs containing a triangle.

Theorem 5. 1 Let G be a connected graph of minimum degree at least three and (A, B) be a non-trivial partition of $V(G)$. For any cycle C in G, there exist A-B paths of every length less than $|V(C)|$ in G, unless G is bipartite with the bipartition (A, B).

Theorem 5.2 Let $k \geqslant 3$ be an integer and G be a 2-connected graph of the minimum degree at least k. If G is K_3-free, then G contains a cycle of length at least $2k+2$, except that $G=K_{k,n}$ for some $n \geqslant k$.

Theorem 5.3 Let $k \geqslant 2$ be an integer. Every 2-connected graph G of minimum degree at least k containing a triangle K_3 contains k cycles of consecutive lengths, except that $G=K_{k+1}$.

Now, we are in a position to prove Theorem 1.2 for the case $k=5$, which we rephrase as follows.

Theorem 5.4 Every 6-critical non-complete graph G contains cycles of all lengths modulo 5.

Proof Suppose that G does not contain cycles of all lengths modulo 5 and G is not K_6. It is well-known that G is a 2-connected graph of the minimum degree at least 5. By Theorem 5.3, we may assume that G is K_3-free. Fix a vertex r and let T be a breadth-first search tree in G with root r. Let $L_0=\{r\}$ and L_i be the set of vertices of T at distance i from its root r.

Claim 1 Every component of $G[L_i]$ has chromatic number at most 3, for all $i \geqslant 0$.

Proof of Claim 1 Suppose to the contrary that there exists a component D of $G[L_t]$ which has chromatic number at least 4 for some t. Let H be a 4-critical subgraph of D. It is clear that H is a 2-connected non-bipartite graph of minimum degree at least 3. By Theorem 5.2, H contains a cycle of length at least 8. Let T' be the minimal subtree of T whose set of leaves is precisely $V(H)$, and let r' be the root of T'. Let h denote the distance between r' and vertices in H in T'. Since G is K_3-free, $h \geqslant 2$. By the minimality of T', r' has at least two children in T'. Let x be one of its children. Let A be the set of vertices in H which are the descendants of x in T' and let $B=V(H)-A$. Then both A, B are nonempty and for any $a \in A$ and $b \in B$, $T_{a,b}$ has the same length $2h$. By Theorem 5.1, there are 7 subpaths of H from a vertex of A to a vertex of B of lengths 1, 2, $\cdots$, 7, respectively. It follows that G contains 7 cycles of consecutive lengths, a contradiction. This completes the proof of Claim 1.

For a connected graph D, a vertex in D is called good if it is not contained in the minimal connected subgraph of D which contains all 2-connected blocks of D, and bad otherwise.

We now prove a claim which is key for the proof of Theorem 5.4.

Claim 2 Let H_1 be a non-bipartite component of $G[L_i]$ and H_2 be a non-bipartite component of $G[L_{i+1}]$ for some $i \geqslant 1$. If $N_{H_1}(H_2) \neq \emptyset$, then every vertex in $N_{H_1}(H_2)$ is a good vertex of H_1.

Proof of Claim 2 Suppose that there exists a bad vertex v of H_1 which has a neighbor in H_2. Let T' be the minimal subtree of T whose set of leaves is precisely $V(H_1)$, and let r' be the root of T'. Let h denote the distance between r' and vertices in H_1 in T'. Since G is K_3-free, $h \geqslant 2$. By the minimality of T', r' has at least two children in T'. Let (X, Y) be a non-trivial partition of all children of r' in T'. Let A be the set of vertices in H_1 which are the descendants of X in T' and let B be the set of vertices in H_1 which are the descendants of Y in T'. Note that (A, B) is a non-trivial partition of $V(H_1)$. Note that every vertex in B is the descendants of Y in T'. Let A' be the set of vertices in L_i-A which are the descendants of X in T. Let B' be the set of vertices in L_i-B which are the descendants of Y in T. Let $M:=L_i-(A \cup A' \cup B \cup B')$. Note that A, A', B, B' and M form a partition of L_i. Note that every vertex of H_2 has a neighbor in L_i.

Suppose that there exists a vertex $m \in V(H_2)$ which has a neighbor m' in M. Recall that H_1 is non-bipartite and K_3-free. There exists a path $z_1z_2z_3z_4z_5$ of length 4 in H_1 with $z_1=v$. It is easy to see that $T_{z_i, m}$ contains r' for $i \in [5]$, so they have the same length. Since v has a neighbor in H_2, there is a path P from v to m in $G[H_2 \cup \{v\}]$. Then $P \cup z_1z_2\cdots z_i \cup T_{z_i,m'} \cup m'm$, for $i \in [5]$ are 5 cycles of consecutive lengths in G, a contradiction. Therefore $N_M(H_2)=\emptyset$, that is every vertex in H_2 has a neighbor in $A \cup A' \cup B \cup B'$. For a vertex in $V(H_2)$, we call it type-A if it has a neighbor in $A \cup A'$ and it type-B if it has a neighbor in $B \cup B'$①.

Let $C=v_0v_1\cdots v_n$ be an odd cycle of H_1, where $n \geqslant 4$. Suppose that $V(C) \subseteq A$. Since B is non-empty, we choose an arbitrary vertex b in B. Since H_1 is connected, there exists a path P from b to $V(C)$ internal disjoint from $V(C)$. Without loss of generality, we assume that $V(P) \cap V(C)=\{v_0\}$. Then $P \cup C[v_0, v_i] \cup T_{b,v_i}$ for $i=0, 1, \cdots, 4$ give 5 cycles of consecutive lengths, a contradiction. Therefore, $B \cap V(C) \neq \emptyset$, and similarly, $A \cap V(C) \neq \emptyset$. Then there must be an A-B path of length 4 in C (otherwise, since 4 and $|C|$ are co-prime and $|C| \geqslant 5$, one can deduce that all vertices of C are contained in one of the two parts A and B, a contradiction).

Without loss of generality, we may assume that v_0, $v_1 \in A$ and $v_2 \in B$. Then $T_{v_1,v_2} \cup v_2v_1$ and $T_{v_0,v_2} \cup v_2v_1v_0$ are two cycles of lengths $2h+1$ and $2h+2$, respectively. We have showed that there exists some A-B path of length 4 in C which gives a cycle of length $2h+4$, so we may assume that there is no A-B path of length 3 or 5 in

① We remark that a vertex can be both type-A and type-B.

C. This would force that one of the following holds.

5.1 There is no A-B path of length 3 in H_1

This would force that for any path $P'=u_0u_1\cdots u_s$ in H_1 with $u_1=v_0$, $u_2=v_1$, $u_3=v_2$, we can derive that $u_j\in B$ if $j\equiv 0 \bmod 3$ and $u_j\in A$ if $j\equiv 1$ or $2 \bmod 3$. Moreover, we have that v_{3i}, $v_{3i+1}\in A$ and $v_{3i+2}\in B$ for each possible $i\geqslant 0$. So $|C|\geqslant 9$ and G contains a cycle of length $l\in\{2h+1, 2h+2, 2h+4, 2h+5, 2h+7, 2h+8\}$. In particular, since H_1 is connected, for any vertex $b\in B$, there exists a path of length 2 in H_1 from b to some vertex in A. And for any bad vertex $a\in A$, there exists a path $b_1aa_1b_2$ satisfying b_1, $b_2\in B$ and a, $a_1\in A$.

Suppose that $N_{A\cup A'}(H_2)\neq\emptyset$ and $N_{B\cup B'}(H_2)\neq\emptyset$. Since H_2 is connected and every vertex of H_2 has a neighbor in $A\cup A'\cup B\cup B'$, there exist two adjacent vertices p, q of H_2 such that p has a neighbor p' in $A\cup A'$ and q has a neighbor q' in $B\cup B'$. Then $p'pqq'\cup T_{p',q'}$ is a cycle of length $2h+3$. It follows that G contains 5 cycles of lengths $2h+1$, $2h+2$, $2h+3$, $2h+4$ and $2h+5$, respectively, a contradiction.

Suppose that $N_{L_i}(H_2)\subseteq B\cup B'$. Since $N_{A\cup B}(H_2)\neq\emptyset$, we have that $v\in B$. Let u be any vertex in $N_{H_2}(v)$. Choose $w_1\in V(H_2)$ such that there exists a path Q of length 2 from u to w_1 in H_2. Since any vertex in H_2 has a neighbor in L_i, by our assumption, w_1 has a neighbor in $B\cup B'$. Let w_2 be a neighbor of w_1 in $B\cup B'$. Suppose that $w_2\neq v$. Note that there is a path $R:=vv''v'$ such that v', $v''\in A$. Then $R\cup vu\cup Q\cup w_1w_2\cup T_{w_2,v'}$ is a cycle of length $2h+6$. So G contains cycles of lengths $2h+4$, $2h+5$, $2h+6$, $2h+7$ and $2h+8$, a contradiction. Therefore $w_2=v$ and $w_1\in N_{H_2}(v)$. That says, every vertex in H_2 of distance 2 from a neighbor of v is a neighbor of v. Continuing to apply this along with a path from u to an odd cycle C_0 in H_2, we could obtain that v is adjacent to all vertices of C_0, which contradicts that G is K_3-free. Therefore, $N_{B\cup B'}(H_2)=\emptyset$.

Now we see that $N_{L_i}(H_2)\subseteq A\cup A'$. This forces that $v\in A$. For any neighbor u' of v in H_2, let $w_3\in V(H_2)$ satisfies that there exists a path Q' of length 2 from u' to w_3 in H_2. Note that $v\in A$ is bad in H_1, we can infer that there exists a path $b_2va_1b_1$ in H_1 such that $a_1\in A$ and b_1, $b_2\in B$. Note that v and a_1 are symmetric. Let w_4 be a neighbor of w_3 in $A\cup A'$. Suppose that $w_4\notin\{v, a_1\}$. Then $vu'\cup Q'\cup w_3w_4\cup T_{w_4,b_1}\cup b_1a_1v$ is a cycle of length $2h+6$. So again, G contains cycles of lengths $2h+4$, $2h+5$, $2h+6$, $2h+7$ and $2h+8$, a contradiction. Therefore, $w_4\in\{v, a_1\}$. That is, every vertex in H_2 of distance 2 from a neighbor of v or a_1 is adjacent to one of v, a_1. Continuing to apply this along with a path from u' to an odd cycle C_1 in H_2, we could obtain that every vertex of C_1 is adjacent to one of v, a_1. But this would force a copy of K_3 containing a_1v in G. This final contradiction completes the proof of this subsection.

5.2 There is an A-B path of length 3 in H_1

Therefore, we may assume that there is no A-B paths of length 5 in H_1.

We first show that for any path $t_1t_2t_3$ in H_1 satisfying that t_1 and t_3 are in different parts, t_2 does not have a neighbor in $V(H_2)$; call this Property ★. Suppose to the contrary that t_2 has a neighbor in H_2. Without loss of generality, we may assume that $t_1, t_2\in A$ and $t_3\in B$. Let s be any vertex in $N_{H_2}(t_2)$. Choose $s'\in V(H_2)$ such that there exists a path Q of length 2 from s to s' in H_2. Let t be a neighbor of s' in L_i-M. Suppose that $t\neq t_2$. If $t\in A\cup A'$, then $t_3t_2s\cup Q\cup s't\cup T_{t,t_3}$ is a cycle of length $2h+5$. So G contains cycles of lengths $2h+1$, $2h+2$, $2h+3$, $2h+4$ and $2h+5$, a contradiction. Therefore $t\in B\cup B'$, then $t_1t_2s\cup Q\cup s't\cup T_{t,t_1}$ is a cycle of length $2h+5$. So G contains cycles of lengths $2h+1$, $2h+2$, $2h+3$, $2h+4$ and $2h+5$, a contradiction. Therefore $t=t_2$ and s' is the neighbor of t_2. That says, every vertex in H_2 of distance 2 from a neighbor of t_2 is a neighbor of t_2. Continuing to apply this along with a path from s to an odd cycle C_2 in H_2, we could obtain that t_2 is adjacent to all vertices of C_2, which contradicts that G is K_3-free.

Suppose that $N_{A\cup A'}(H_2)\neq\emptyset$ and $N_{B\cup B'}(H_2)\neq\emptyset$. Suppose that there exists a path $p_0p_1p_2p_3$ in H_2 such that p_0 is type-A and p_3 is type-B. Let q be the neighbor of p_0 in $A\cup A'$ and q' be the neighbor of p_3 in $B\cup B'$. Then $qp_0p_1p_2p_3q'\cup T_{q',q}$ is a cycle of length $2h+5$. So G contains cycles of lengths $2h+1$, $2h+2$, $2h+3$, $2h+4$ and $2h+5$, a contradiction. This forces that every two vertices which are linked by a path of length 3 in H_2 have the same type. Note that $N_{A\cup A'}(H_2)\neq\emptyset$ and $N_{B\cup B'}(H_2)\neq\emptyset$. By symmetry between $A\cup A'$ and $B\cup B'$, there exists a path $z_0z_1z_2$ in H_2 such that z_0 and z_1 are type-A and z_2 is type-B. Moreover, for any path $P'':=u_0u_1\cdots u_s$ in H_2 with $u_0=z_0$, $u_1=z_1$, $u_2=z_2$, we can derive that u_j is type-A if $j\equiv 0$ or $1 \bmod 3$ and u_j is type-B if $j\equiv 2 \bmod 3$. Moreover, for any path $P''':=u_0u_1\cdots u_s$ in H_2 with $u_0=z_2$, $u_1=z_1$, $u_2=z_0$, we can derive that u_j is type-B if $j\equiv 0 \bmod 3$ and u_j is type-A if $j\equiv 1$ or $2 \bmod 3$. This forces that every cycle in H_2 has length 0 modulo 3. Since H_2 is non-bipartite and K_3-free, there is an odd cycle $C_3:=w_0w_1\cdots w_mw_0$ of length at least 9. Note that w_0 and w_8 have different types. If follows that there is a cycle of length $2h+10$. So G contains cycles of lengths $2h+1$, $2h+2$, $2h+3$, $2h+4$ and $2h+10$, a contradiction.

Therefore, all vertices in H_2 have the same type.

Without loss of generality, we may assume that $N_{L_i}(H_2) \subseteq A \cup A'$. Therefore $v \in A$ and let f_0 be a neighbor of v in H_2. Since H_2 is K_3-free and non-bipartite, there is a path $f_0 f_1 f_2$ in H_2. Since H_1 is a K_3-free non-bipartite graph and v is a bad vertex in H_1, there is a path $a_0 a_1 v a_2 a_3$ in H_1. Since there is no A-B path of length 5 in H_1, we have that for any path $Q' := u_0 u_1 \cdots u_s$ in H_1 with $u_0 = a_0$, $u_1 = a_1$, $u_2 = v$, $u_3 = a_2$, $u_4 = a_3$, we can derive that u_j and u_k are in the same part if $j \equiv k \bmod 5$. Also, we have that for any path $Q' := u_0 u_1 \cdots u_s$ in H_1 with $u_0 = a_3$, $u_1 = a_2$, $u_2 = v$, $u_3 = a_1$, $u_4 = a_0$, we can derive that u_j and u_k are in the same part if $j \equiv k \bmod 5$. By Property ★, we have that a_1 and a_2 are in the same part of H_1.

Suppose that a_1, $a_2 \in A$. Since $V(H_1) \cap B \neq \emptyset$, we have that one of a_0 and a_3 is in B. Without loss of generality, we may assume that $a_0 \in B$. Let w be a neighbor of f_1 in H_1. We have that $w \in A \cup A'$. Since G is K_3-free, $w \neq v$. Note that $a_0 a_1 v$ satisfying that a_0 and v are in different parts of H_1. By Property ★, we have that $w \neq a_1$. Therefore, $w f_1 f_0 v a_1 a_0 \cup T_{a_0, w}$ is a cycle of length $2h+5$. So G contains cycles of lengths $2h+1$, $2h+2$, $2h+3$, $2h+4$ and $2h+5$, a contradiction.

Therefore, a_1, $a_2 \in B$. Let w' be a neighbor of f_2 in H_1. We have that $w' \in A \cup A'$. Suppose that $w' \neq v$. Then $w' f_2 f_1 f_0 v a_1 \cup T_{a_1, w'}$ is a cycle of length $2h+5$. So G contains cycles of lengths $2h+1$, $2h+2$, $2h+3$, $2h+4$ and $2h+5$, a contradiction. Therefore $w' = v$. That says, every vertex in H_2 of distance 2 from a neighbor of v is a neighbor of v. Continuing to apply this along with a path from f_0 to an odd cycle C_4 in H_2, we could obtain that v is adjacent to all vertices of C_4, which contradicts that G is K_3-free.

This completes the proof of Claim 2.

Now, we define a coloring $c: V(G) \to \{1, 2, 3, 4, 5\}$ as follows. Let D be any bipartite component of $G[L_i]$ for some i. If i is even, we color one part of D with color 1 and the other part with color 2, and if i is odd, we color one part of D with color 4 and the other part with color 5. Let F be any non-bipartite component of $G[L_j]$ for some j. If j is even, by using the block structure of F, we can properly color $V(F)$ with colors 1, 2 and 3 by coloring bad vertices with colors 1, 2 and 3 and coloring good vertices with colors 1 and 2. If j is odd, then we also can properly color $V(F)$ with colors 3, 4 and 5 by coloring bad vertices with colors 3, 4 and 5 and coloring good vertices with colors 4 and 5.

Next, we argue that c is a proper coloring on G. Let H_1 be a component of $G[L_i]$ and H_2 be a component of $G[L_{i+1}]$ for $i \geqslant 0$ such that there exists an edge between H_1 and H_2. If one of them is bipartite, then c is proper on $V(H_1) \cup V(H_2)$. Therefore, both H_1 and H_2 are non-bipartite. By the above claim, all vertices of H_2 are not adjacent to vertices of color 3 in H_1. It follows that c is proper on $V(H_1) \cup V(H_2)$. Therefore, c is a proper 5-coloring of G, which contradicts that G is 6-critical. This completes the proof of Theorem 5.4.

Proof of Theorem 1.2 Let G be a $(k+1)$-critical non-complete graph, for $k \in \{4, 5\}$. Suppose that $k=4$. It is well-known that G is a 2-connected graph of minimum degree at least 4. Then by Theorem 4.1, G contains cycles of all lengths modulo 4. Suppose that $k=5$. By Theorem 5.4, G contains cycles of all lengths modulo 5.

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Conflict of interest

The author declares no conflict of interest.

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染色数为 5 和 6 的图中的圈长

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摘要: Moore 和 West 提出问题:每一个$(k+1)$-临界的非完全图中是否存在一个模 k 的意义下长度为 2 的圈. 这里证明了更强的结论:对于 $k=4, 5$,每一个$(k+1)$-临界的非完全图中一定存在模 k 的意义下所有长度的圈.
关键词: 圈长;染色数;最小度;广度优先搜索树